Elasticity and Seismic Waves

- Concepts of macroscopic mechanics of solids
- Rock as 'elastic continuum'
  - Elastic body is deformed in response to stress
  - Two types of deformation (strain): Changes in volume and shape
- Equations of motion
- Wave equations
- Plane and spherical waves

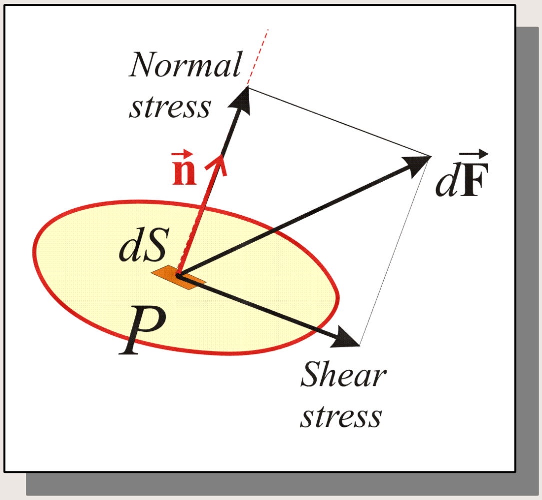
- Reading:
  - Shearer, Sections 2, 3
  - Telford *et al*., Section 4.2.

Mechanical properties of continuous solids (or fluids)

- In seismic waves, we consider only small deformations (one part out million or less)
  - Particles only oscillate slightly near equilibrium points, and the behaviour of fluids/gases is similar to solids
- I try explaining mechanics of continuous solid media through analogies with a mass suspended on a spring:
  - Mass of the body m is analogous to density ρ
  - Force vector **F** is analogous to stress tensor $\boldsymbol{\sigma}$
  - Extension of the spring **x** – to strain tensor $\boldsymbol{\varepsilon}$
  - Hooke's law $\mathbf{F} = -k\mathbf{x}$ – to Hooke's law for solids
  - Elastic constant k – to elastic "moduli" for solids
  - Newton's law $m\mathbf{a} = \mathbf{F}$ <u>also applies in both cases</u>
  - Oscillations of mass m correspond to <u>waves</u> and multiple forms of <u>free oscillations</u> of solid bodies
- In the following slides, we go through these concepts one by one

Stress

- Consider the interior of a deformed body:



At point *P*, force *d***F** acts on any infinitesimal area *dS*

*d***F** is proportional to *dS*
(shown on the next slide)

Stress, *with respect to direction* **n**, is a vector equal:

lim(*d***F**/*dS*) (as $dS \to 0$)

- Stress is measured in [*Newton/m*^{2}], or Pascal
  - Note that this is a unit of pressure
- *d***F** can be decomposed in two components relative to the surface or **n**:
  - Orthogonal to the surface (parallel to **n**; this called normal stress)
  - Tangential to the surface (shear stress)

Stress

- Stress, in general, is a *tensor*:
  - It is described in terms of 3 force components acting across each of 3 mutually orthogonal surfaces
  - 6 independent parameters
  - Force $d\mathbf{F}/dS$ depends on the orientation **n**, but stress *does not*
  - Stress is best described by a matrix:

$$\begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = dS \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix},$$

Normal stress components

$$\sigma_{xy} = \sigma_{yx},$$

$$\sigma_{xz} = \sigma_{zx},$$

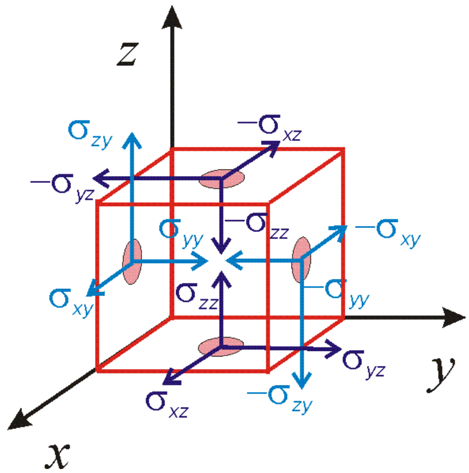
$$\sigma_{yz} = \sigma_{zy}$$

Shear stress components are symmetric

- In a continuous medium, stress depends on (x,y,z,t) and thus it is a *field*

Forces acting on a small cube

- Consider a small cube within the elastic body. Assume dimensions of the cube equal '1'
- Both the *forces* and *torque* acting on the cube from the outside are balanced:



$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

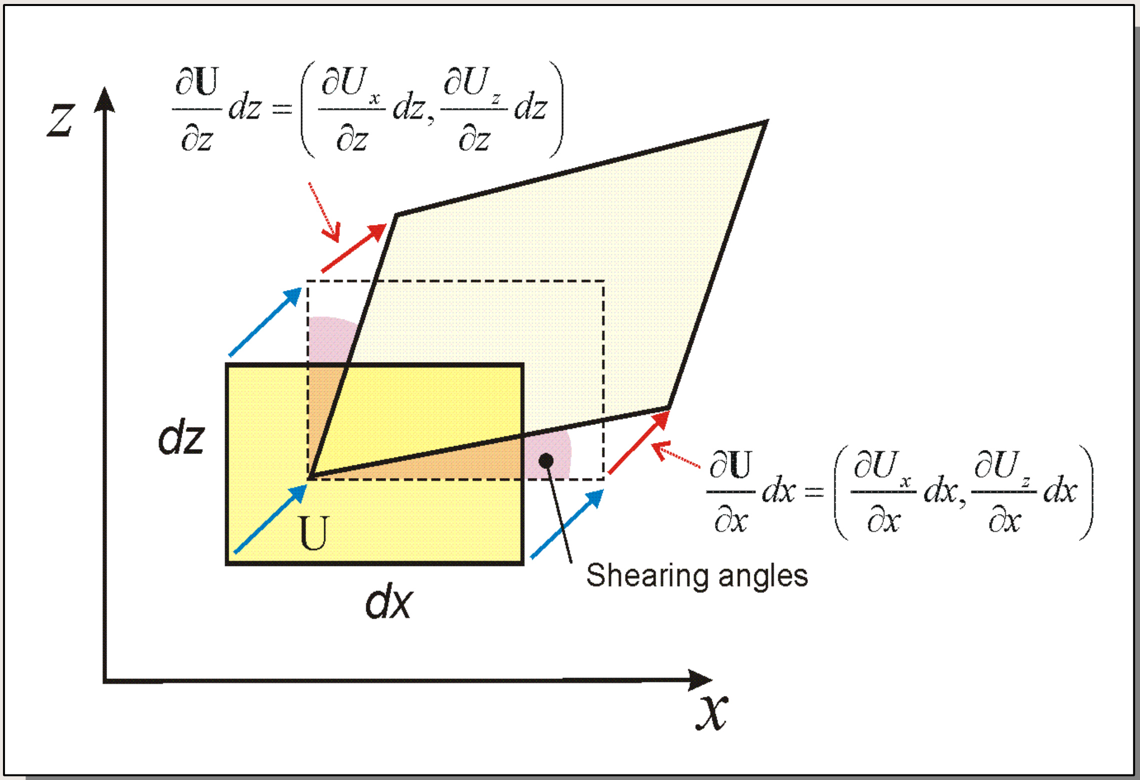
- In consequence, the stress tensor is *symmetric:* $\sigma_{ij} = \sigma_{ji}$
- The stress tensor is given by just <u>6 independent parameters</u> out of 9

Strain within a deformed body

- Strain is a measure of deformation, *i.e.*, *variation of relative displacement* as associated with a *particular direction* within the body
- It is, therefore, also a *tensor*
  - Represented by a matrix
  - Like stress, it is decomposed into *normal* and *shear* components
- Seismic waves yield strains of 10^{-10}-10^{-6}
  - So we rely on infinitesimal strain theory

Elementary Strain

- When a body is deformed, *displacements* (**U**) of its points are dependent on (x,y,z), and consist of:
  - Translation (blue arrows below)
  - Deformation (red arrows)
- Elementary strain is simply $$e_{ij} = \frac{\partial U_i}{\partial x_j}$$



Strain Components

- However, certain forms of **U**(x,y,z) dependencies correspond to simple rotations of the body without changing its shape:
  - Deformation in which $\frac{\partial U_z}{\partial x} = -\frac{\partial U_x}{\partial z}$ is actually a rotation about the 'y' axis.
  - So, the case of $\frac{\partial U_z}{\partial x} = \frac{\partial U_x}{\partial z}$ is called *pure shear* (no rotation)
- To characterize <u>deformation without rotations</u>, only the <u>symmetric combination</u> of the elementary strains is used:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right),$$

$$\varepsilon_{ij} = \varepsilon_{ji}, \text{ where } \mathrm{i, j} = x, \mathrm{y}, \text{ or } \mathrm{z}$$

$$\varepsilon = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{1}{2}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y}\right) & \frac{\partial U_y}{\partial y} & \frac{1}{2}\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial U_z}{\partial y} + \frac{\partial U_y}{\partial z}\right) & \frac{\partial U_z}{\partial z} \end{pmatrix}$$

Dilatational Strain
(relative volume change during deformation)

- Original volume: $V=\delta x \delta y \delta z$
- Deformed volume:
  $V+\delta V=(1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\delta x \delta y \delta z$
- Dilatational (volumetric) strain:

$$\Delta = \frac{\delta V}{V} = \left(1+\varepsilon_{xx}\right)\left(1+\varepsilon_{yy}\right)\left(1+\varepsilon_{zz}\right)-1 \approx \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}$$

$$\Delta = \varepsilon_{ii} = \partial_i U_i = \nabla_i U_i = \operatorname{div} \mathbf{U}$$

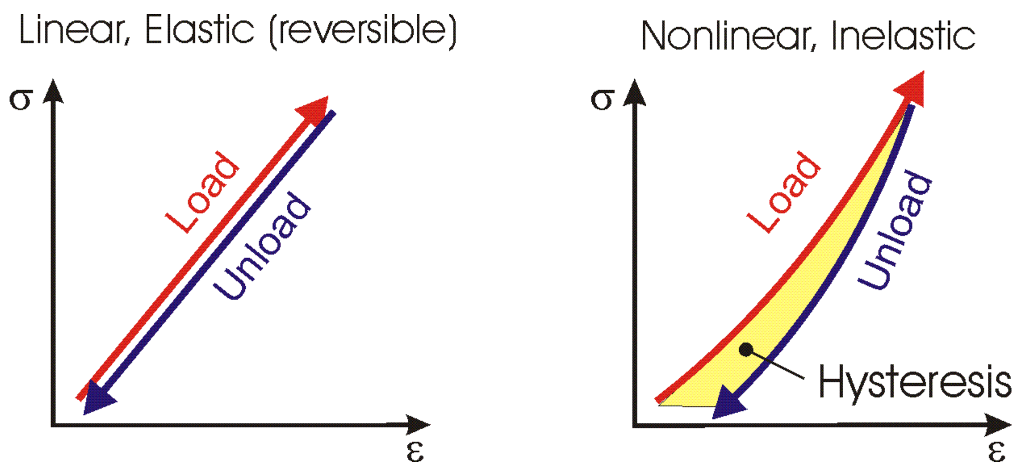
- Note that (as expected) shearing strain does not change the volume

Hooke's Law (general)

- Describes the stress developed in a deformed body:

  F = -k**x** for an ordinary spring (1-D)

  $\sigma \sim \varepsilon$ (in some sense) for a '*linear*', '*elastic*' 3-D solid. This is what it means:



During loading and unloading, material passes through the same (ε,σ) points (reversible process)

During unloading, material passes through larger strains (ε) and does not return to the initial state (irrreversible process)

Hooke's Law (general)

- For a general (*anisotropic*) medium, there are 36 coefficients of proportionality between six independent values of ε_{ij}. and six σ_{ij}. These coefficients form the "rigidity" matrix **C**:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \mathbf{C} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{pmatrix}$$

- Matrix **C** also turns out to be symmetric, and so there exist 21 independent elastic (rigidity) constants for an anisotropic solid

Hooke's Law (isotropic medium)

- For <u>isotropic</u> medium, the strain/stress relation is described by just two constants:

  $\sigma_{ij} = \lambda\Delta + 2\mu\varepsilon_{ij}$ for normal strain/stress ($i=j$, where $i,j = x,y,z$)

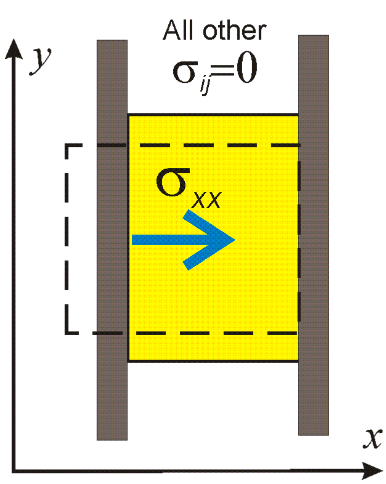
  $\sigma_{ij} = 2\mu\varepsilon_{ij}$ for shear components ($i \neq j$)

  - λ and μ are called the *Lamé constants*.

Empirical Elastic Moduli

Young's (extensional) modulus and Poisson's ratio

- Lamé modulus λ practically never acts alone and is not observed in experiments
- Depending on boundary conditions (*i.e.*, on experimental setup), different combinations of λ and μ are measured. These combinations are called (empirical) *elastic constants*, or moduli
- Similar to λ and μ, elastic constants come in pairs:
  - Young's modulus and Poisson's ratio:
    - Consider a cylindrical sample uniformly squeezed or stretched along axis X:



$$\sigma_{xx} = \lambda'\Delta + 2\mu\varepsilon_{xx},$$

$$\sigma_{yy} = \lambda'\Delta + 2\mu\varepsilon_{yy} = 0,$$

$$\sigma_{zz} = \lambda'\Delta + 2\mu\varepsilon_{zz} = 0 \Rightarrow \varepsilon_{yy} = \varepsilon_{zz} = \frac{-\lambda'\Delta}{2\mu}.$$

Young's modulus: $$E = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{2\mu(3\lambda'+2\mu)}{\lambda'+\mu}$$

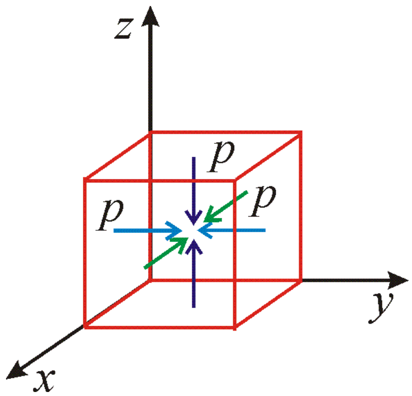
Poisson's ratio: $$\nu = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda'}{2(\lambda'+\mu)}$$

Often denoted σ.

Empirical Elastic Moduli

Bulk and Shear

- Bulk modulus, *K*
  - Consider a cube subjected to hydrostatic pressure



$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p,$$

$$-3p = 3\lambda'\Delta + 2\mu\Delta$$

Bulk modulus: $K = \dfrac{-p}{\Delta} = \lambda' + \dfrac{2}{3}\mu$

- Finally, the constant μ complements *K* in describing the shear rigidity of the medium, and thus it is also called the '*rigidity modulus*'
- For rocks:
  - Generally, 10 GPa < μ < *K* < *E* < 200 GPa
  - 0 < ν < ½ always; for rocks, 0.05 < ν < 0.45, for most "hard rocks", ν is near 0.25
- For fluids, $\nu = ½$ and $\mu = 0$ (no shear resistance)

Empirical Elastic Moduli

P-wave and S-wave

- From seismic waves, another pair of empirical moduli is obtained from measured wave velocities:

$$V_P = \sqrt{\frac{M}{\rho}} \qquad V_S = \sqrt{\frac{\mu}{\rho}}$$

  - "P-wave modulus" *M* corresponds to compression-extension deformations in one direction only
  - "S wave modulus" corresponds to shear deformations (without volume change) transversely to wave propagation
    - This modulus is the same as μ

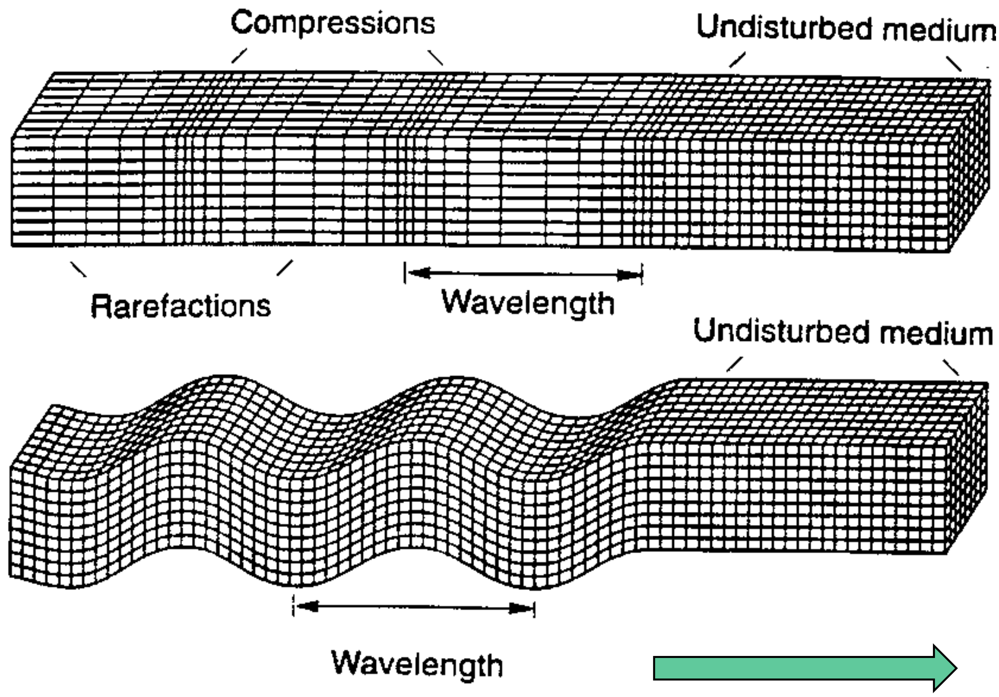
P wave
("primary")

Modulus *M*

$$M = \lambda + 2\mu$$

S wave
("secondary", "shear")

Modulus μ



Strain/Stress
Energy Density

- Mechanical work is required to deform an elastic body; as a result, elastic energy is accumulated in the strain/stress field
- When released, this energy gives rise to earthquakes and seismic waves
- For a loaded spring (1-D elastic body), $E = \frac{1}{2}kx^2 = \frac{1}{2}Fx$
- Similarly, for a deformed elastic medium, *energy density* is:

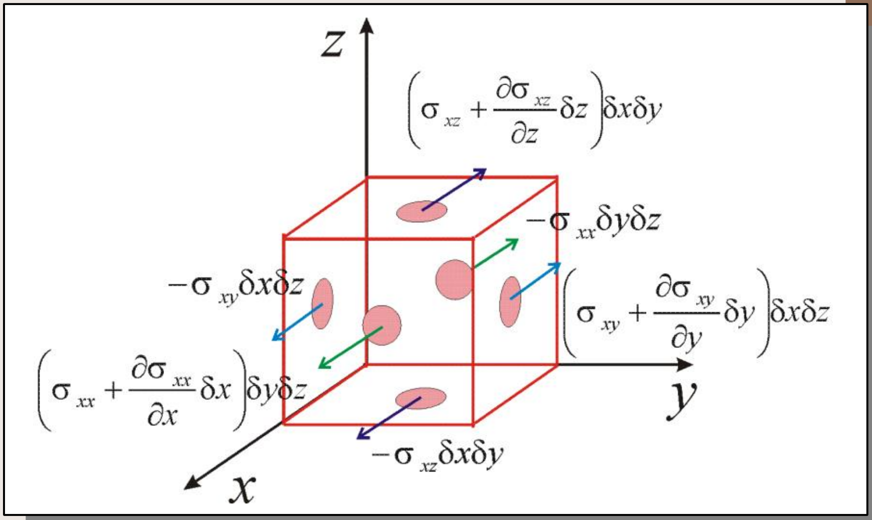
$$\boxed{E = \frac{1}{2}\sum_{i,j=x,y,z}\sigma_{ij}\varepsilon_{ij}}$$

- Energy density (per unit volume) is thus measured in:

$$\left[\frac{\text{Newton}\times\text{m}}{\text{m}^3}\right]=\left[\frac{\text{Newton}}{\text{m}^2}\right]=[\text{Pa}]$$

Inhomogeneous Stress

If stress is inhomogeneous (variable in space), its derivatives result in a *net force* acting on an infinitesimal volume:



$$
\begin{aligned}
F_x = &\left[\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\delta x\right) - \sigma_{xx}\right]\delta y \delta z \\
&+ \left[\left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y}\delta y\right) - \sigma_{xy}\right]\delta x \delta z \\
&+ \left[\left(\sigma_{xz} + \frac{\partial \sigma_{xz}}{\partial z}\delta z\right) - \sigma_{xz}\right]\delta x \delta y = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right)\delta y \delta y \delta z
\end{aligned}
$$

Thus, for $i = x, y, z$:

$$
F_i = \left(\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z}\right)\delta V
$$

Equations of Motion

(Govern motion of the elastic body with time)

Uncompensated net force will result in *acceleration* (Newton's law):

Newton's law: $$\rho\delta V\frac{\partial^2 U_i}{\partial t^2}=F_i$$

$$\rho\frac{\partial^2 U_i}{\partial t^2}=\left(\frac{\partial\sigma_{ix}}{\partial x}+\frac{\partial\sigma_{iy}}{\partial y}+\frac{\partial\sigma_{iz}}{\partial z}\right)$$

$$\rho\frac{\partial^2 U_x}{\partial t^2}=\frac{\partial}{\partial x}\left(\lambda'\Delta+2\mu\frac{\partial U_x}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial U_x}{\partial y}+\frac{\partial U_y}{\partial x}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial U_x}{\partial z}+\frac{\partial U_z}{\partial x}\right)$$

$$=\lambda'\frac{\partial\Delta}{\partial x}+\mu\frac{\partial}{\partial x}\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)+\mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right)$$

$$=(\lambda'+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 U_x$$

These are the *equations of motion* for each of the components of **U**:

$$\rho\frac{\partial^2 U_x}{\partial t^2}=(\lambda'+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 U_x$$

$$\rho\frac{\partial^2 U_y}{\partial t^2}=(\lambda'+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 U_y$$

$$\rho\frac{\partial^2 U_z}{\partial t^2}=(\lambda'+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 U_z$$

Wave Equation
(Propagation of compressional/acoustic waves)

- To show that these three equations describe several types of waves, first let's apply *divergence operation* to them:

$$\rho \frac{\partial^2 \Delta}{\partial t^2} = (\lambda' + \mu)\nabla^2 \Delta + \mu \nabla^2 \Delta = (\lambda' + 2\mu)\nabla^2 \Delta$$

- This *is* a wave equation; compare to the general form of equation describing wave processes:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] f(x, y, z, t) = 0$$

- Above, c is the wave velocity.
- We have:

$$\left[\frac{\rho}{(\lambda' + 2\mu)}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\Delta = 0$$

- This equation describes *compressional* (P) waves
- P-wave velocity:

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

Wave Equation

(Propagation of shear waves)

- Similarly, let's apply the *curl operation* to the equations for **U** (remember, **curl**(**grad**) = 0 for any field:

$$\rho \frac{\partial^2}{\partial t^2} \operatorname{curl} \mathbf{U} = \mu \nabla^2 \operatorname{curl} \mathbf{U}$$

- This is also a wave equation; again compare to the general form:

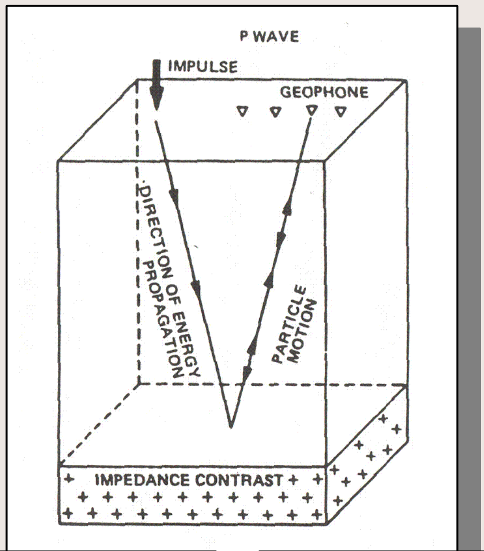
$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] f(x, y, z, t) = 0$$

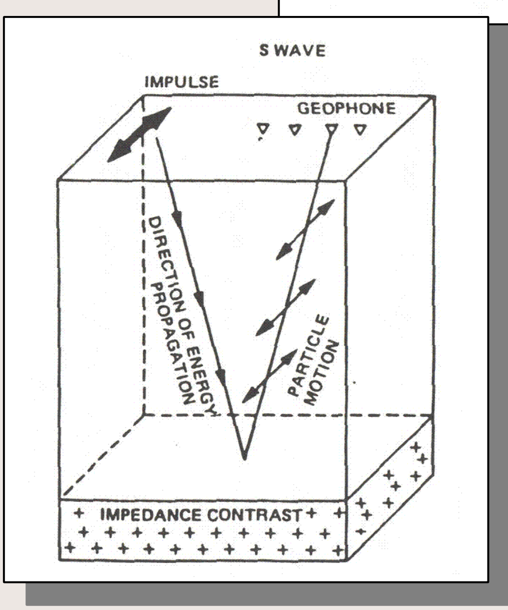
- This equation describes *shear* (*S*), or *transverse* waves.
- Since it involves rotation, there is no associated volume change, and particle motion is *across* the wave propagation direction.
- Its velocity: $V_S < V_P$,
- For $\nu = 0.25$, $\frac{V_P}{V_S} = \sqrt{3}$

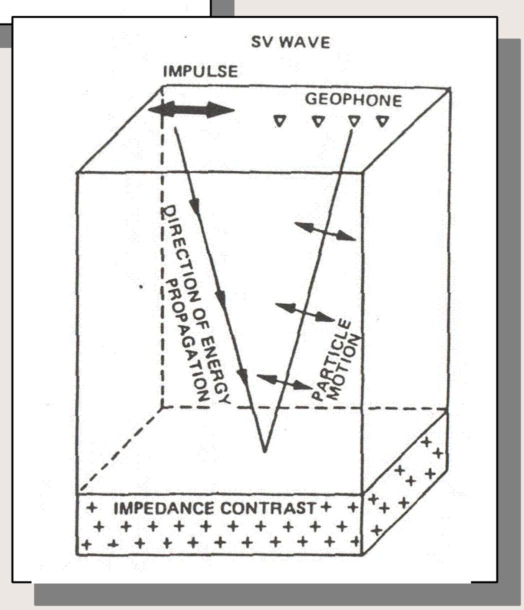
$$V_S = \sqrt{\frac{\mu}{\rho}}$$

Wave Polarization

- Thus, elastic solid supports two types of *body waves*:







$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}, \qquad v_S = \sqrt{\frac{\mu}{\rho}}$$

Waveforms and wave fronts

Plane waves

- Consider the wave equation:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] f(x, y, z, t) = 0$$

- Why does it describe a wave? Note that it is satisfied with any function of the form:

$$f(x, y, z, t) = \varphi(x - ct) \qquad f(x, y, z, t) = \varphi(x + ct)$$

- The function $\varphi()$ is *the waveform*. Note that the entire waveform propagates with time to the right or left along the x-axis , $x=\pm ct$. This is what is called the *wave process*.
- The argument of $\varphi(...)$ is called *phase*
- Surfaces of constant phase are called *wavefronts*
  - In our case, the wavefronts are planes:

$\boxed{x = \text{phase} \pm ct}$ for any (y,z).

  - For this reason, the above solutions are *plane waves*

Waveforms and wave fronts

Non-planar waves

- The wave equation is also satisfied by such solutions (*spherical waves*):

$$f(\mathbf{r},t)=\frac{1}{|\mathbf{r}|}\phi\left(|\mathbf{r}|-ct\right)$$

Spreading away from point **r** = 0

$$f(\mathbf{r},t)=\frac{1}{|\mathbf{r}|}\phi\left(|\mathbf{r}|+ct\right)$$

Converging to **r** = 0

...and by such (cylindrical *waves*):

$$f(\rho,t)=\frac{1}{\sqrt{\rho}}\phi(\rho-ct)$$

Spreading away from ρ = 0

$$f(\rho,t)=\frac{1}{\sqrt{\rho}}\phi(\rho+ct)$$

Converging to ρ = 0

...and by various other solutions

Question: what is the problem with the second solution in each pair?

Waves and sources

- Homogeneous wave equation describes free waves:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}-\nabla^2\right)f=0$$

  - plane, spherical, cylindrical...
  - incoming, outgoing...

- Inhomogeneous equation describes waves generated by a source:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}-\nabla^2\right)f=source$$

- Note that this also includes all of the free waves, and so one also needs boundary conditions to specify a unique solution
  - For example, no waves usually come from infinity toward the source (this is called the "*radiation condition*")