Elasticity and Seismic Waves

- Concepts of macroscopic mechanics of solids
- Rock as 'elastic continuum'
  - Elastic body is deformed in response to stress
  - Two types of deformation (strain): Changes in volume and shape
- Equations of motion
- Wave equations
- Plane and spherical waves

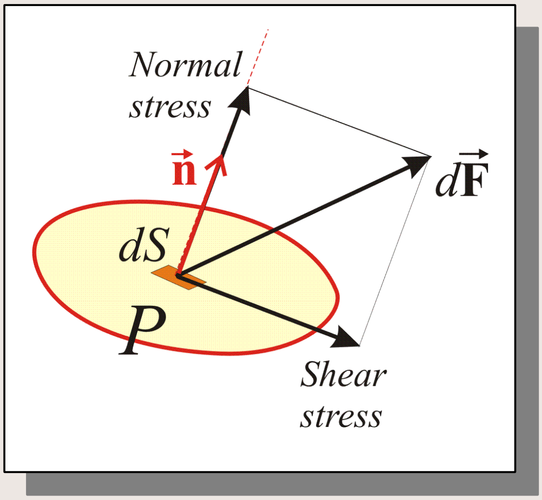
- Reading:
  - Shearer, Sections 2, 3
  - Telford *et al.*, Section 4.2.

Mechanical properties of continuous solids (or fluids)

- In seismic waves, we consider only small deformations (one part out million or less)
  - Particles only oscillate slightly near equilibrium points, and the behaviour of fluids/gases is similar to solids
- I try explaining mechanics of continuous solid media through analogies with a mass suspended on a spring:
  - Mass of the body m is analogous to density ρ
  - Force vector **F** is analogous to stress tensor $\boldsymbol{\sigma}$
  - Extension of the spring **x** – to strain tensor $\boldsymbol{\varepsilon}$
  - Hooke's law $\mathbf{F} = -k\mathbf{x}$ – to Hooke's law for solids
  - Elastic constant k – to elastic "moduli" for solids
  - Newton's law $m\mathbf{a} = \mathbf{F}$ also applies in both cases
  - Oscillations of mass m correspond to waves and multiple forms of free oscillations of solid bodies
- In the following slides, we go through these concepts one by one

Stress

- Consider the interior of a deformed body:



At point *P*, force *d***F** acts on any infinitesimal area *dS*

*d***F** is proportional to *dS* *(shown on the next slide)*

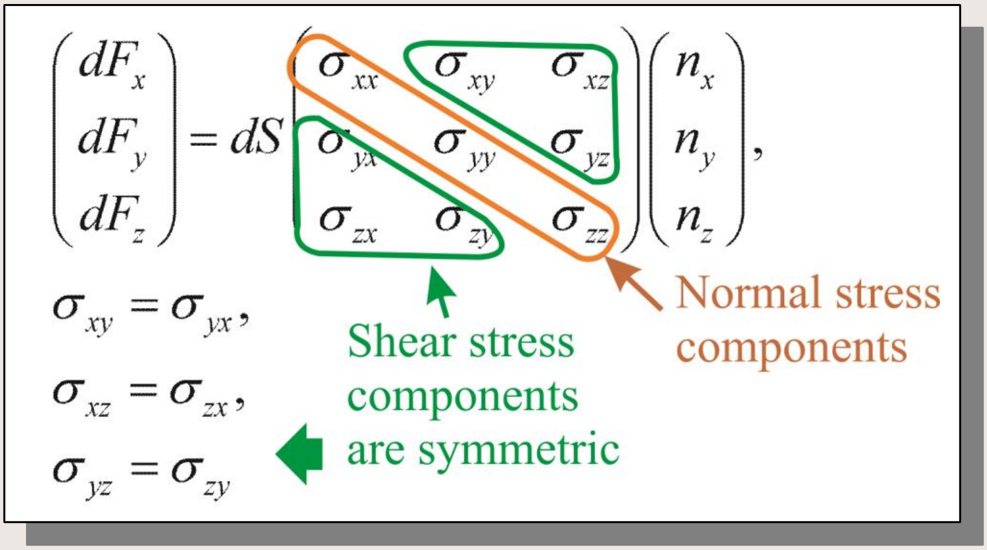
Stress, *with respect to direction* **n**, is a vector equal:

lim(*d***F**/*dS*) (as $dS \to 0$)

- Stress is measured in [*Newton/m*^{2}], or Pascal
  - Note that this is a unit of pressure
- *d***F** can be decomposed in two components relative to the surface or **n**:
  - Orthogonal to the surface (parallel to **n**; this called normal stress)
  - Tangential to the surface (shear stress)

Stress

- Stress, in general, is a *tensor*:
  - It is described in terms of 3 force components acting across each of 3 mutually orthogonal surfaces
  - 6 independent parameters
  - Force $d\mathbf{F}/dS$ depends on the orientation $\mathbf{n}$, but stress *does not*
  - Stress is best described by a matrix:



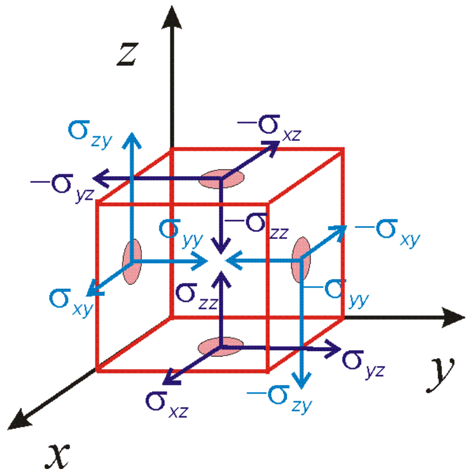
$$\begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = dS \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix},$$

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy}$$

- In a continuous medium, stress depends on (x,y,z,t) and thus it is a *field*

Forces acting on a small cube

- Consider a small cube within the elastic body. Assume dimensions of the cube equal '1'
- Both the *forces* and *torque* acting on the cube from the outside are balanced:



$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

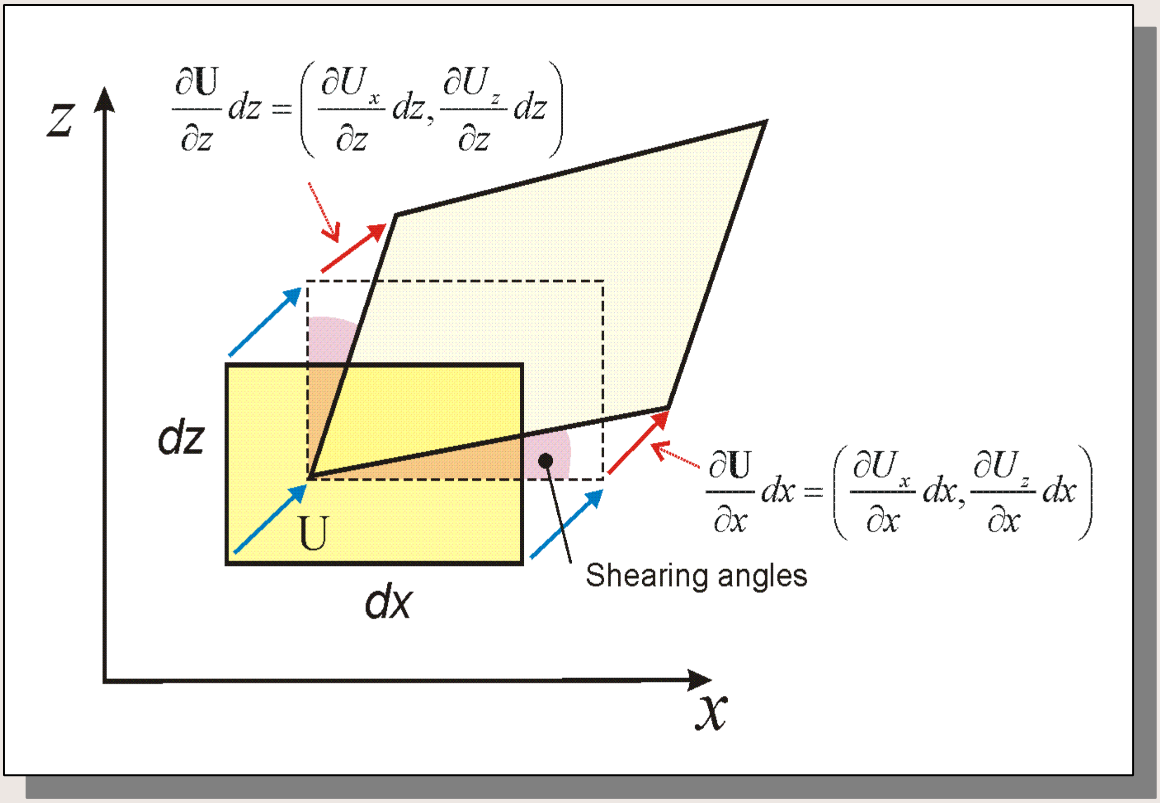
- In consequence, the stress tensor is *symmetric:* $\sigma_{ij} = \sigma_{ji}$
- The stress tensor is given by just 6 independent parameters out of 9

Strain within a deformed body

- Strain is a measure of deformation, *i.e.*, *variation of relative displacement* as associated with a *particular direction* within the body
- It is, therefore, also a *tensor*
  - Represented by a matrix
  - Like stress, it is decomposed into *normal* and *shear* components
- Seismic waves yield strains of 10^{-10}-10^{-6}
  - So we rely on infinitesimal strain theory

Elementary Strain

- When a body is deformed, *displacements* (**U**) of its points are dependent on (x,y,z), and consist of:
  - Translation (blue arrows below)
  - Deformation (red arrows)
- Elementary strain is simply $$e_{ij} = \frac{\partial U_i}{\partial x_j}$$



Strain Components

- However, certain forms of **U**(*x*,*y*,*z*) dependencies correspond to simple rotations of the body without changing its shape:
  - Deformation in which $\frac{\partial U_z}{\partial x} = -\frac{\partial U_x}{\partial z}$ is actually a rotation about the '*y*' axis.
  - So, the case of $\frac{\partial U_z}{\partial x} = \frac{\partial U_x}{\partial z}$ is called *pure shear* (no rotation)
- To characterize deformation without rotations, only the symmetric combination of the elementary strains is used:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right),$$

$$\varepsilon_{ij} = \varepsilon_{ji}, \text{ where } \mathrm{i, j} = x, \mathrm{y, or\ z}$$

$$\varepsilon = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{1}{2}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y}\right) & \frac{\partial U_y}{\partial y} & \frac{1}{2}\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial U_z}{\partial y} + \frac{\partial U_y}{\partial z}\right) & \frac{\partial U_z}{\partial z} \end{pmatrix}$$

Dilatational Strain
(relative volume change during deformation)

- Original volume: $V=\delta x \delta y \delta z$
- Deformed volume: $V+\delta V=(1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\delta x \delta y \delta z$
- Dilatational (volumetric) strain:

$$\Delta = \frac{\delta V}{V} = \left(1+\varepsilon_{xx}\right)\left(1+\varepsilon_{yy}\right)\left(1+\varepsilon_{zz}\right)-1 \approx \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}$$

$$\Delta = \varepsilon_{ii} = \partial_i U_i = \nabla_i U_i = \operatorname{div}\mathbf{U}$$

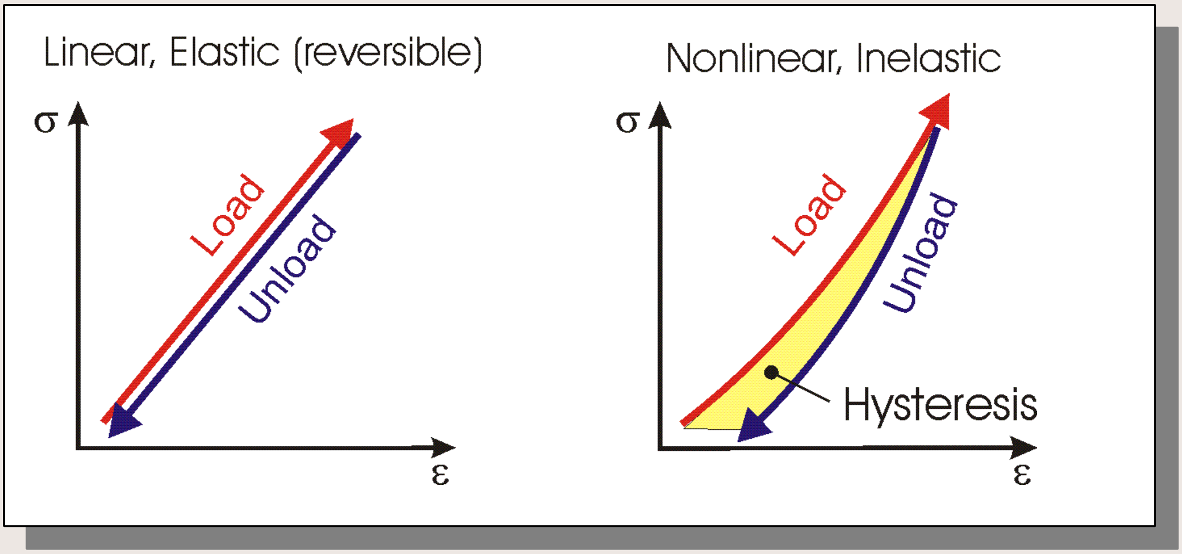
- Note that (as expected) shearing strain does not change the volume

Hooke's Law (general)

- Describes the stress developed in a deformed body:

  F = -*k***x** for an ordinary spring (1-D)

  $\sigma \sim \varepsilon$ (in some sense) for a '*linear*', '*elastic*' 3-D solid. This is what it means:



- For a general (*anisotropic*) medium, there are 36 coefficients of proportionality between six independent σ_{ij} and six ε_{ij}

Hooke's Law (isotropic medium)

- For isotropic medium, the strain/stress relation is described by just two constants:

  $\sigma_{ij} = \lambda\Delta + 2\mu\varepsilon_{ij}$ for normal strain/stress (i=j, where $i,j = x,y,z$)

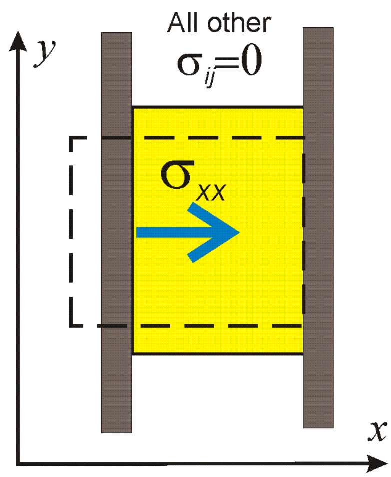
  $\sigma_{ij} = 2\mu\varepsilon_{ij}$ for shear components ($i \neq j$)

  - λ and μ are called the *Lamé constants*.

Empirical Elastic Moduli

Young’s (extensional) modulus and Poisson’s ratio

- Lamé modulus λ practically never acts alone or is observed in experiments
- Depending on boundary conditions (*i.e.*, experimental setup), combinations of λ and μ are measured. These combinations are called *elastic constants*, or moduli:
  - Young's modulus and Poisson's ratio:
    - Consider a cylindrical sample uniformly squeezed or stretched along axis *X*:



$$\sigma_{xx} = \lambda'\Delta + 2\mu\varepsilon_{xx},$$
$$\sigma_{yy} = \lambda'\Delta + 2\mu\varepsilon_{yy} = 0,$$
$$\sigma_{zz} = \lambda'\Delta + 2\mu\varepsilon_{zz} = 0 \Rightarrow \varepsilon_{yy} = \varepsilon_{zz} = \frac{-\lambda'\Delta}{2\mu}.$$

Young's modulus: $E = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{2\mu(3\lambda'+2\mu)}{\lambda'+\mu}$

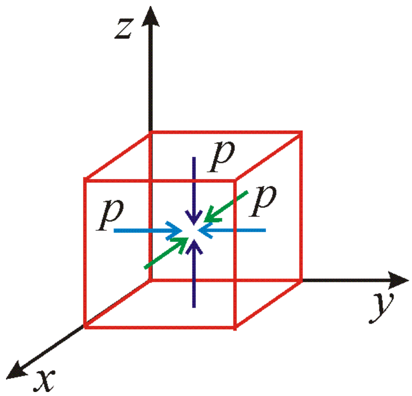
Poisson's ratio: $\nu = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda'}{2(\lambda'+\mu)}$

Often denoted σ.

Empirical Elastic Moduli

Bulk and Shear

- Bulk modulus, *K*
  - Consider a cube subjected to hydrostatic pressure



$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p,$$

$$-3p = 3\lambda'\Delta + 2\mu\Delta$$

Bulk modulus: $K = \frac{-p}{\Delta} = \lambda' + \frac{2}{3}\mu$

- Finally, the constant μ complements *K* in describing the shear rigidity of the medium, and thus it is also called '*rigidity modulus*'
- For rocks:
  - Generally, 10 GPa < μ < *K* < *E* < 200 GPa
  - 0 < ν < ½ always; for rocks, 0.05 < ν < 0.45, for most "hard rocks", ν is near 0.25
- For fluids, ν = ½ and μ = 0 (no shear resistance)

Empirical Elastic Moduli

P-wave and S-wave

- From seismic waves, another pair of empirical moduli is obtained from measured wave velocities:

$$V_P = \sqrt{\frac{M}{\rho}} \qquad V_S = \sqrt{\frac{\mu}{\rho}}$$

  - "P-wave modulus" M corresponds to compression-extension deformations in one direction only
  - "S wave modulus" corresponds to shear deformations (without volume change) transversely to wave propagation
    - This modulus is the same as μ

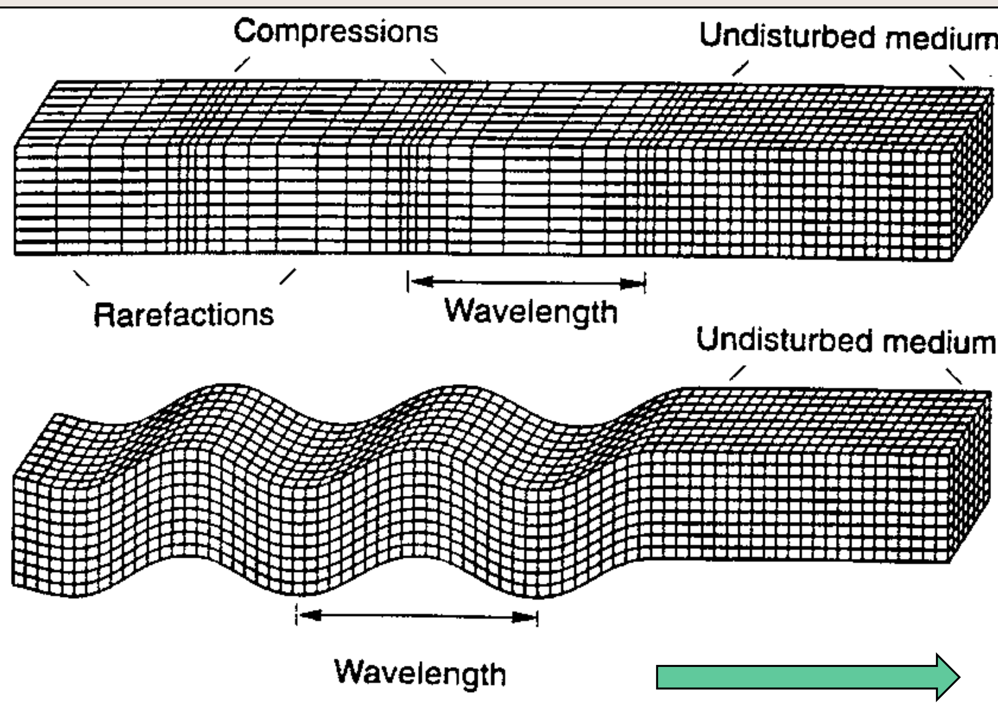
P wave
("primary")

Modulus M

$$M = \lambda + 2\mu$$

S wave
("secondary", "shear")

Modulus μ



Strain/Stress Energy Density

- Mechanical work is required to deform an elastic body; as a result, elastic energy is accumulated in the strain/stress field
- When released, this energy gives rise to earthquakes and seismic waves
- For a loaded spring (1-D elastic body), $E = \frac{1}{2}kx^2 = \frac{1}{2}Fx$
- Similarly, for a deformed elastic medium, *energy density* is:

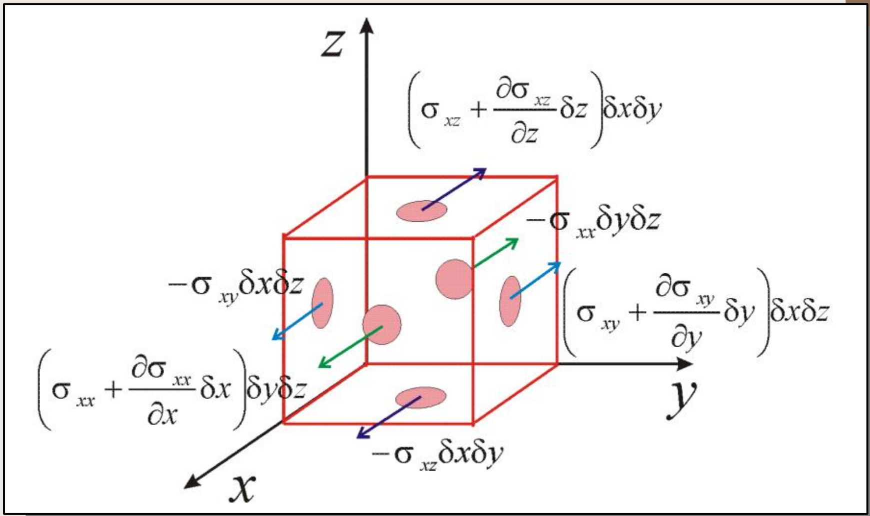
$$E = \frac{1}{2} \sum_{i,j=x,y,z} \sigma_{ij} \varepsilon_{ij}$$

- Energy density (per unit volume) is thus measured in:

$$\left[\frac{\text{Newton} \times \text{m}}{\text{m}^3}\right] = \left[\frac{\text{Newton}}{\text{m}^2}\right] = [\text{Pa}]$$

Inhomogeneous Stress

If stress is inhomogeneous (variable in space), its derivatives result in a *net force* acting on an infinitesimal volume:



$$
\begin{aligned}
F_x &= \left[\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}\delta x\right)-\sigma_{xx}\right]\delta y\delta z \\
&+\left[\left(\sigma_{xy}+\frac{\partial\sigma_{xy}}{\partial y}\delta y\right)-\sigma_{xy}\right]\delta x\delta z \\
&+\left[\left(\sigma_{xz}+\frac{\partial\sigma_{xz}}{\partial z}\delta z\right)-\sigma_{xz}\right]\delta x\delta y = \left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}\right)\delta y\delta y\delta z
\end{aligned}
$$

Thus, for $i = x, y, z$:

$$
F_i = \left(\frac{\partial\sigma_{ix}}{\partial x}+\frac{\partial\sigma_{iy}}{\partial y}+\frac{\partial\sigma_{iz}}{\partial z}\right)\delta V
$$

Equations of Motion

(Govern motion of the elastic body with time)

Uncompensated net force will result in *acceleration* (Newton's law):

Newton's law: $$\rho\delta V\frac{\partial^2 U_i}{\partial t^2}=F_i$$

$$\rho\frac{\partial^2 U_i}{\partial t^2}=\left(\frac{\partial\sigma_{ix}}{\partial x}+\frac{\partial\sigma_{iy}}{\partial y}+\frac{\partial\sigma_{iz}}{\partial z}\right)$$

$$\rho\frac{\partial^2 U_x}{\partial t^2}=\frac{\partial}{\partial x}\left(\lambda'\Delta+2\mu\frac{\partial U_x}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial U_x}{\partial y}+\frac{\partial U_y}{\partial x}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial U_x}{\partial z}+\frac{\partial U_z}{\partial x}\right)$$
$$=\lambda'\frac{\partial\Delta}{\partial x}+\mu\frac{\partial}{\partial x}\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)+\mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right)$$
$$=(\lambda'+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 U_x$$

These are the *equations of motion* for each of the components of **U**:

$$\rho\frac{\partial^2 U_x}{\partial t^2}=(\lambda'+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 U_x$$
$$\rho\frac{\partial^2 U_y}{\partial t^2}=(\lambda'+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 U_y$$
$$\rho\frac{\partial^2 U_z}{\partial t^2}=(\lambda'+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 U_z$$

Wave Equation
(Propagation of compressional/acoustic waves)

- To show that these three equations describe several types of waves, first let's apply *divergence operation* to them:

$$\rho \frac{\partial^2 \Delta}{\partial t^2} = (\lambda' + \mu)\nabla^2 \Delta + \mu \nabla^2 \Delta = (\lambda' + 2\mu)\nabla^2 \Delta$$

- This *is* a wave equation; compare to the general form of equation describing wave processes:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] f(x, y, z, t) = 0$$

- Above, c is the wave velocity.
- We have:

$$\left[\frac{\rho}{(\lambda' + 2\mu)}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\Delta = 0$$

- This equation describes *compressional* (P) waves
- P-wave velocity:

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

Wave Equation

(Propagation of shear waves)

- Similarly, let's apply the *curl operation* to the equations for **U** (remember, **curl**(**grad**) = 0 for any field:

$$\rho \frac{\partial^2}{\partial t^2} \operatorname{curl} \mathbf{U} = \mu \nabla^2 \operatorname{curl} \mathbf{U}$$

- This is also a wave equation; again compare to the general form:

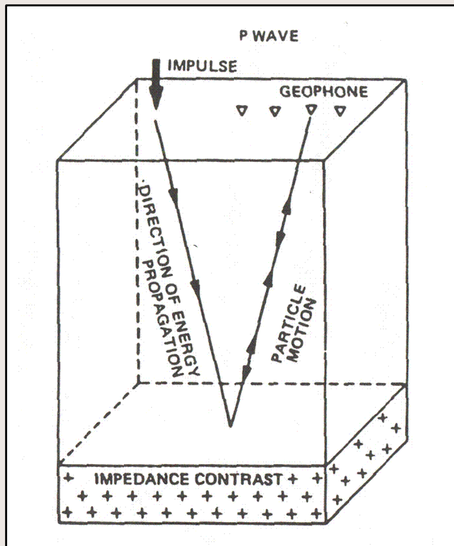
$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] f(x, y, z, t) = 0$$

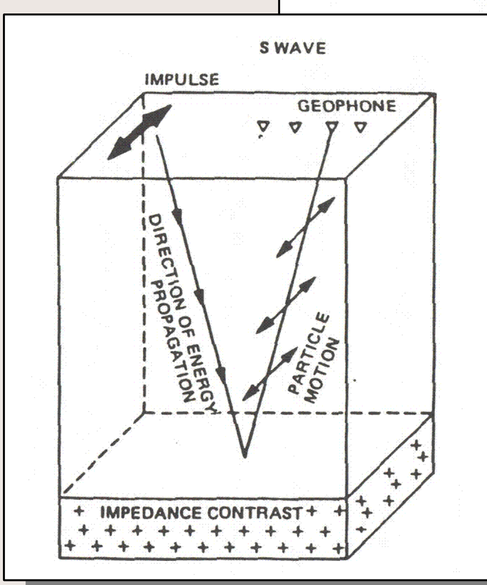
- This equation describes *shear* (*S*), or *transverse* waves.
- Since it involves rotation, there is no associated volume change, and particle motion is *across* the wave propagation direction.
- Its velocity: $V_S < V_P$,
- For $\nu = 0.25$, $\frac{V_P}{V_S} = \sqrt{3}$

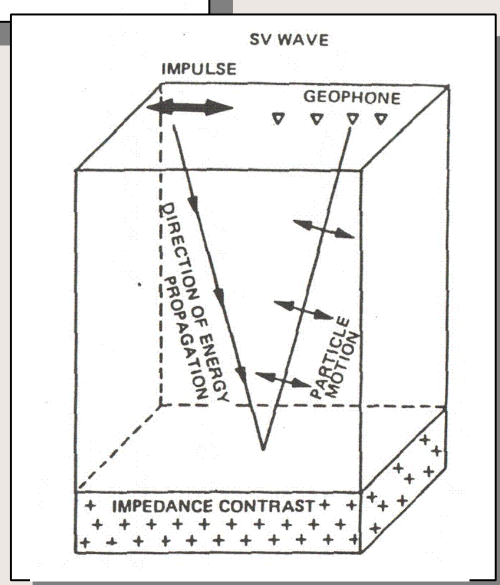
$$V_S = \sqrt{\frac{\mu}{\rho}}$$

Wave Polarization

- Thus, elastic solid supports two types of *body waves*:







$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}} \,, \qquad v_S = \sqrt{\frac{\mu}{\rho}}$$

Waveforms and wave fronts

Plane waves

- Consider the wave equation:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]f(x,y,z,t)=0$$

- Why does it describe a wave? Note that it is satisfied with any function of the form:

$$f(x,y,z,t)=\varphi(x-ct) \qquad f(x,y,z,t)=\varphi(x+ct)$$

- The function $\varphi()$ is *the waveform*. Note that the entire waveform propagates with time to the right or left along the x-axis , $x=\pm ct$. This is what is called the *wave process*.
- The argument of $\varphi(...)$ is called *phase*
- Surfaces of constant phase are called *wavefronts*
  - In our case, the wavefronts are planes: x = phase $\pm ct$ for any (y,z).
  - For this reason, the above solutions are *plane waves*

Waveforms and wave fronts

Non-planar waves

- The wave equation is also satisfied by such solutions (*spherical waves*):

$$f(\mathbf{r},t)=\frac{1}{|\mathbf{r}|}\phi(|\mathbf{r}|-ct)$$

$$f(\mathbf{r},t)=\frac{1}{|\mathbf{r}|}\phi(|\mathbf{r}|+ct)$$

Spreading away from point **r** = 0

Converging to **r** = 0

...and by such (cylindrical *waves*):

$$f(\rho,t)=\frac{1}{\sqrt{\rho}}\phi(\rho-ct)$$

$$f(\rho,t)=\frac{1}{\sqrt{\rho}}\phi(\rho+ct)$$

Spreading away from ρ = 0

Converging to ρ = 0

...and by various other solutions

Question: what is the problem with the second solution in each pair?

Waves and sources

- Homogeneous wave equation describes free waves:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial^2 t} - \nabla^2\right) f = 0$$

  - plane, spherical, cylindrical...
  - incoming, outgoing...

- Inhomogeneous equation describes waves generated by a source:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial^2 t} - \nabla^2\right) f = source$$

- Note that this also includes all of the free waves, and so one also needs boundary conditions to specify a unique solution
  - For example, no waves usually come from infinity toward the source (this is called the "*radiation condition*")