Reflection coefficients

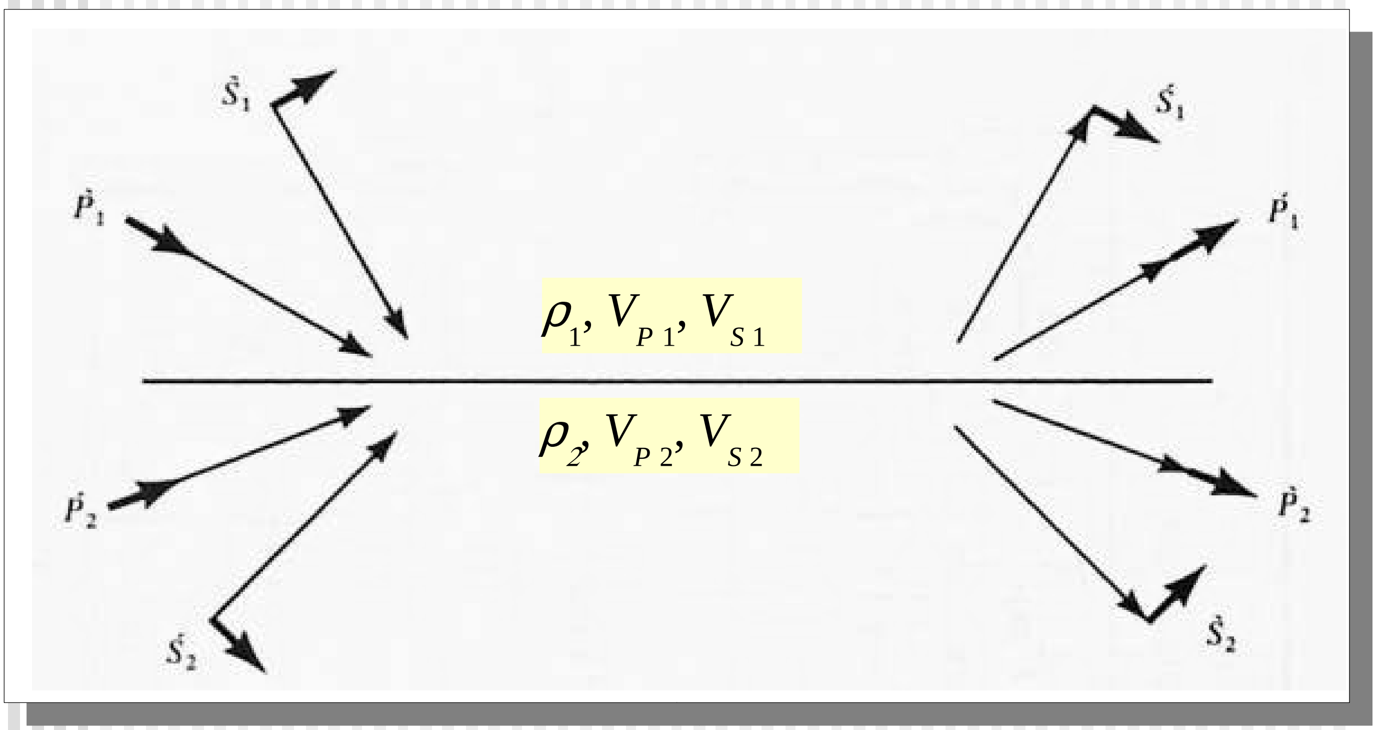
- Reflection and conversion of plane waves
- Snell's law
- P/SV wave conversion
- Scattering matrix
- Zoeppritz equations
- Amplitude vs. Angle and Offset relations

- Reading:
  - Telford et al., Section 4.2.
  - Sheriff and Geldart, Chapter 3

Surface reflection transmission, and conversion

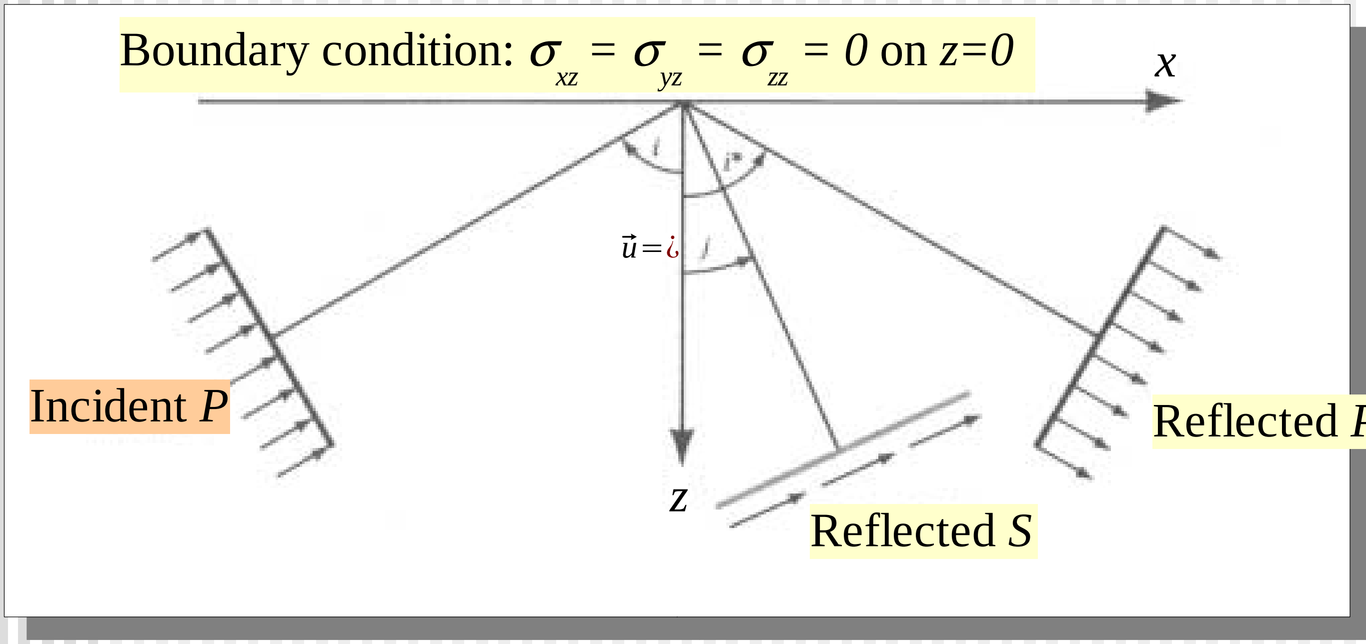
- Consider waves incident on a welded horizontal interface of two uniform half-spaces:
  - Because of their vertical motion, *P* and *SV* waves *couple* to each other on the interface,
  - therefore, there are 8 possible waves interacting with each other at the boundary.

...what about *SH* waves?



Free-surface reflection and conversion

- Consider a *P* wave incident on a free surface:



- Each of the *P*- or *S*-waves is described by potentials:

$$\vec{u}_P(\vec{x},\vec{z}) = \left(\frac{\partial\phi}{\partial x}, 0, \frac{\partial\phi}{\partial z}\right), \quad \phi = \phi^{inc} + \phi^{refl} \qquad P\text{-waves}$$

$$\vec{u}_S(\vec{x},\vec{z}) = \left(\frac{-\partial\psi}{\partial z}, 0, \frac{\partial\psi}{\partial x}\right), \quad \psi = \psi^{refl} \qquad SV\text{-wave}$$

Free-surface reflection and conversion (2)

- Traction (force acting on the surface):

$$\vec{F}_P(\vec{x},\vec{z}) = \left(2\mu\frac{\partial^2\phi}{\partial x\partial z}, 0, \lambda\nabla^2\phi + 2\mu\frac{\partial^2\phi}{\partial z^2}\right),$$ *P*-waves

$$\vec{F}_S(\vec{x},\vec{z}) = \left(\mu\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right), 0, \lambda\nabla^2\phi + 2\mu\frac{\partial^2\psi}{\partial x\partial z}\right),$$ *SV*-wave

- Consider *plane harmonic* waves:

$$\phi^{inc} = A_P^{inc}\exp\left[i\omega\left(\frac{\vec{x}\vec{n}_{incP}}{V_P} - t\right)\right]$$ incident *P*

$$\phi^{refl} = A_P^{refl}\exp\left[i\omega\left(\frac{\vec{x}\vec{n}_{reflP}}{V_P} - t\right)\right]$$ reflected *P*

$$\psi^{refl} = A_S^{refl}\exp\left[i\omega\left(\frac{\vec{x}\vec{n}_{reflS}}{V_S} - t\right)\right]$$ reflected *SV*

- What are the dependencies of ϕ and ψ above on coordinate *x*?

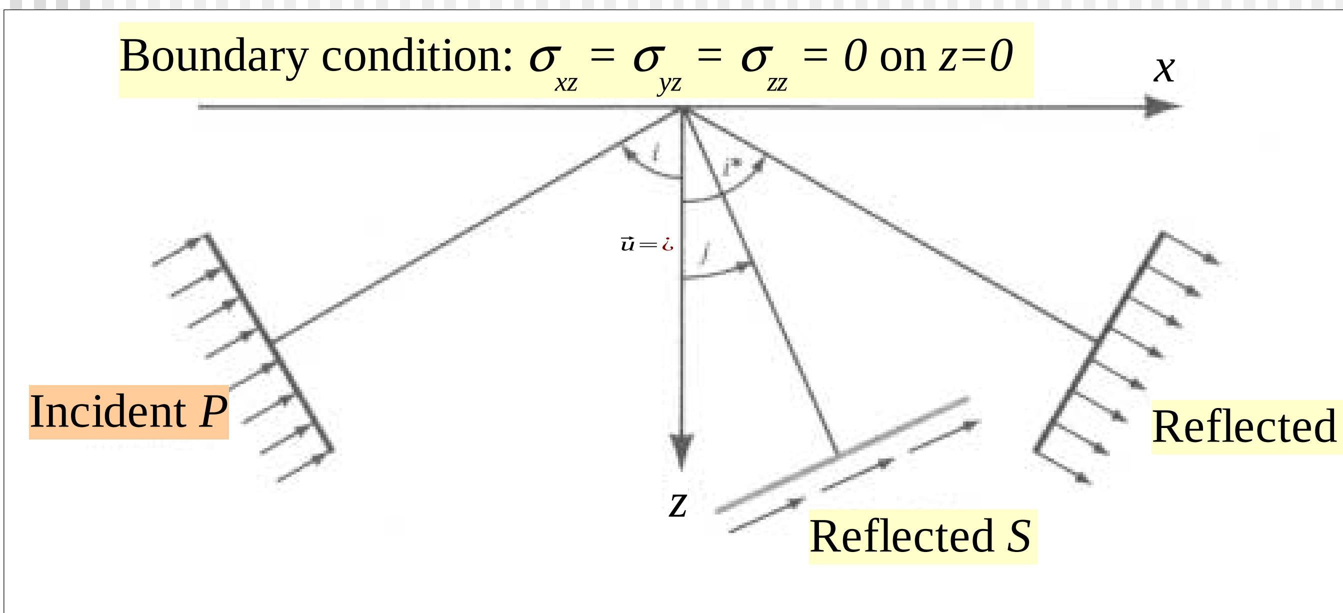
Free-surface reflection and conversion (3)

- The boundary condition is: ***Force***(x,t)=0
- Note that functional dependences of ϕ and ψ on (x,t) are:

$$\exp\left[i\omega\left(\frac{\sin i}{V_P}-t\right)\right],$$
$$\exp\left[i\omega\left(\frac{\sin i^*}{V_P}-t\right)\right],$$
$$\exp\left[i\omega\left(\frac{\sin j}{V_S}-t\right)\right],$$

- These must satisfy for <u>any *x*</u> , consequently, the *Snells law*:

$$\frac{\sin i}{V_P}=\frac{\sin i^*}{V_P}=\frac{\sin j}{V_S}=p$$



Free-surface reflection and conversion (4)

- Displacement in plane waves is thus:

$$\vec{u}_P(\vec{x},\vec{z}) = \left(i\omega p\phi, 0, \pm i\omega \frac{\cos j}{V_P}\phi\right),$$ *P*-waves

$$\vec{u}_S(\vec{x},\vec{z}) = \left(\mp i\omega \frac{\cos j}{V_P}\psi, 0, i\omega p\psi\right),$$ *SV*-wave

- ...and traction:

$$\vec{F}_P(\vec{x},\vec{z}) = \left(-2\rho V_S^2 p\phi, 0, -\rho(1-2V^2p^2) i\omega^2 V_S \phi\right),$$

$$\vec{F}_S(\vec{x},\vec{z}) = \left(\rho(1-2V^2p^2) i\omega^2 V_S\psi, 0, 2\rho V_S^2 p\psi, 0\right).$$

Free-surface reflection and conversion (5)

- Traction vector at the surface must vanish:

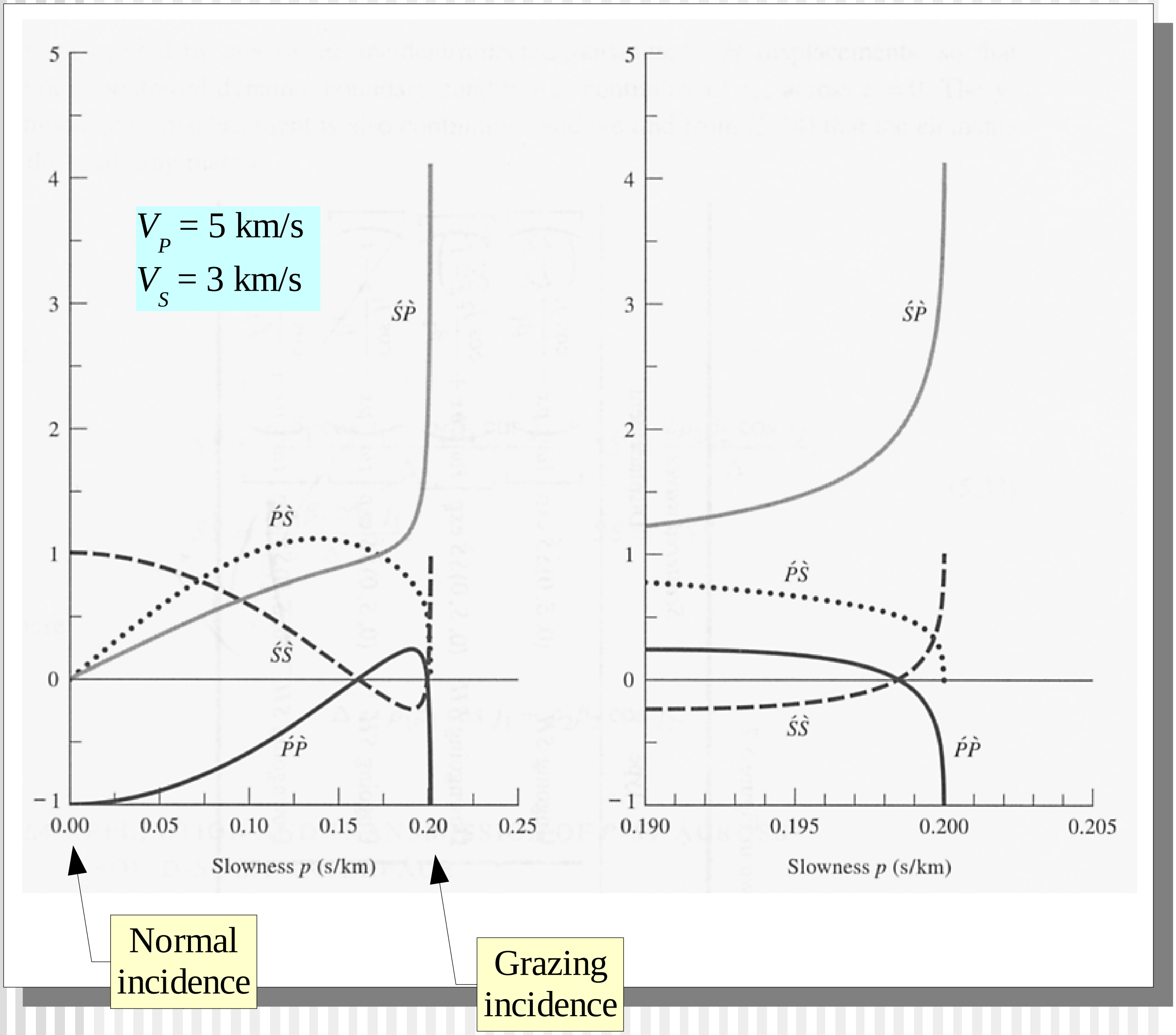
  $F_x = F_z = 0$

  - Therefore, we have two equations to constrain the amplitudes of the two reflected waves;
  - Their solution:

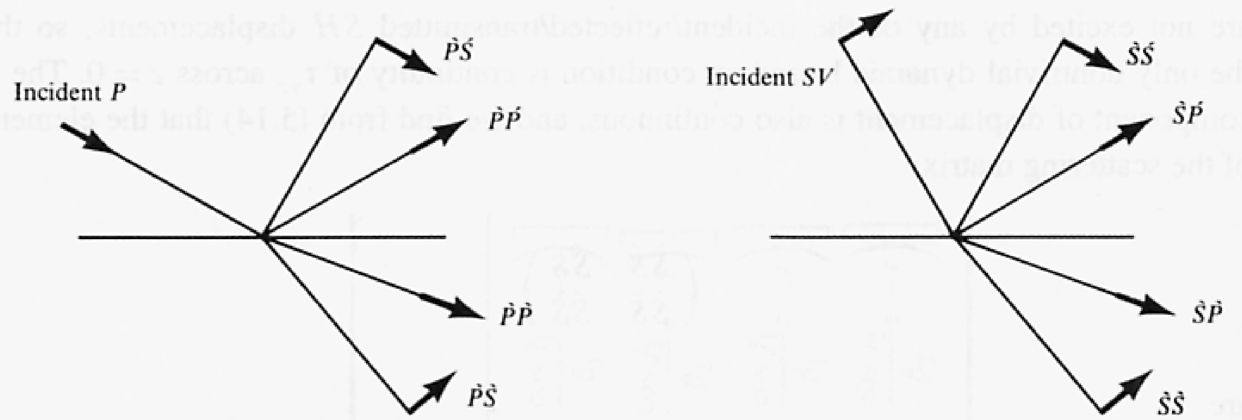
$$\frac{A_P^{refl}}{A_P^{inc}} = \frac{4V_S^4 p^2 \frac{\cos i}{V_P}\frac{\cos j}{V_S} - (1-2V_S^2 p^2)^2}{4V_S^4 p \frac{\cos i}{V_P}\frac{\cos j}{V_S} + (1-2V_S^2 p^2)^2},$$

$$\frac{A_S^{refl}}{A_P^{inc}} = \frac{-4V_S^2 p \frac{\cos i}{V_P}(1-2V_S^2 p^2)}{4V_S^4 p \frac{\cos i}{V_P}\frac{\cos j}{V_S} + (1-2V_S^2 p^2)^2}.$$

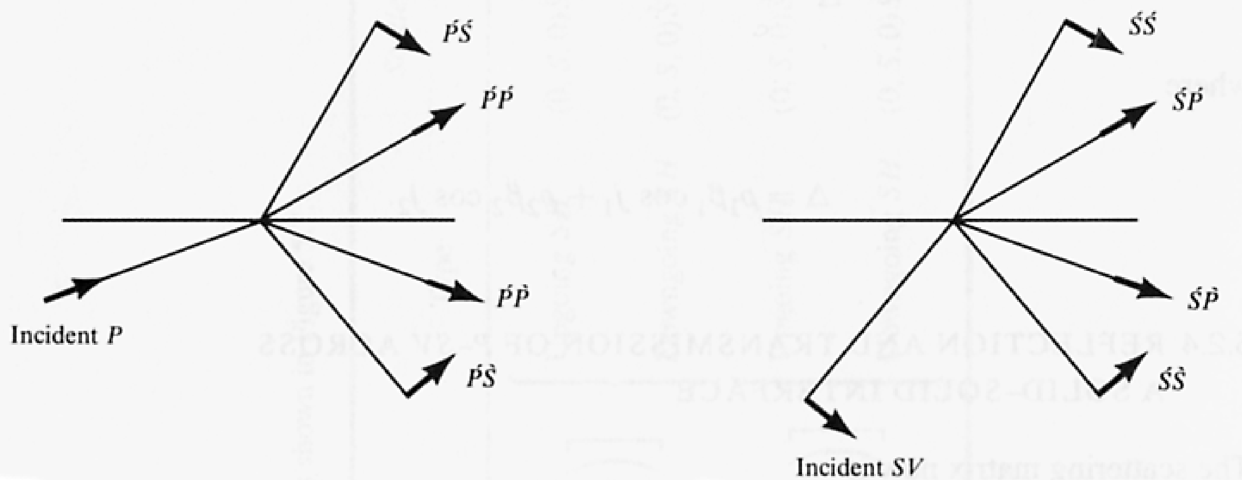
Free-surface reflection and conversion (5)



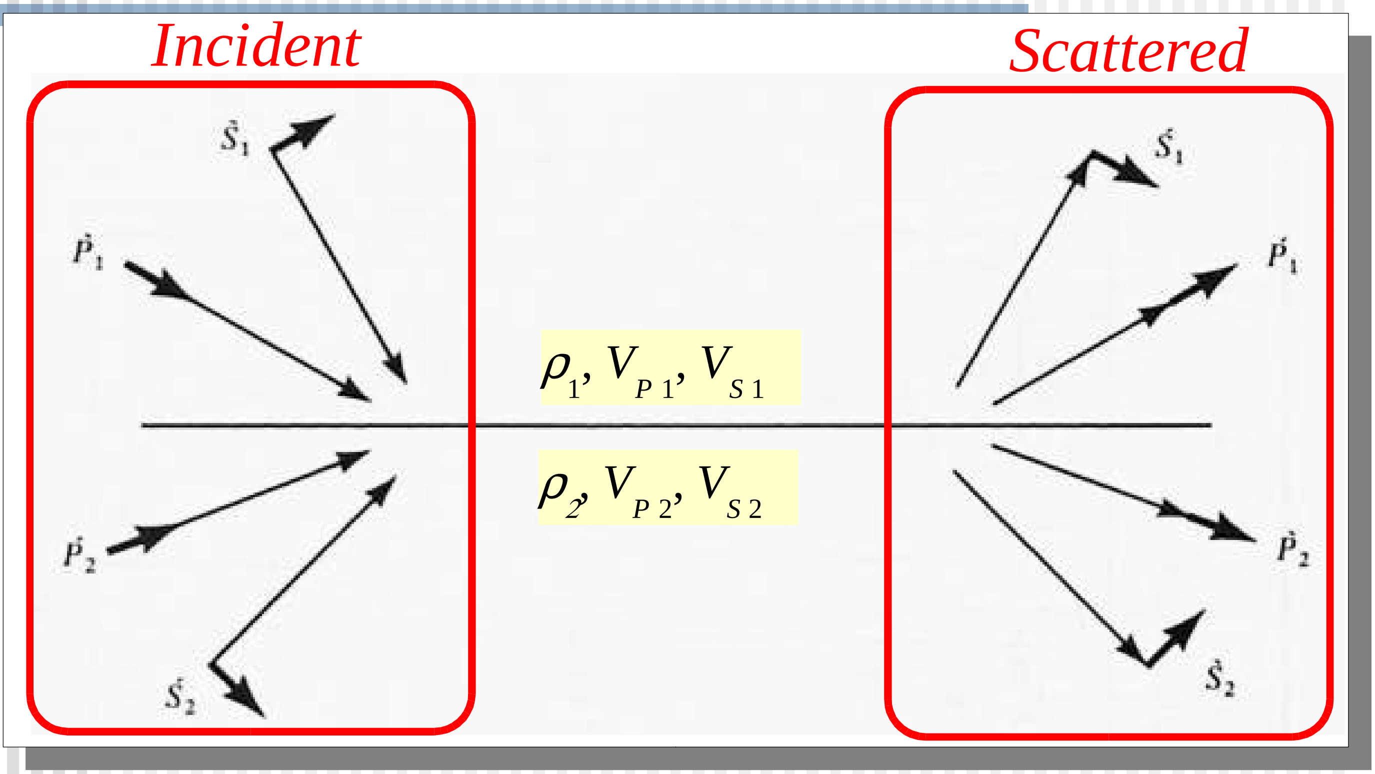
Complete reflection/transmission problem



- There are 16 possible reflection/transmission coefficients on a welded contact of two half-spaces



Scattering matrix



- All 16 possible reflection coefficients can be summarized in the *scattering matrix*:

$$\boldsymbol{S} = \begin{pmatrix} \grave{P}\acute{P} & \grave{S}\acute{P} & \acute{P}\acute{P} & \acute{S}\acute{P} \\ \grave{P}\acute{S} & \grave{S}\acute{S} & \acute{P}\acute{S} & \acute{S}\acute{S} \\ \grave{P}\grave{P} & \grave{S}\grave{P} & \acute{P}\grave{P} & \acute{S}\grave{P} \\ \grave{P}\grave{S} & \grave{S}\grave{S} & \acute{P}\grave{S} & \acute{S}\grave{S} \end{pmatrix}$$

$$\begin{pmatrix} \acute{P}_1 \\ \acute{S}_1 \\ \grave{P}_2 \\ \grave{S}_2 \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} \grave{P}_1 \\ \grave{S}_1 \\ \acute{P}_2 \\ \acute{S}_2 \end{pmatrix}.$$

All reflection and refraction amplitudes at an interface
(Derivation of the Scattering Matrix)

- The scattering matrix can be used to easily derive all possible reflection and refraction amplitudes at once:
  - consider matrix **N** that is giving displacement and traction at the interface for the incident field, and a similar matrix **M** for the scattered field:

$$\begin{pmatrix} u_x \\ u_y \\ \sigma_{xz} \\ \sigma_{zz} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} \acute{P}_1 \\ \acute{S}_1 \\ \grave{P}_2 \\ \grave{S}_2 \end{pmatrix} = \boldsymbol{N} \begin{pmatrix} \grave{P}_1 \\ \grave{S}_1 \\ \acute{P}_2 \\ \acute{S}_2 \end{pmatrix}.$$

  - This is a general (matrix) form of *Zoeppritz' equations* (relating the incident, reflected, and converted wave amplitudes).
  - Their general solution: $\boldsymbol{S} = \boldsymbol{M}^{-1}\boldsymbol{N}$

M and N

- The matrices **M** and **N** consist of the coefficients of plane-wave amplitudes and tractions for *P*- and *SV*-waves:

$$\boldsymbol{M}=\begin{pmatrix} -V_{P1}p & -\cos j_1 & V_{P2}p & \cos j_2 \\ \cos i_1 & -V_{S1}p & \cos i_2 & -V_{S2}p \\ 2\rho_1 V_{S1}^2 p\cos i_1 & \rho_1 V_{S1}(1-2V_{S1}^2p^2) & 2\rho_2 V_{S2}^2 p\cos i_2 & \rho_2 V_{S2}(1-2V_{S2}^2p^2) \\ -\rho_1 V_{P1}(1-2V_{S1}^2p^2) & 2\rho_1 V_{S1}^2 p\cos j_1 & \rho_2 V_{P2}(1-2V_{S2}^2p^2) & -2\rho_2 V_{S1}^2 p\cos j_2 \end{pmatrix},$$

$$\boldsymbol{N}=\begin{pmatrix} V_{P1}p & \cos j_1 & -V_{P2}p & -\cos j_2 \\ \cos i_1 & -V_{S1}p & \cos i_2 & -V_{S2}p \\ 2\rho_1 V_{S1}^2 p\cos i_1 & \rho_1 V_{S1}(1-2V_{S1}^2p^2) & 2\rho_2 V_{S2}^2 p\cos i_2 & \rho_2 V_{S2}(1-2V_{S2}^2p^2) \\ \rho_1 V_{P1}(1-2V_{S1}^2p^2) & -2\rho_1 V_{S1}^2 p\cos j_1 & -\rho_2 V_{P2}(1-2V_{S2}^2p^2) & 2\rho_2 V_{S1}^2 p\cos j_2 \end{pmatrix},$$

$$\boldsymbol{S}\equiv\begin{pmatrix} \grave{P}\acute{P} & \grave{S}\acute{P} & \acute{P}\acute{P} & \acute{S}\acute{P} \\ \grave{P}\acute{S} & \grave{S}\acute{S} & \acute{P}\acute{S} & \acute{S}\acute{S} \\ \grave{P}\grave{P} & \grave{S}\grave{P} & \acute{P}\grave{P} & \acute{S}\grave{P} \\ \grave{P}\grave{S} & \grave{S}\grave{S} & \acute{P}\grave{S} & \acute{S}\grave{S} \end{pmatrix}=\boldsymbol{M}^{-1}\boldsymbol{N}\,.$$

This is matrix form of *Knott' equations* (solutions for reflected and refracted amplitudes)

Partitioning at normal incidence

- At normal incidence, $i_1 = i_2 = j_1 = j_2 = 0$, and $p = 0$:

$$\begin{array}{cc} & \begin{array}{cccc} P & S & P & S \end{array} \\ \boldsymbol{M} = & \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & \rho_1 V_{S1} & 0 & \rho_2 V_{S2} \\ -\rho_1 V_{P1} & 0 & \rho_2 V_{P2} & 0 \end{pmatrix}, \end{array} \quad \begin{array}{cc} & \begin{array}{cccc} P & S & P & S \end{array} \\ \boldsymbol{N} = & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & \rho_1 V_{S1} & 0 & \rho_2 V_{S2} \\ \rho_1 V_{P1} & 0 & -\rho_2 V_{P2} & 0 \end{pmatrix}, \end{array}$$

- The *P- and S-waves do not interact at normal incidence*, and so we can look, e.g., at *P*-waves only (extract the odd-numbered columns):

$$\boldsymbol{M} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ -\rho_1 V_{P1} & \rho_2 V_{P2} \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ \rho_1 V_{P1} & -\rho_2 V_{P2} \end{pmatrix},$$

Note that these two constraints are satisfied automatically

Impedance, $\rho V = Z$

- Drop the two trivial equations (#1 and 3) and obtain:

$$\begin{pmatrix} \grave{P}\acute{P} & \acute{P}\acute{P} \\ \grave{P}\grave{P} & \acute{P}\grave{P} \end{pmatrix} = \boldsymbol{M}^{-1}\boldsymbol{N} = \begin{pmatrix} 1 & 1 \\ -Z_1 & Z_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ Z_1 & -Z_2 \end{pmatrix} = \frac{1}{Z_1 + Z_2} \begin{pmatrix} Z_2 - Z_1 & Z_1 - Z_2 \\ 2Z_1 & 2Z_2 \end{pmatrix}.$$

Reflection and transmission coefficients

Reflection and Transmission at normal incidence

- Thus, at normal incidence (in practice, for angles up to ~15°)
  - Reflection coefficient:

$$R=\frac{Z_2-Z_1}{Z_1+Z_2}\approx\frac{\Delta Z}{2}Z\approx\frac{1}{2}\Delta(lnZ)\approx\frac{1}{2}(\frac{\Delta V_P}{V_P}+\frac{\Delta\rho}{\rho})$$

  - Transmission coefficient:

$$T=\frac{2Z_1}{Z_1+Z_2}$$

  - Energy Reflection coefficient:

$$E_R=R^2$$

  - Energy Transmission coefficient:

$$E_T=1-E_R=\frac{2Z_1Z_2}{Z_1+Z_2}.$$

    - Note that the energy coefficients do not depend on the direction of wave propagation, but *R* changes its sign.
    - *R < 0* leads to *phase reversal* in reflection records.

Typical impedance contrasts and reflectivities

Table 3.1 *Energy reflected at interface between two media*

	First medium		Second medium				
Interface	Velocity	Density	Velocity	Density	Z_1/Z_2	R	E_R
Sandstone on limestone	2.0	2.4	3.0	2.4	0.67	0.2	0.040
Limestone on sandstone	3.0	2.4	2.0	2.4	1.5	−0.2	0.040
Shallow interface	2.1	2.4	2.3	2.4	0.93	0.045	0.0021
Deep interface	4.3	2.4	4.5	2.4	0.97	0.022	0.0005
"Soft" ocean bottom	1.5	1.0	1.5	2.0	0.50	0.33	0.11
"Hard" ocean botom	1.5	1.0	3.0	2.5	0.20	0.67	0.44
Surface of ocean (from below)	1.5	1.0	0.36	0.0012	3800	−0.9994	0.9988
Base of weathering	0.5	1.5	2.0	2.0	0.19	0.68	0.47
Shale over water sand	2.4	2.3	2.5	2.3	0.96	0.02	0.0004
Shale over gas sand	2.4	2.3	2.2	1.8	1.39	−0.16	0.027
Gas sand over water sand	2.2	1.8	2.5	2.3	0.69	0.18	0.034

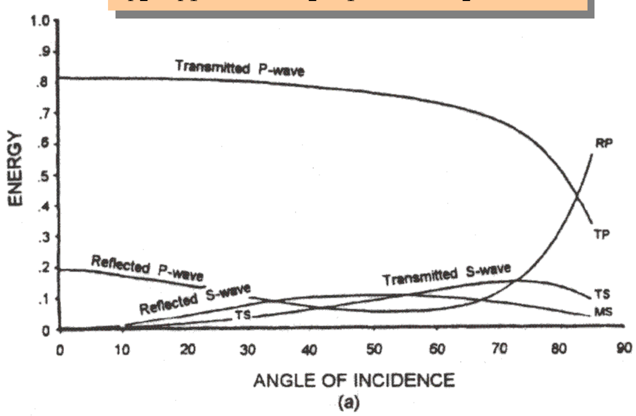
All velocities in km/s, densities in g/cm^3; the minus signs indicate 180° phase reversal.

Oblique incidence

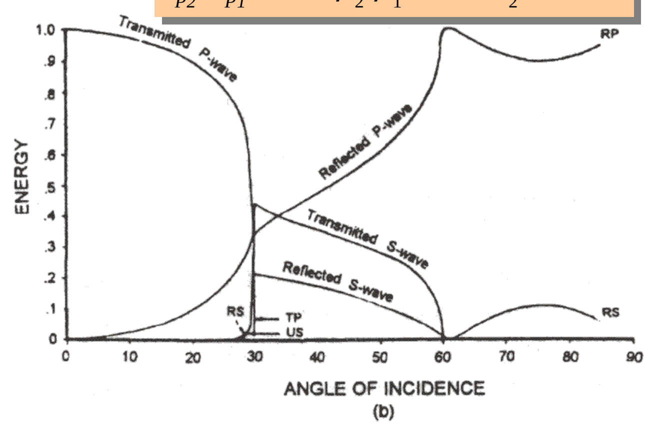
Amplitude versus Angle (AVA) variation

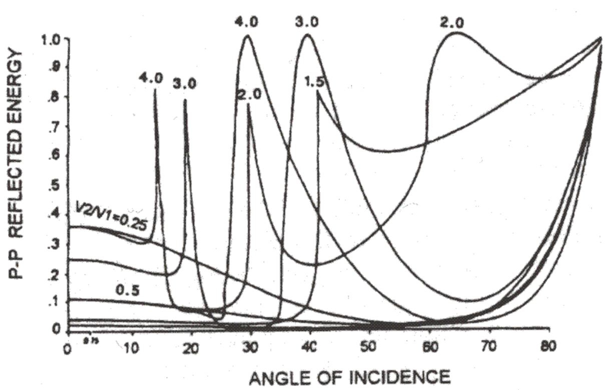
- At oblique incidence, we have to use the full $\mathbf{M}^{-1}\mathbf{N}$ expression for $\mathbf{S}$
  - *Amplitudes* and *polarities* of the reflections vary with incidence angles.

Fast to slow:
$V_{P2}/V_{P1} = 0.5$, $\rho_2/\rho_1 = 0.8$; $\sigma_2 = 0.25$

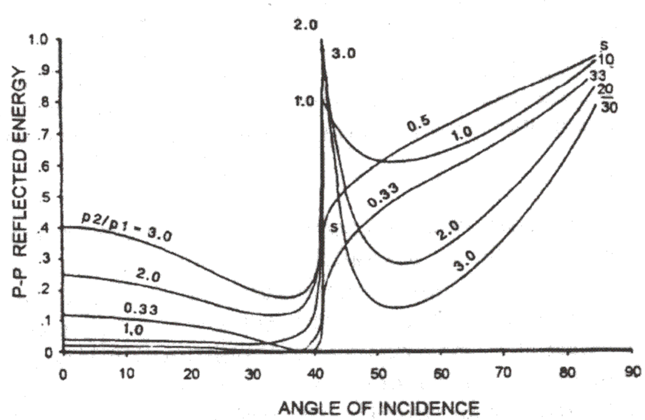


Slow to Fast:
$V_{P2}/V_{P1} = 2.0$, $\rho_2/\rho_1 = 0.5$; $\sigma_2 = 0.3$





Fraction of P-wave reflection energy, for various V_{P2}/V_{P1}
$\rho_2/\rho_1 = 1.0$; $\sigma_1 = \sigma_2 = 0.25$



Fraction of P-wave reflection energy, for various ρ_2/ρ_1
$V_{P2}/V_{P1} = 1.5$; $\sigma_1 = \sigma_2 = 0.25$

Oblique incidence
Small-contrast AVA approximation

- ΔV_P, ΔV_S, $\Delta\rho$, and therefore, ray angle variations are considered small
  - Shuey's (1985) formula gives the variation of R from the cae on normal incidence in terms of ΔV_P and $\Delta\sigma$ (Poisson's ratio):

Important at >~30°

$$\frac{R(\theta)}{R(0)} \approx 1 + \boxed{P\sin^2\theta} + \boxed{Q\left(\tan^2\theta - \sin^2\theta\right)}$$

Important at typical reflection angles

  - where:

$$R(0) \approx \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right),$$

$$P = \left[Q - \frac{2(1+\sigma)(1-2\sigma)}{1-\sigma}\right] + \frac{\Delta\sigma}{R(0)(1-\sigma)^2},$$

$$Q = \frac{\dfrac{\Delta V_P}{V_P}}{\dfrac{\Delta V_P}{V_P} + \dfrac{\Delta\rho}{\rho}} = \frac{1}{1 + \dfrac{\Delta\rho/\rho}{\Delta V_P/V_P}}.$$

Amplitude versus Offset (AVO)

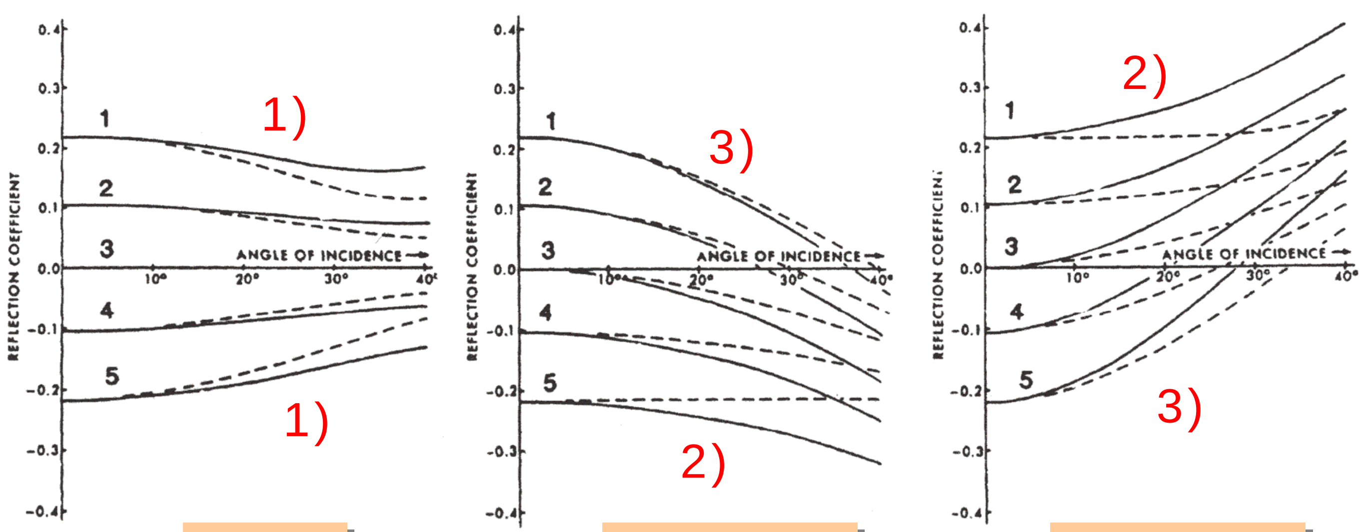
- AVO is a group of processing techniques designed to detect reflection AVA effects:
  - Records processed with *true amplitudes* (preserving proportionality to the actual recorded amplitudes);
  - Source-receiver offsets converted to the incidence angles;
  - From pre-stack (variable-offset) data gathers, parameters $R(0)$, P and Q are estimated:

$$R(\theta) \approx R(0)\left[1 + P\sin^2\theta + Q\left(\tan^2\theta - \sin^2\theta\right)\right].$$

  - Thus, additional attributes are extracted to distinguish between materials with varying σ.

Three practical AVA cases

- Three typical AVA behaviours:
  1) Amplitude decreases with angle without crossing 0;
  2) Amplitude increases;
  3) Amplitude decreases and crosses 0 (reflection polarity changes)



$\sigma_2 = \sigma_1$
=0.3 (solid)
=0.2 (dashed)

$\sigma_2 < \sigma_1$
: 0.4 to 0.1 (solid)
: 0.3 to 0.1 (dashed)

$\sigma_2 > \sigma_1$
: 0.1 to 0.4 (solid)
: 0.1 to 0.2 (dashed)

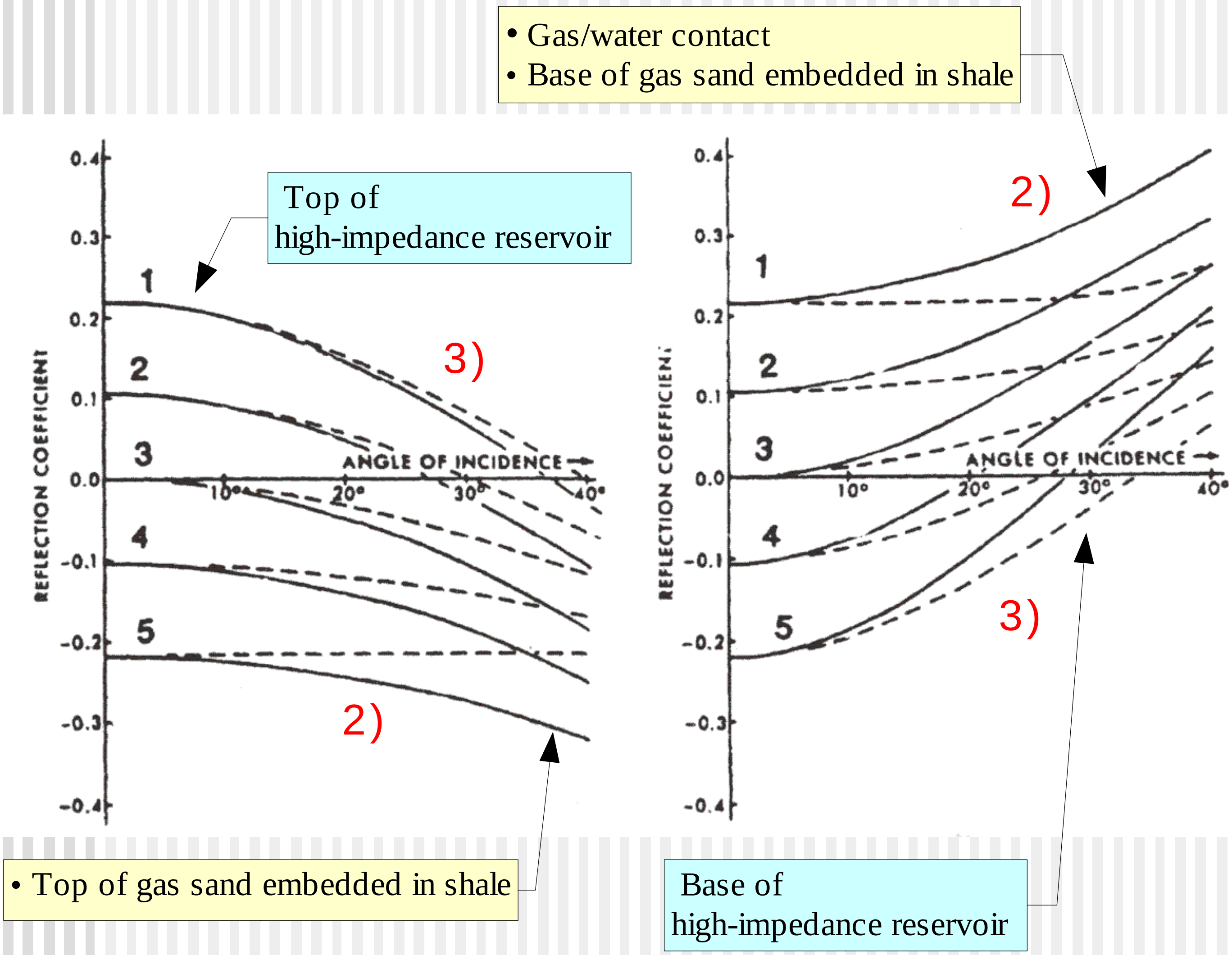
"Normal" case

"AVO (AVA) anomalies"

(Above: $V_{P2}/V_{P1} = \rho_2/\rho_1$ =1.25; 1.11; 1.0; 0.9, and 0.8)

From Ostrander, 1984

AVA (AVO) anomalies

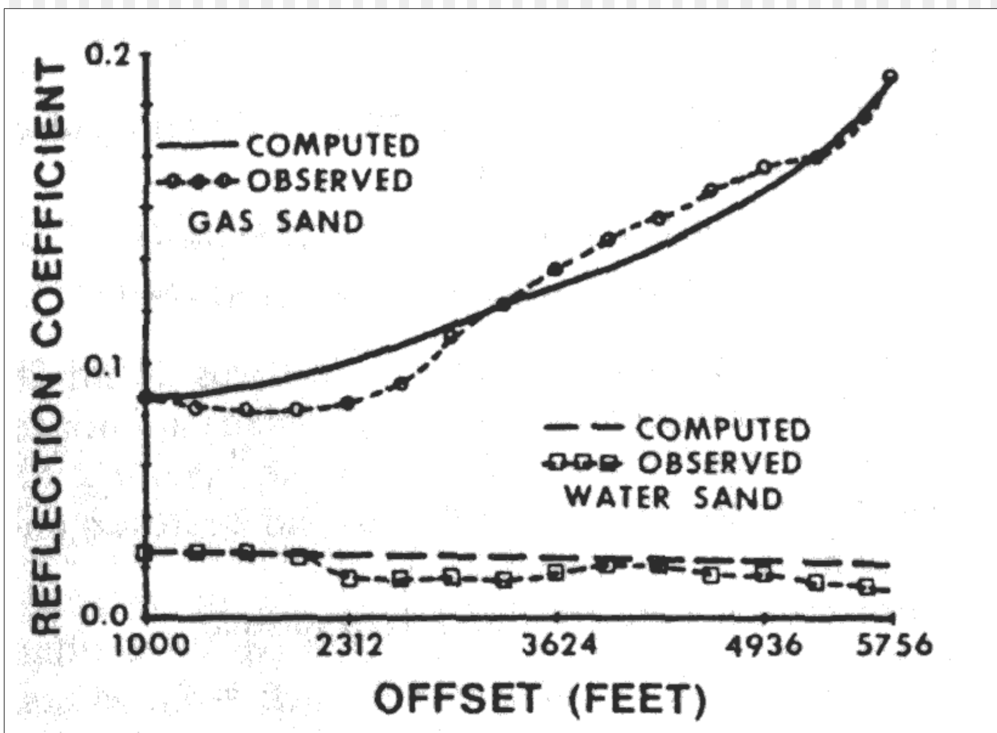


From Ostrander, 1984

Amplitude versus Offset (AVO)

Gas sand vs. wet sand

- Gas-filled pores tend to reduce V_P more than V_S, and as a result, the Poisson's ratio (σ) is reduced.
- Negative ΔV_P *and* $\Delta\sigma$ thus cause negative-polarity bright reflection ("bright spot") *and* an AVO effect (increase in reflection amplitude with offset) that are regarded as hydrocarbon indicators.
  - However, not every AVO anomaly is related to a commercial reservoir...



From Yu, 1985