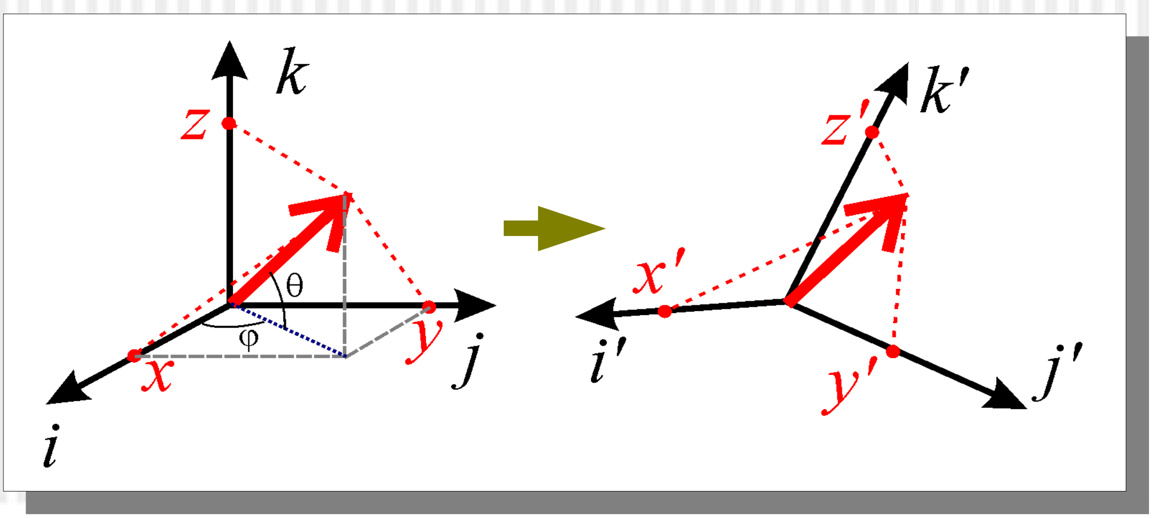
Key Concepts

- Scalars, Vectors, Tensors
- Matrices, Determinants
- Fields
- Wave equation
- Principle of superposition
- Boundary conditions
- Reading:
  - Telford et al., Sections A.2-3, A.5, A.7
  - Sheriff and Geldart, Chapter 15.1

Vector

- Directional quantity
  - Possesses 'amplitude' and 'direction' and nothing else...
    - Thus it can be described by its amplitude and two directional angles (e.g., *azimuth* and *dip,* or *tilt*).
  - Characterized by projections onto three selected axes: (x,y,z)
  - When axes are rotated, the projections are transformed via an *axes rotation* matrix **R**:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

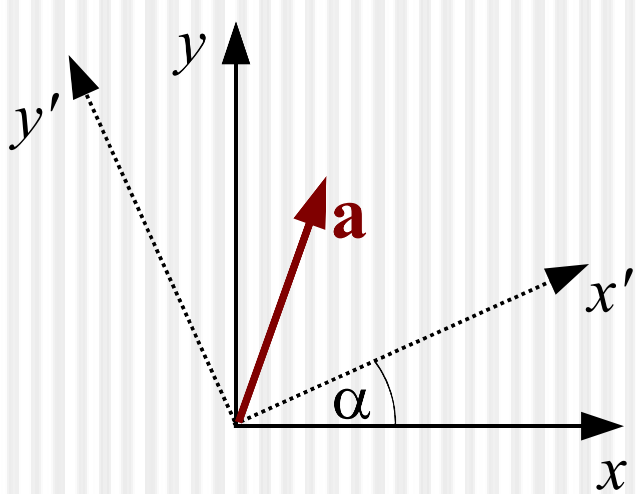


Exercise: Two dimensional (2D) rotation

- Derive the transformation for a counter-clockwise axes rotation by angle α:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Note that the matrix is anti-symmetric
- What is the matrix $\mathbf{R}^{-1}$ of inverse transformation?
- What is the result of application or **R** and $\mathbf{R}^{-1}$ to a vector **a**?



Tensor

- Bi-Directional quantity:
  - Operator transforming one vector (say, **a**) into another (**b**);
  - Represented by a *matrix:*

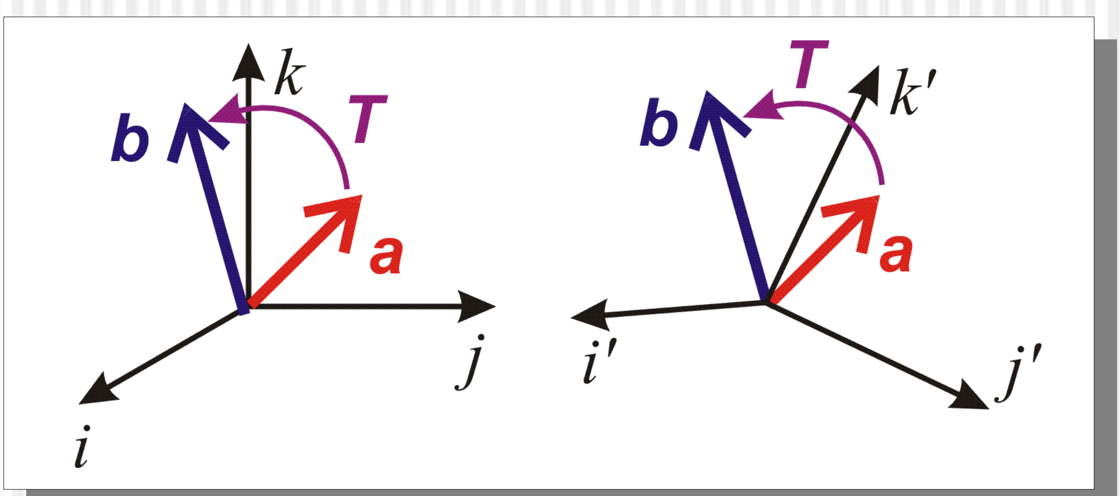
$$b_i = \sum_{j=1}^{3} T_{ij} a_j \equiv T_{ij} a_j$$

Summation is assumed for repeated index (*j*) (Einstein's notation)

    - 3×3 in three-dimensional space, 2×2 in two dimensions, etc.
    - Transformed whenever the frame of reference is rotated:

$$T'_{ij} = \sum_{k,m} R_{ik} R^{-1}_{jm} T_{km}$$

- Examples:



Vector and tensor operations
and their Einstein's representations

- Summation: $\boldsymbol{c}=\boldsymbol{a}+\boldsymbol{b}$

  $$c_i=a_i+b_i$$

- Scaling: $\boldsymbol{c}=\lambda\boldsymbol{b}$

  $$c_i=\lambda b_i$$

- Scalar (dot) product:

  $$c=\boldsymbol{a}\cdot\boldsymbol{b}=a_i b_{i.}$$

  Vector by vector is scalar

  $$\boldsymbol{d}=\tau\cdot\boldsymbol{b}=\tau_{ij} b_{j.}$$

  Tensor by vector is vector

- Vector (cross-) product:

  $$\boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{b}=\left\|\begin{matrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{matrix}\right\|.$$

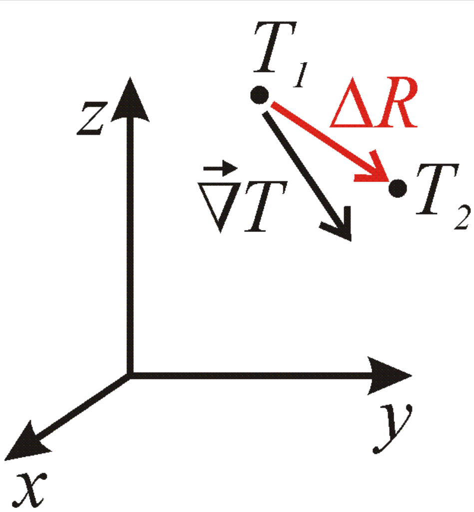
  $$c_i=\epsilon_{ijk} a_j b_k$$

Field

- Physical quantity which takes values at a continuum of points in space and/or time
  - Represented by a function of coordinates and/or time:
    - Scalar: $f(x, y, z, t)$ or $f(\boldsymbol{r}, t)$
      - Examples: temperature, density, seismic velocity, pressure, gravity, electric potential
    - Vector: $\boldsymbol{F}(\boldsymbol{r}, t)$
      - Examples: particle displacement, velocity, or acceleration, electric or magnetic field, current
    - Tensor:
      - Examples: strain and stress;
      - The only way to describe *anisotropy*
  - Always associated with some *source*, carries some kind of *energy*, and often able to propagate *waves*

Scalar Fields
Some characteristics

- Gradient
  - Spatial derivative of a scalar field (say, temperature, *T*(x,y,z,t))
  - It is a Vector field, denoted ∇T ('*nabla*' T):



$$\Delta T = T_2 - T_1$$

$$= \vec{\nabla} T \cdot \Delta \vec{R}$$

$$= \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z$$

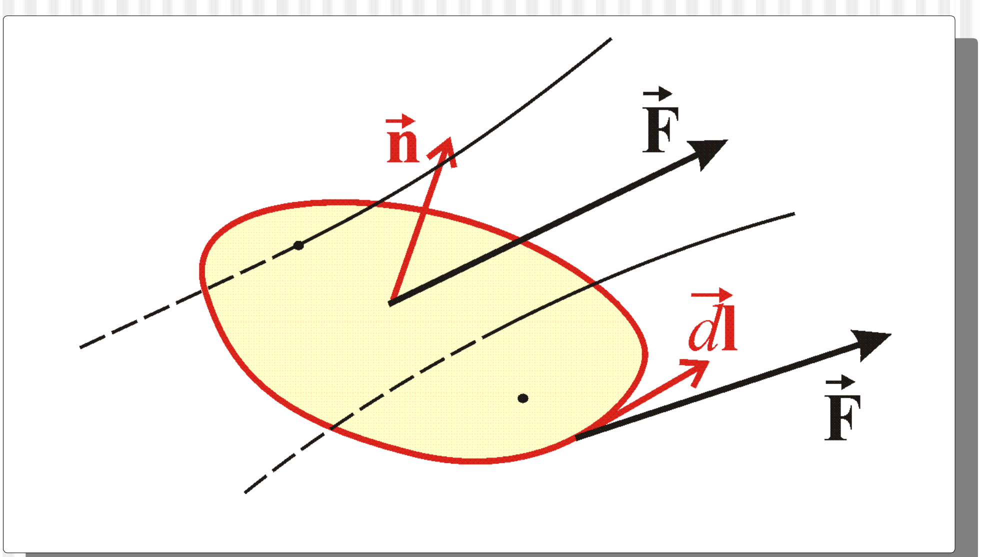
$$\Delta \vec{R} = \vec{i}\,\Delta x + \vec{j}\,\Delta y + \vec{k}\,\Delta z$$

$$\vec{\nabla} T = \vec{i}\,\frac{\partial T}{\partial x} + \vec{j}\,\frac{\partial T}{\partial y} + \vec{k}\,\frac{\partial T}{\partial z}$$

or: $$(\vec{\nabla} T)_i = \partial_i T \equiv \frac{\partial T}{\partial x_i}$$

Vector Fields
Some characteristics

- Flux through a loop =
  - (Average normal component of **F**) times (Surface area of the loop)
  - $Flux(\boldsymbol{F}) = \iint \boldsymbol{F} \cdot \boldsymbol{n}\, dS$
- Circulation along a loop =
  - (Average tangential component of F) times (Length of the loop)
  - $Circulation(\boldsymbol{F}) = \int \boldsymbol{F} \cdot \boldsymbol{dl}$



Vector Fields
Two Key Theorems

- Gauss's Theorem - relates a flux out of a closed surface to a 'volume' integral:

$$\oiint_{\text{Surface}} \boldsymbol{E} \cdot \boldsymbol{n}\, ds = \iiint_{\text{Volume}} \nabla \cdot \boldsymbol{E}\, dV$$

div **E**

  - Divergence is associated with sources and sinks of the field

- Stoke's theorem: relates a circulation around a closed loop to a surface integral:

$$\oint_{\text{Loop}} \boldsymbol{A} \cdot d\,\boldsymbol{l} = \iint_{\text{Surface}} (\nabla \times \boldsymbol{A})\, dS$$

curl **A**

Vector Fields
Two Important Identities

- Divergence of a curl is always zero:

  div(**curl**(**U**)) ≡ 0.

- Curl of a gradient is zero:

  curl(**grad**(ϕ)) ≡ 0.

- Exercise: prove these identities using Einstein's' notation:

$$\nabla_i \phi = \partial_i \phi$$

$$\mathrm{div}\, \vec{U} = \partial_i U_i$$

$$(\mathrm{curl}\, \vec{U})_i = \epsilon_{ijk} \partial_j U_k$$

Fields and Waves

- Fields in geophysics typically exhibit either *static* or *wave* behaviour:
  - Static fields – independent on time: $\frac{\partial T}{\partial t} = 0.$

    Stationary temperature distribution (geotherm).
  - Seismology utilizes *WAVES* – stable spatial field pattern propagating in space with time:

$$u = f(\vec{r} \cdot \vec{n} - ct)$$

Plane wave propagating along direction vector **n.**

$$u = f(|\vec{r}| - ct)$$

Spherical wave

$$u = f(|\vec{r}| - ct)$$

Cylindrical wave

The argument of *f*() is called *phase*

f() is the waveform,
at time *t*, it is centered at $x = ct$

Wave equation and the principle of superposition

- Wave equation:

$$\frac{1}{c^2(\vec{r})}\frac{\partial^2 u}{\partial t^2}-\nabla^2 u=source(\vec{r},t).$$ Scalar

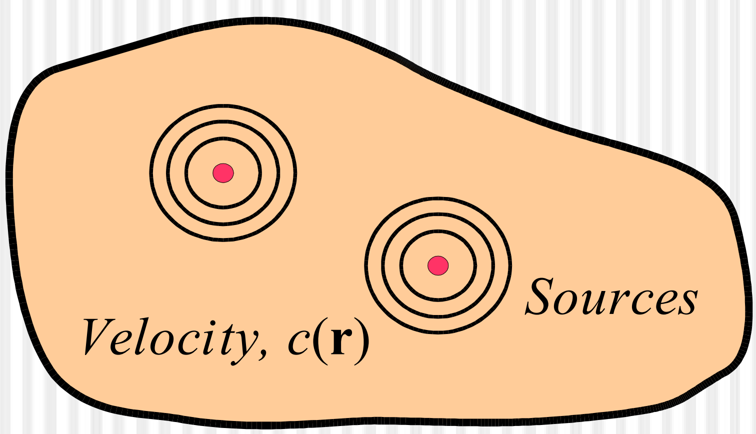
$$\frac{1}{c^2(\vec{r})}\frac{\partial^2 \vec{u}}{\partial t^2}-\nabla^2 \vec{u}=\overrightarrow{source}(\vec{r},t).$$ Vector

- Note that the wave equation is *linear*: if $u_1(\mathbf{r},t)$ and $u_2(\mathbf{r},t)$ are its solutions then $u_1(\mathbf{r},t) + u_2(\mathbf{r},t)$ is also a solution.
  - This property is known as the *principle of superposition.*
  - Because of it, the total wavefield can always be *decomposed* into field generated by elementary sources:
    - Point sources – spherical waves;
    - Linear sources – cylindrical waves;
    - Planar sources – plane waves.

(in a uniform velocity field)

Boundary conditions

- Boundaries (sharp contrasts) in the velocity field $c(\mathbf{r})$ result in *secondary sources* that produce reflected, converted, or scattered waves.
- The amplitudes of these sources and waves are determined through the appropriate *boundary conditions*
  - e.g., *zero displacement* at a rigid boundary (*kinematic* boundary condition);
  - ...or *zero force* at a free boundary (*dynamic* boundary condition).



Boundary conditions

Three factors determining the wave field