Time and Spatial Series and Transforms

- Z- and Fourier transforms
- Gibbs' phenomenon
- Convolution
- Cross- and Auto-correlation
- Multidimensional transforms

- Reading:
  - Sheriff and Geldart, Chapter 15

Z-Transform

- Consider a digitized record of *N* readings: U={u_0, u_1, u_2, ..., u_{N-1}}. How can we represent this series differently?
- The *Z* transform simply associates with this time series a *polynomial*:

$$Z(U) = u_0 + u_1 z + u_2 z^2 + u_3 z^3 + \ldots$$

  - For example, a 3-sample record of {1,2,5} is represented by a quadratic polynomial:
    $1 + 2z + 5z^2$.
- In *Z*-domain, the all-important operation of *convolution* of time series becomes simple multiplication of their Z-transforms:

$$U_1 * U_2 \leftrightarrow Z(U_1)Z(U_2)$$

Fourier Transform

- To describe a polynomial of order *N*-1, it is sufficient to specify its values at *N* points in *z*.
- The *Discrete Fourier transform* is obtained by taking the *Z*-transform at *N* points uniformly distributed around a unit circle on the complex plane of *z*:

$$U(k) = \sum_{m=0}^{N-1} e^{-i\frac{2\pi k}{N}m} u_m, \qquad k = 0,1,\ldots,N-1$$

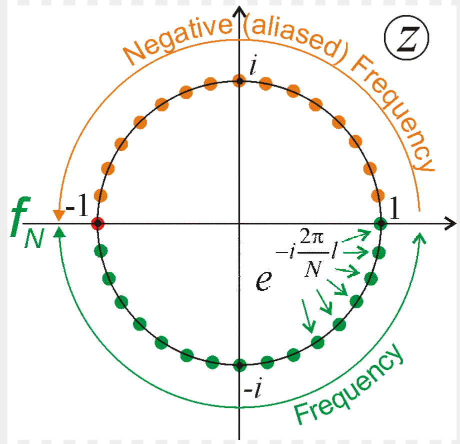
- Each term (*k*>0) in the sum above is a *periodic function* (a combination of *sin* and *cos*), with a period of *N*/*k* sampling intervals:

$$e^{-i\alpha} = \cos\alpha - i\sin\alpha$$

- Thus, the Fourier transform expresses the signal in terms of its frequency components,
  - and also has the nice property of the *Z*-transform regarding convolution

Relation of Fourier to Z-transform

- z-points used to construct the Fourier transform:



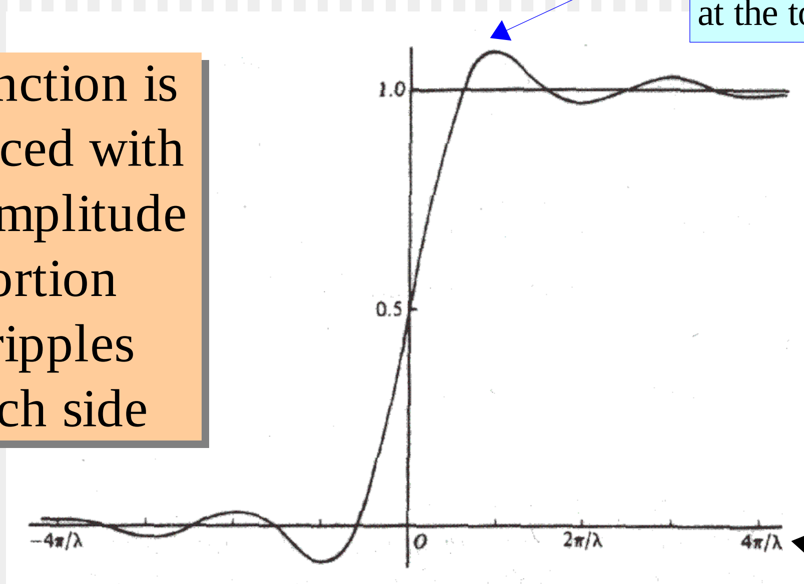
- **Aliasing**: for a real-valued signal, the values of FT at frequencies below and above the Nyquist (orange and green dots) are complex conjugate. Thus, only a half of the frequency band describes the process uniquely.
- This ambiguity is the source of aliasing.
- For this reason, frequencies above f_N should not be used.
- **Note**: forward and inverse FT result in a signal whose N samples are repeated periodically in time.

Gibbs' phenomenon

- At a discontinuity, application of the Fourier forward and inverse transform (with a limited bandwidth), results in ringing.

Note the ~9% "overshoot" at the top and at the bottom

Step function is reproduced with ~18% amplitude distortion and ripples on each side



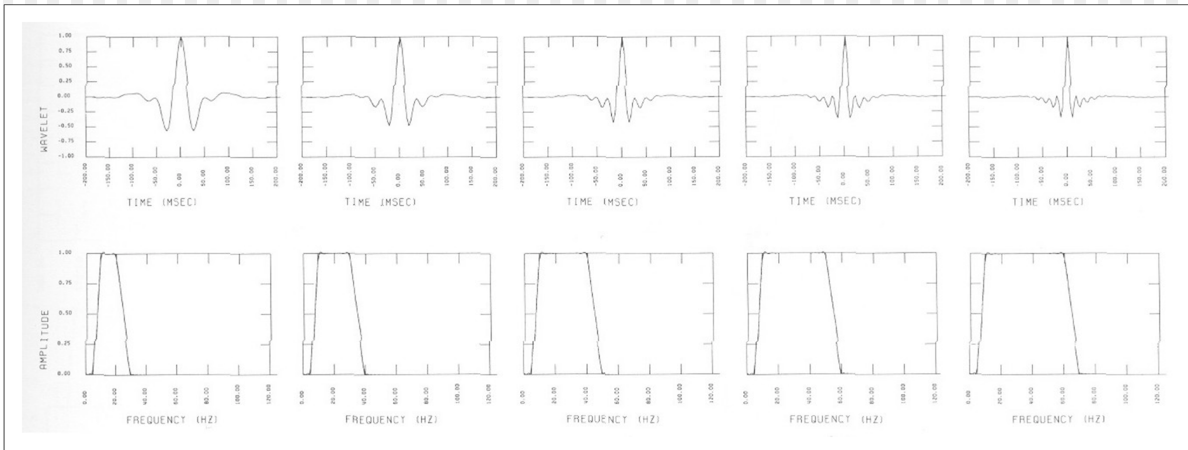
λ is our ω_{max}

- This is important for constructing time and frequency windows
  - Boxcar windows create ringing at their edges.
  - "Hanning" (cosine) windows are often used to reduce ringing:

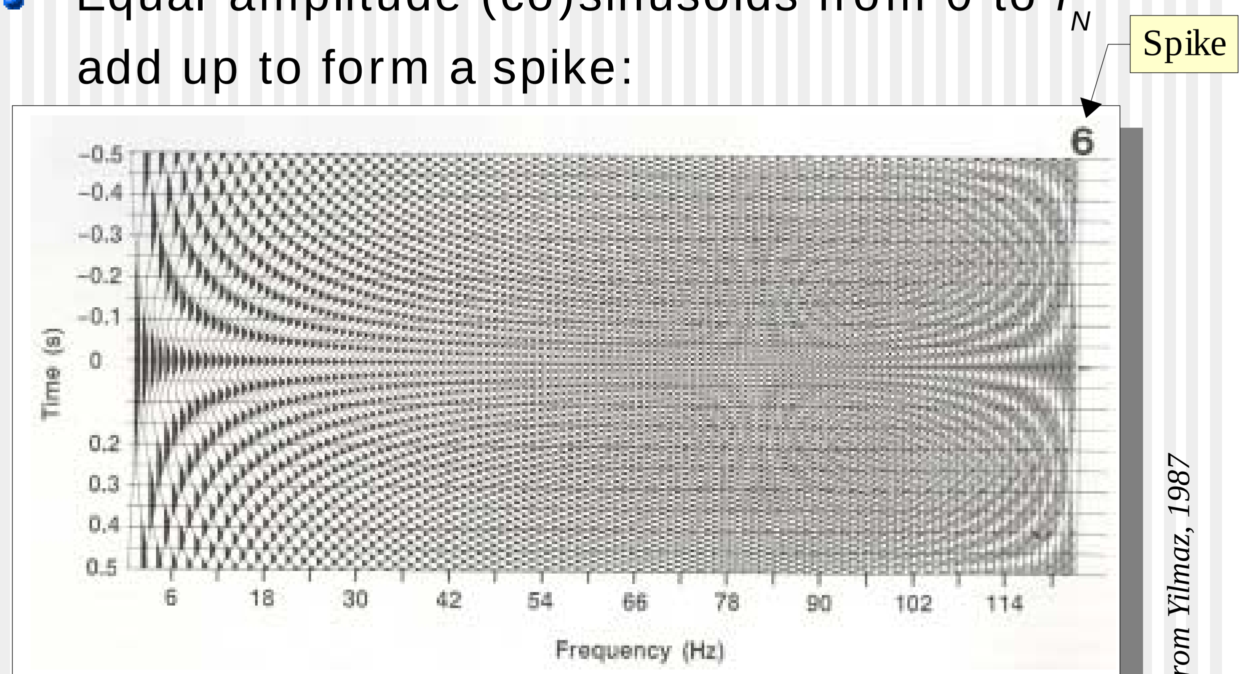
$$H_{\Delta t}(t) = \frac{1}{2}\left(1 - \cos\frac{\pi t}{\Delta t}\right).$$

Spectra of Pulses

- For a pulse of width T s, its spectrum is about $1/T$ Hz in width:



- Equal-amplitude (co)sinusoids from 0 to f_N add up to form a spike:



From Yilmaz, 1987

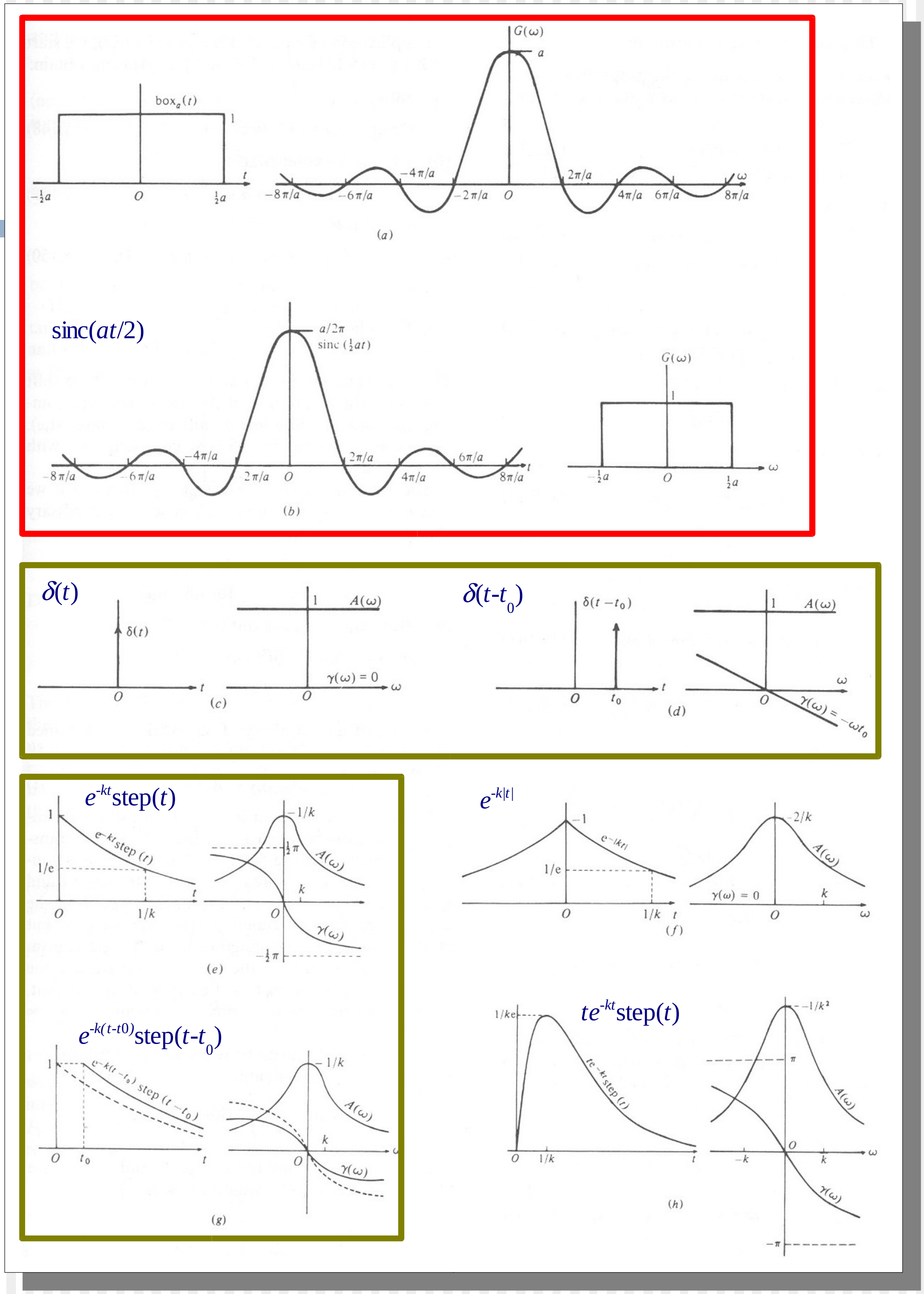
Integral Fourier Transform

- For continuous time and frequency (infinitesimal sampling interval and infinite recording time), Fourier transform reads:
  - Forward: $$U(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, u(t) e^{-i\omega t}.$$
  - Inverse: $$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, U(\omega) e^{i\omega t}.$$
- In practice, the bandwidth (and time) is always limited, and so the actual combination of the forward and inverse transforms is rather:

$$u(t) = \frac{1}{2\pi} \int_{-\omega_{max}}^{\omega_{max}} d\omega \left[\int_{-\infty}^{\infty} d\tau\, u(\tau) e^{-i\omega\tau} \right] e^{i\omega t}.$$

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, u(\tau) \left[\int_{-\omega_{max}}^{\omega_{max}} d\omega\, e^{i\omega(t-\tau)} \right].$$

Sample Fourier Transforms



From Sheriff, Geldart, 1995

- Compare the transforms within the boxes...

Convolution

- Convolution of two series, u_i, and w_i, denoted $u*w$, is:

$$\left(u*w\right)_i = \sum_k u_k w_{i-k}$$

For each i, the result is dot product of *u* and shifted and "reflected", or "folded" (i..e., running backwards) *w*

- In integral form:

$$u(t)*w(t) = \int_{-\infty}^{+\infty} u(\tau)w(t-\tau)d\tau$$

- As multiplication, it is symmetric (commutative):

$$u*w = w*u$$

- Note that to multiply two polynomials, with coefficients u_k and w_k, we would use exactly the first formula above. Therefore, **in *Z* or *frequency* domains, convolution becomes simple multiplication** of polynomials (show this!):

$$u*w \leftrightarrow Z(u)Z(w) \leftrightarrow F(u)F(w)$$

- This property facilitates efficient digital filtering.

Cross-Correlation

- Cross-correlation of two series, u_i, and w_i, is:

$$\left(u*w\right)_i=\sum_k u_k w_{i+k}$$

Unlike in convolution, no "folding" of *w*

- In integral form:

$$u(t)*w(t)=\int_{-\infty}^{+\infty} u(\tau)w(t+\tau)d\tau$$

- It is anti-symmetric in the following sense (show this!):

$$\left(u*w\right)(t)=\left(w*u\right)(-t)$$

- **In *Z* or *frequency* domains**, cross-correlation is:

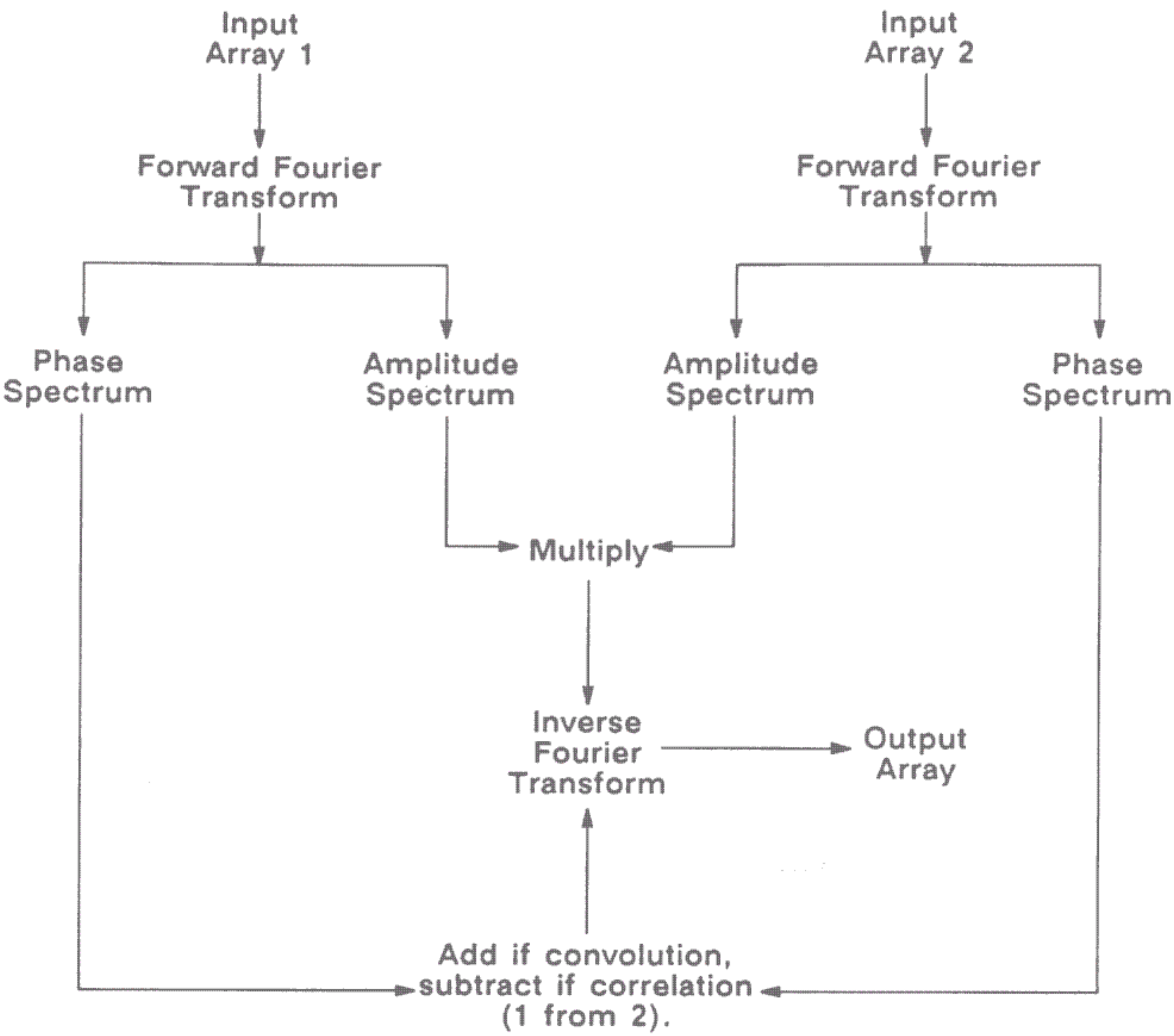
$$\left(u*w\right)(z)=\overline{U(z)}W(z)$$

Complex conjugate

- Cross-correlation is used as a *measure of similarity* between time series.

Practical convolution and cross-correlation

- Is attained via (Fast) Fourier Transform



From Yilmaz, 1987

Autocorrelation

- Cross-correlation of a time function with itself is called *Autocorrelation*:

$$\left(u*u\right)_i=\sum_k u_k u_{i+k}$$

- In integral form:

$$Auto_u(t)=\int_{-\infty}^{+\infty} u(\tau)u(t+\tau)d\tau$$

- It always is an even function (show this!):

$$Auto_u(t)=Auto_u(-t)$$

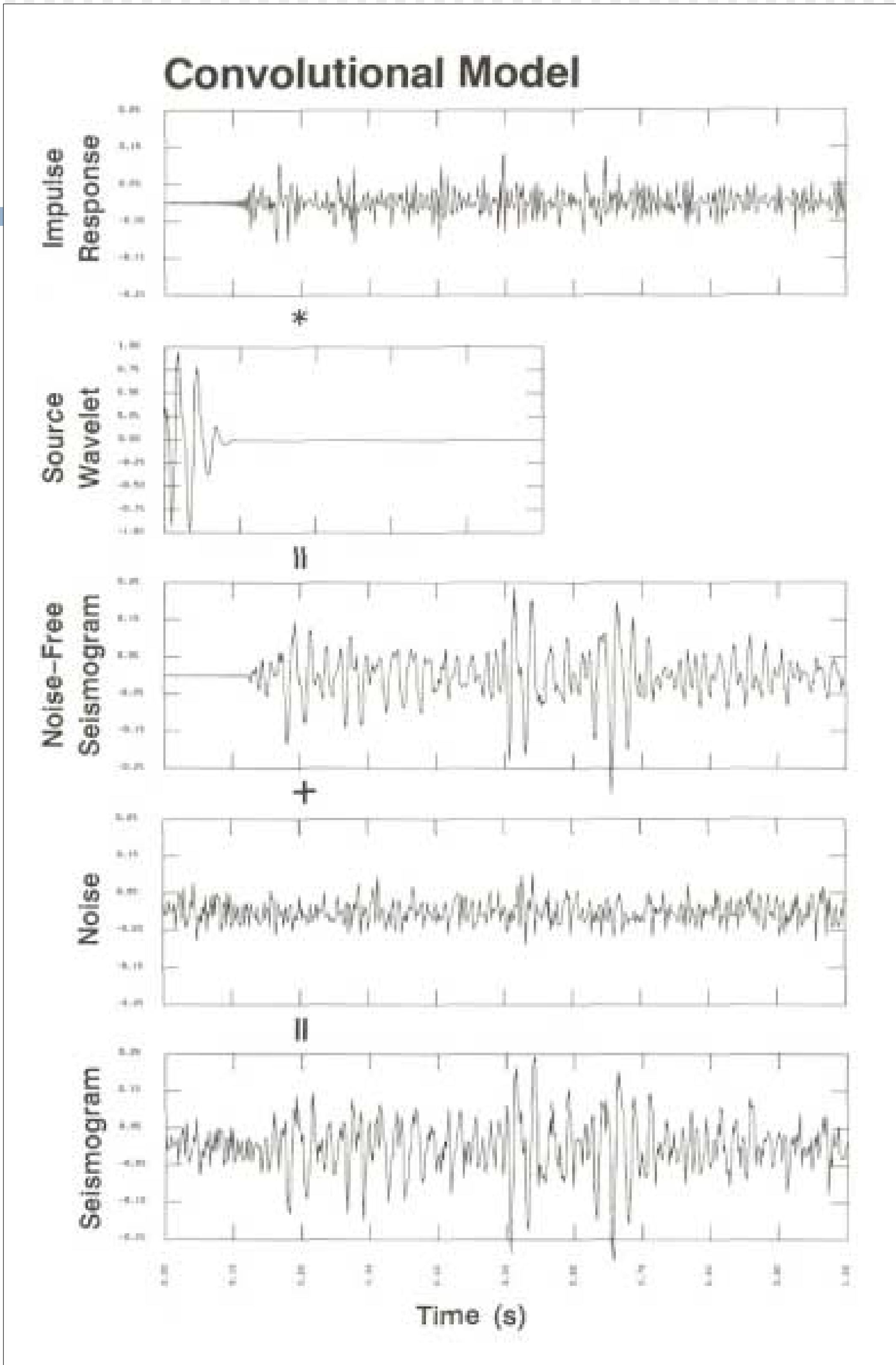
- **In *Z* or *frequency* domains**, autocorrelation is:

$$Auto_u(z)=\overline{U(z)}U(z)=\left|U(z)\right|^2$$

Always real value - **Energy Spectrum**

- Therefore, autocorrelation is also the Fourier transform of the energy spectrum of the signal
  - It is independent of the phase spectrum!
- Autocorrelation is used as a *measure of self-similarity* within a time series.

Convolutional model



From Yilmaz, 1987