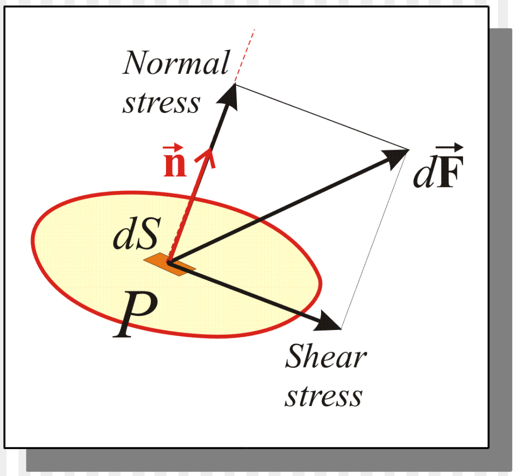
Theory of Elasticity

- Macroscopic theory
- Equations of motion
- Wave equations
- Energy of a seismic wave

- Reading:
  - Telford et al., Section 4.2
  - Sheriff and Geldart, Sections 2.1-4

Stress

- Consider the interior of a deformed body:



At point P, force $d\mathbf{F}$ acts on any infinitesimal area dS. $d\mathbf{F}$ is a projection of *stress tensor*, σ, onto **n**:

$$dF_i = \sigma_{ij} n_j dS$$

- Stress σ_{ij} is measured in [*Newton/m²*], or Pascal (unit of pressure).
- $d\mathbf{F}$ can be decomposed into two components relative to the orientation of the surface, **n**:
  - Parallel (normal stress)

$$(dF_n)_i = n_i \cdot (\text{projection of F onto n}) \;\; = n_i \sigma_{kj} n_k n_j dS$$

Note summation over k and j

  - Tangential (shear stress, *traction*)

$$d\vec{F}_\tau = d\vec{F} - d\vec{F}_n$$

Forces acting on a small cube

- Consider a small parallelepiped ($dx \times dy \times dz = dV$) within the elastic body.
- Exercise 1: show that the *force* applied to the parallelepiped from the outside is:

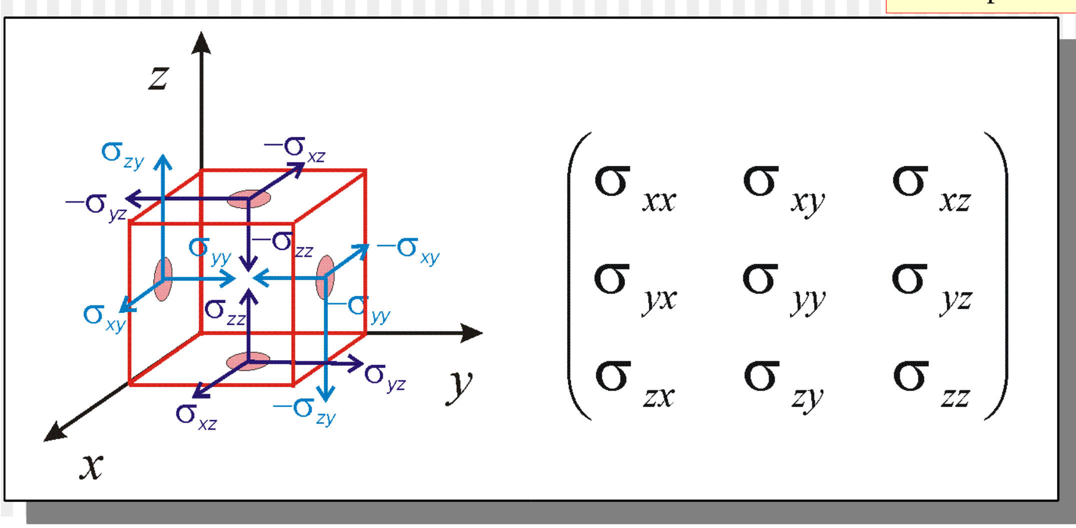
$$F_i = -\partial_j \sigma_{ij} dV$$

(This is simply minus divergence (convergence?) of stress!)

- Exercise 2: Show that *torque* applied to the cube from the outside is:

$$L_i = -\epsilon_{ijk} \sigma_{jk} dV$$

Again, keep in mind implied summations over repeated indices!



Symmetry of stress tensor

- Thus, L is proportional to dV: $L = O(dV)$
- The *moment of inertia* for any of the axes is proportional to $dV \cdot length^2$:

$$I_x = \int_{dV} (y^2 + z^2)\,\rho\, dV$$

and so it tends to 0 faster than dV:

$I = o(dV)$.

- Angular acceleration: $\theta = L/I$, must be finite as $dV \to 0$, and therefore:

$$L_i / dV = -\epsilon_{ijk}\sigma_{jk} = 0.$$

- Consequently, the stress tensor is symmetric: $\sigma_{ij} = \sigma_{ji}$
- σ_{ji} has only 6 independent parameters out of 9.

$$\begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = dS \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_x \\ n_x \end{pmatrix},$$

$$\sigma_{xy} = \sigma_{yx},$$
$$\sigma_{xz} = \sigma_{zx},$$
$$\sigma_{yz} = \sigma_{zy}$$

Shear stress components are symmetric

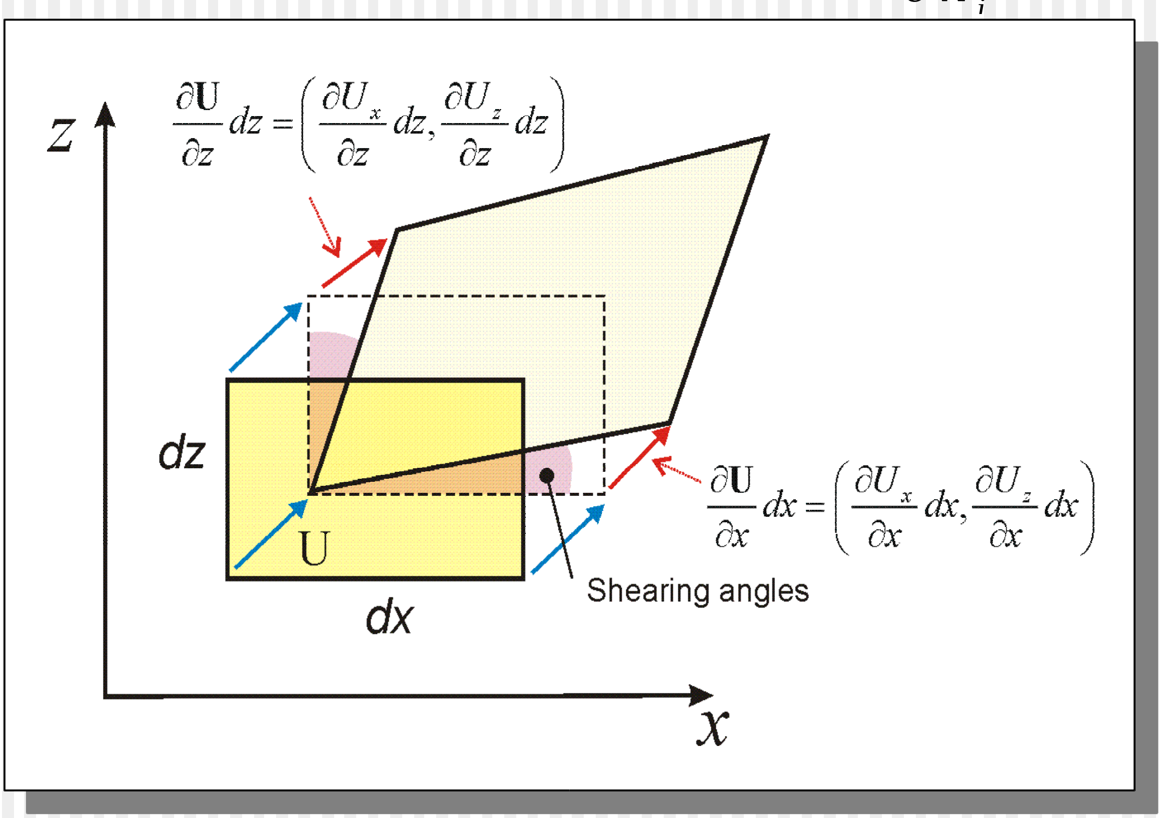
Normal stress components

Strain

- Strain is a measure of deformation, i.e., *variation of relative displacement* as associated with a *particular direction* within the body
- It is, therefore, also a *tensor*
  - Represented by a matrix
  - Like stress, it is decomposed into *normal* and *shear* components
- Seismic waves yield strains of 10^{-10}-10^{-6}
  - So we can rely on *infinitesimal* strain theory

Elementary Strain

- When a body is deformed, *displacements* (**U**) of its points are dependent on (x,y,z), and consist of:
  - Translation (blue arrows below)
  - Deformation (red arrows)
- Elementary strain is: $e_{ij} = \frac{\partial U_i}{\partial x_j}$



Stretching and Rotation

- Exercise 1: Derive the elementary strain associated with unidirectional stretching of the body along the *X* axis:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1+\gamma & 0 \\ 0 & 1+\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Exercise 2: Derive the elementary strain associated with rotation by a small angle α:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Strain Components

- *Anti-symmetric* combinations of e_{ij} yield rotations of the body without changing its shape:
  - e.g., $\frac{1}{2}\left(\frac{\partial U_z}{\partial x} - \frac{\partial U_x}{\partial z}\right)$ yields rotation about the 'y' axis.
  - So, the case of $\frac{\partial U_z}{\partial x} = \frac{\partial U_x}{\partial z}$ is called *pure shear* (no rotation).
- To characterize *deformation*, only the *symmetric* component of the elementary strain is used:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right),$$

$$\varepsilon_{ij} = \varepsilon_{ji}, \text{ where } i, j = x, y, \text{ or } z$$

$$\varepsilon = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{1}{2}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y}\right) & \frac{\partial U_y}{\partial y} & \frac{1}{2}\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial U_z}{\partial y} + \frac{\partial U_y}{\partial z}\right) & \frac{\partial U_z}{\partial z} \end{pmatrix}$$

Dilatational Strain (relative volume change during deformation)

- Original volume: $V = \delta x \delta y \delta z$
- Deformed volume: $V + \delta V = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz}) \delta x \delta y \delta z$
- Dilatational strain:

$$\Delta = \frac{\delta V}{V} = (1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz}) \approx \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$\Delta = \varepsilon_{ii} = \partial_i U_i = \vec{\nabla} \vec{U} = \mathrm{div}\ \vec{U}$$

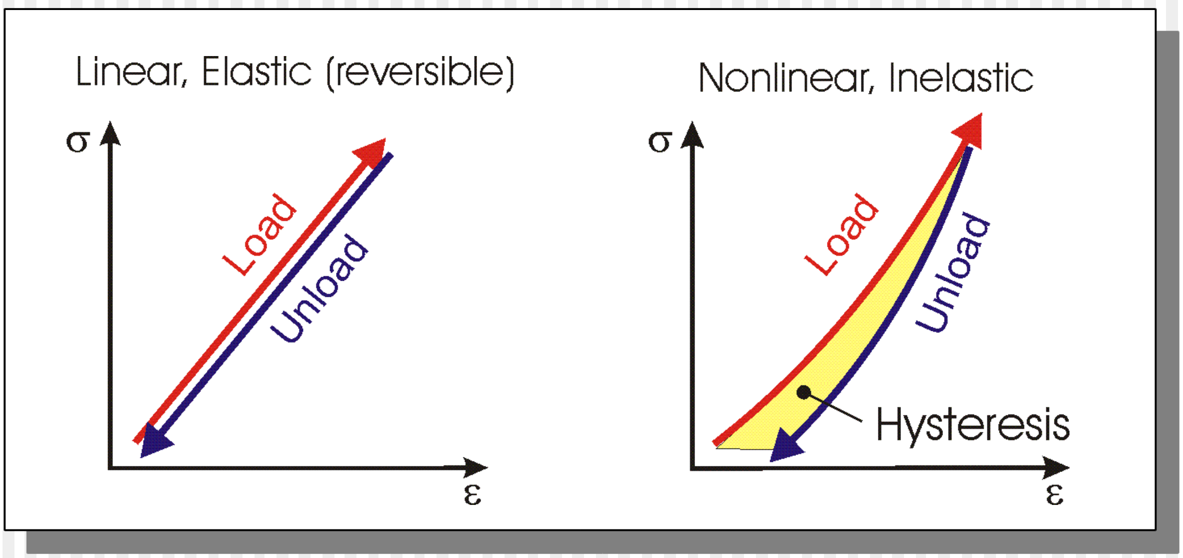
- Note that *shearing strain does not change the volume*.

Hooke's Law (general)

- Hookes' law describes the stress developed in a deformed body:

  F = -k**x** for an ordinary spring (1-D)

  $\sigma \sim \varepsilon$ (in some sense) for a '*linear*', '*elastic*' 3-D solid. This is what these terms mean:



- For a general (*anisotropic*) medium, there are 36 coefficients of proportionality between six independent σ_{ij} and six ε_{ij}:

$$\sigma_{ij} = \Lambda_{ij,kl} \epsilon_{kl} .$$

Hooke's Law (isotropic medium)

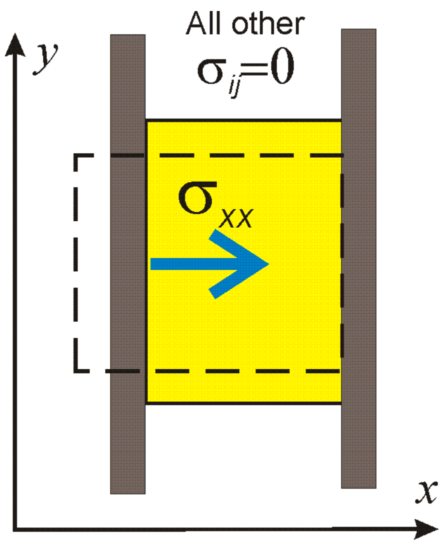
- For isotropic medium, the strain/stress relation is described by just 2 constants:

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\varepsilon_{ij}$$

  - δ_{ij} is the "Kronecker symbol" (unit tensor) equal 1 for $i=j$ and 0 otherwise;
  - λ and μ are called the *Lamé constants*.
- Although λ and μ provide a natural mathematical parametrization for $\sigma(\varepsilon)$, they are always intermixed in experimental environments.
  - Their combinations, called "*elastic modulae*" are typically directly measured.
- Question: what are the units for λ and μ?

Four Elastic Modulae

- Depending on boundary conditions (i.e., experimental setup), different combinations of λ' and μ may be convenient. These combinations are called *elastic constants*, or modulae:
  - Young's modulus and Poisson's ratio:
    - Consider a cylindrical sample uniformly squeezed along axis X:



$$\sigma_{xx} = \lambda'\Delta + 2\mu\varepsilon_{xx},$$
$$\sigma_{yy} = \lambda'\Delta + 2\mu\varepsilon_{yy} = 0,$$
$$\sigma_{zz} = \lambda'\Delta + 2\mu\varepsilon_{zz} = 0 \Rightarrow \varepsilon_{yy} = \varepsilon_{zz} = \frac{-\lambda'\Delta}{2\mu}.$$

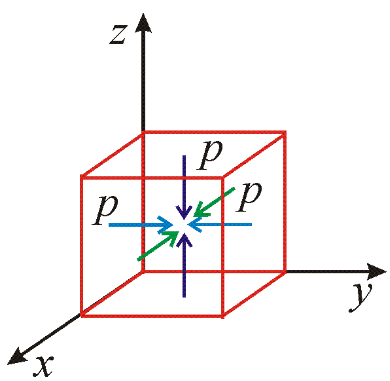
Young's modulus: $$E = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{2\mu(3\lambda'+2\mu)}{\lambda'+\mu}$$

Poisson's ratio: $$\nu = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda'}{2(\lambda'+\mu)}$$

- Note: The Poisson's ratio is more often denoted σ
- It measures the ratio of λ and μ: $\frac{\mu}{\lambda} = \frac{1}{2\sigma} - 1$

Four Elastic Modulae

- Bulk modulus, K
  - Consider a cube subjected to hydrostatic pressure



$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p,$$

$$-3p = 3\lambda'\Delta + 2\mu\Delta$$

Bulk modulus: $K = \frac{-p}{\Delta} = \lambda' + \frac{2}{3}\mu$

- The constant μ complements K in describing the shear rigidity of the medium. Thus, μ is also called the '*rigidity modulus*'
- For rocks:
  - Generally, 10 Gpa $< \mu < K < E <$ 200 Gpa
  - $0 < \nu < ½$ always; for rocks, $0.05 < \nu < 0.45$, for most, ν is near 0.25.
- For liquids, $\nu = ½$ and $\mu = 0$ (no shear resistance)

Equations of Motion

(Motion of the elastic body with time)

- Uncompensated net force will result in *acceleration* (Newton's law):

Newton's law: $$\rho\,\delta V \frac{\partial^2 U_i}{\partial t^2} = F_i$$

$$\rho \frac{\partial^2 U_i}{\partial t^2} = \left(\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z}\right)$$

$$\rho \frac{\partial^2 U_x}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda'\Delta + 2\mu \frac{\partial U_x}{\partial x}\right) + \mu \frac{\partial}{\partial y}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) + \mu \frac{\partial}{\partial z}\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right)$$

$$= \lambda' \frac{\partial \Delta}{\partial x} + \mu \frac{\partial}{\partial x}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}\right) + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right)$$

$$= (\lambda' + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 U_x$$

- Therefore, the *equations of motion* for the components of **U**:

$$\rho \frac{\partial^2 U_i}{\partial t^2} = (\lambda + \mu)\frac{\partial \Delta}{\partial x_i} + \mu \nabla^2 U_i$$

Wave potentials

Compressional and Shear waves

- These equations describe two types of waves.
- The general solution has the form ("Lame theorem"):

$$\vec{U} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{\psi}. \qquad (\text{or} \quad U_i = \partial_i \phi + \epsilon_{ijk}\partial_j \psi_k \quad)$$

$$\vec{\nabla}\cdot\vec{\psi} = 0.$$

Because there are 4 components in ψ and ϕ only 3 in **U**, we need to constrain ψ.

- Exercise: substitute the above into the equation of motion:

$$\rho \frac{\partial^2 U_i}{\partial t^2} = (\lambda+\mu)\frac{\partial \Delta}{\partial x_i} + \mu \nabla^2 U_i$$

and show:

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu)\nabla^2 \phi,$$

P-wave (scalar) potential.

$$\rho \frac{\partial^2 \psi_i}{\partial t^2} = \mu \nabla^2 \psi_i,$$

S-wave (vector) potential.

Wave velocities

Compressional and Shear waves

- These are wave equations; compare to the general form of equation describing wave processes:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]f\left(x,y,z,t\right)=0$$

- *Compressional* (*P*) wave velocity:

$$v_P=\sqrt{\frac{\lambda+2\mu}{\rho}}=\sqrt{\frac{K+\frac{4}{3}\mu}{\rho}}$$

- *Shear* (*S*) wave velocity:

$$v_S=\sqrt{\frac{\mu}{\rho}}$$

  - $V_S < V_P$,
  - for σ=0.25: $V_P/V_P=\sqrt{3}$

- Note that the V_P/V_S depends on the Poisson ratio alone:

$$\frac{V_P}{V_S}=\sqrt{\frac{\mu}{\lambda+2\mu}}=\sqrt{\frac{1/2-\sigma}{1-\sigma}}.$$

Notes on the use of potentials

- Wave potentials are very useful for solving elastic wave problems
- Just take ϕ or ψ satisfying the wave equation, e.g.:

$$\phi(\vec{r},t) = Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}.$$

(plane wave)

...and use the equations for potentials to derive the displacements:

$$\vec{U} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{\psi}.$$

...and stress from Hooke's law:

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\varepsilon_{ij}$$

- *Displacement amplitude =*

  $\omega\times$(potential amplitude)/V

- *Power = ½ρ[$\omega^2\times$(potential amplitude)/V]^2*

This is *velocity* amplitude

Example:
Compressional (*P*) wave

- Scalar potential for *plane harmonic* wave:

$$\phi(\vec{r},t)=Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}.$$

- Displacement:

$$u_i(\vec{r},t)=\partial_i\phi(\vec{r},t)=\frac{-i\omega n_i}{V_P}Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}.$$

  note that the displacement is always <u>along **n**</u>.

- Strain:

$$\varepsilon_{ij}(\vec{r},t)=\partial_i u_j(\vec{r},t)=\frac{\omega^2 n_i n_j}{V_P^2}Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}.$$

- Dilatational strain:

$$\Delta=\varepsilon_{ii}(\vec{r},t)=\frac{\omega^2}{V_P^2}Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}=\frac{\omega^2}{V_P^2}\phi(\vec{r},t).$$

- Stress:

$$\sigma_{ij}(\vec{r},t)=\frac{\omega^2}{V_P^2}(\lambda\delta_{ij}+2\mu n_i n_j)\phi(\vec{r},t).$$

- <u>Question</u>: what wavefield would we have if used cos() or sin() function instead of complex exp() in the expression for potential above?

Impedance

- *Impedance, Z,* is a measure of the amount of resistance to particle motion.
- In elasticity, impedance is a *ratio of stress to particle velocity*.
  - Thus, for a given applied stress, particle velocity is inversely proportional to impedance.
  - For *P* wave, in the direction of its propagation:

$$Z(\vec{r},t)=\frac{\sigma_{nn}(\vec{r},t)}{\dot{u}_n(\vec{r},t)}=\frac{\lambda+2\mu}{V_P}=\rho V_{P}.$$

    - impedance does not depend on frequency but *depends on the wave type and incidence direction.*

Elastic Energy Density

- Mechanical work is required to deform an elastic body; as a result, elastic energy is accumulated in the strain/stress field
- When released, this energy gives rise to earthquakes and seismic waves
- For a loaded spring (1-D elastic body), $E = \frac{1}{2}kx^2 = \frac{1}{2}Fx$
- Similarly, for a deformed elastic medium, *energy density* is:

$$E = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$$

Elastic Energy Density
in a plane wave

- For a plane wave:

$$u_i = u_i(t - \vec{p}\cdot\vec{x})$$

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) = -\frac{1}{2}(\dot{u}_i p_j + \dot{u}_j p_i).$$

- ...and therefore:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}[(\lambda+\mu)(\vec{p}\cdot\vec{u})^2 + \mu(\vec{u}\cdot\vec{u})(\vec{p}\cdot\vec{p})]$$

- For *P*- and *S*-waves, this gives:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}(\lambda+2\mu)p^2\vec{u}^2 = \frac{1}{2}\rho\dot{\vec{u}}^2 \qquad P\text{-wave}$$

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}(\mu)p^2\vec{u}^2 = \frac{1}{2}\rho\dot{\vec{u}}^2 \qquad S\text{-wave}$$

- Thus, in a wave, *strain energy equals the kinetic energy*

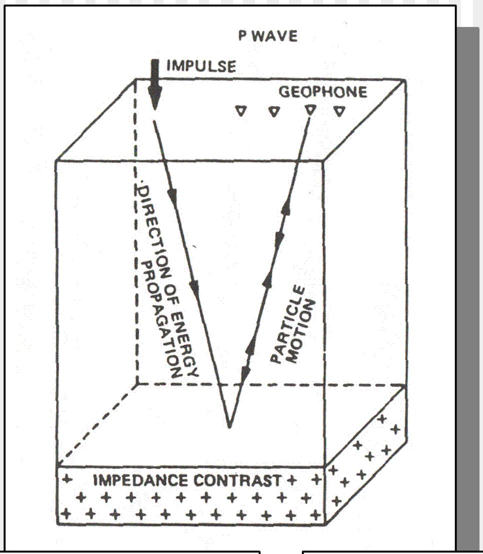
Energy is *NOT* conserved locally!

- *Energy propagates at the same speed* as the wave pulse

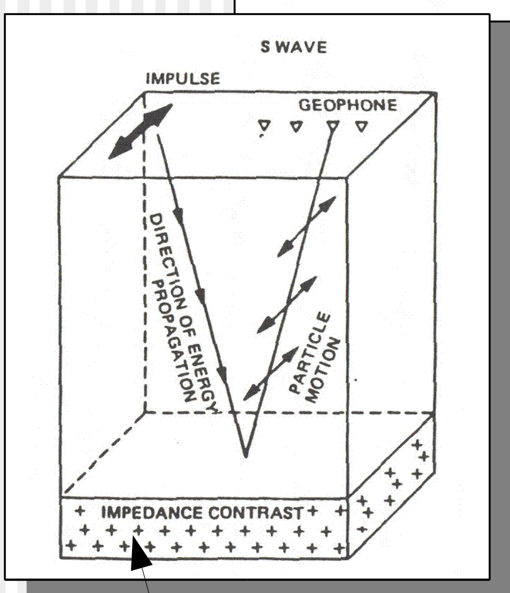
Wave Polarization

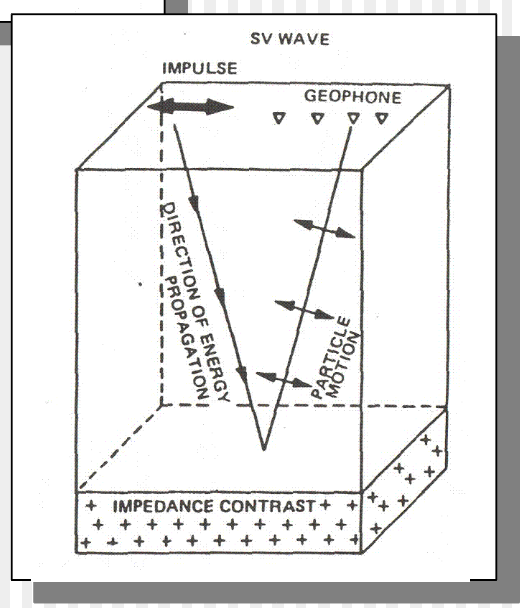
- Elastic solid supports two types of *body waves*:

P



S





Note that this is still an **ISOTROPIC** reflector.
In general, reflection will intermix
the S-wave polarization modes,
and P-wave will convert into SV upon reflection.