Note that both upward and downward continuation only apply to regions with zero densities(sources).

Here is the integral form of upward continuation again.

$$\Delta g(x, y, -h) = \frac{h}{2\pi} \int_{-\infty}^{\infty} \int \frac{\Delta g(\xi, \eta)}{((x-\xi)^2 + (y-\eta)^2 + h^2)^{3/2}} d\xi d\eta$$

Can we change the sign of h and get a downward continuation formula?

We derived the upward continuation formula from the equivalent stratum. ~~The equivalent stratum only works above the plane on which it is defined.~~ Changing the sign in the above formula will therefore not result in a valid downward continuation formula.

Upward continuation integral assumes an (infinite) equivalent density layer immediately below the survey plane. For DOWNWARD continuation, we need a formula without such a layer.

Downward continuation as in inverse problem

$$\Delta g(x,y,0) = \frac{h}{2\pi}\int_{-\infty}^{\infty}\int \frac{\Delta g(\xi,\eta,h)}{((x-\xi)^2+(y-\eta)^2+h^2)^{3/2}}d\xi d\eta$$

What values of $\Delta g(\xi,\eta,h)$ on a plane h below the survey plane will produce the observed values on the survey plane?

Fourier transforming both sides

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int \Delta g(x,y,0)e^{-i(\omega_x x+\omega_y y)}dxdy$$
$$=\frac{h}{2\pi}\int_{\xi}\int \frac{1}{2\pi}\int_{\infty}\frac{\Delta g(\xi,\eta,h)}{((x-\xi)^2+(y-\eta)^2+h^2)^{3/2}}dxdyd\xi d\eta$$

and changing variables on the RHS side so that

$$x-\xi = rcos\theta \quad \text{and} \quad y-\eta = rsin\theta$$

$$G_o(\omega_x,\omega_y) = \frac{h}{4\pi^2}[\int_{\xi}\int_{\eta}\Delta g(\xi,\eta,h)e^{-i(w_x\xi+\omega_y\eta)}d\xi d\eta$$
$$\int_r\int_{\theta}\frac{e^{-ir(\omega_x cos\theta+\omega_y sin\theta)}}{(r^2+h^2)^{3/2}}rdrd\theta]$$

$$G_o(\omega_x, \omega_y) = \frac{h}{2\pi} G_h(\omega_x, \omega_y) \int_r \int_\theta \frac{e^{-ir(\omega_x cos\theta + \omega_y sin\theta)}}{(r^2+h^2)^{3/2}} rdrd\theta]$$

and

$$\int_0^{2\pi} e^{-ir}(w_x cos\theta + w_y sin\theta) d\theta = 2\pi J_0(ur)$$

and

$$u = -w_x/sin\theta = -\omega_y/cos\theta$$

so

$$\begin{aligned} G_o(\omega_x, \omega_y) =& hG_h(\omega_x, \omega_y) \int (r^2+h^2)^{-3/2} J_0(ur) dr \\ =& hG_h(\omega_x, \omega_y) \frac{e^{-uh}}{h} \\ =& e^{-(\omega_x^2+\omega_y^2)^{1/2}} G_h(\omega_x, \omega_y) \end{aligned}$$

and this equation is easy to invert

$$G_h(\omega_x, \omega_y) = G_o(\omega_x, \omega_y) e^{(\omega_x^2+\omega_y^2)^{1/2} h}$$

and then taking the inverse transform

$$\Delta g(x, y, h) = \int_\infty^\infty G_o(\omega_x, \omega_y) e^{(\omega_x^2+\omega_y^2)^{1/2} h + i(\omega_x x + \omega_y y)} d\omega_x d\omega_y$$

and this is a FOURIER TRANSFORM DOWNWARD CONTINUATION formula

$$\Delta g(x, y, \pm h) = \int_{-\infty}^{\infty} G_o(\omega_x, \omega_y) e^{\pm(\omega_x^2+\omega_y^2)^{1/2} h + i(\omega_x x + \omega_y y)} d\omega_x d\omega_y$$

~~Surprisingly,~~ this formula works for either up (choose $-$) or down (choose $+$)

So, to continue a potential field, Fourier transform, multiply by the CONTINUATION FACTOR

$$e^{\pm(\omega_x^2+\omega_y^2)^{1/2}(h)}$$

and inverse Fourier transform

This form is actually the integral Fourier transform. To convert to the discrete transform all we need to do is recognize the discrete frequencies

f_nx and f_ny here are the Nyquist frequencies along X and Y;
f_nx/(N/2)and f_ny/(N/2) are the increments in frequency

$$\omega_x = 2\pi \frac{n f_{nx}}{N/2} \quad n = -N/2 \cdots 0 \cdots N/2 - 1$$

$$\omega_y = 2\pi \frac{m f_{ny}}{M/2} \quad m = -M/2 \cdots 0 \cdots M/2 - 1$$

The 2π is there because $\omega_{x,y}$ are radial frequencies and $f_{nx,ny}$ are circular frequencies. We can also write these as

$$\omega_x = 2\pi \frac{1}{2\Delta x} \frac{n}{N/2} \quad \text{and} \quad \omega_y = 2\pi \frac{1}{2\Delta y} \frac{m}{M/2}$$

and equal grid spacings

With N=M the continuation factor then becomes

$$=e^{\frac{2\pi(n^2+m^2)^{1/2}}{N/2}\frac{S}{2}} \quad \text{CONTINUE S GRID INCREMENTS}$$

$$=e^{\frac{2\pi(n^2+m^2)^{1/2}}{N/2}S} \quad \text{CONTINUE S NYQUIST WAVELENGTHS}$$

Let's look at a case where N=M=16, then n,m=0 to 8

DOWNWARD

CONTINUATION FACTORS

	S		
n,m	.25	.50	1.00
0	0	0	0
1	1.32	1.74	3.04
2	1.74	3.04	9.22
3	2.30	5.29	27.99
4	3.03	9.22	85.02
5	4.01	16.06	258.2
6	5.29	27.99	883.2
7	6.98	48.99	2380
8	9.22	85.02	7228

So, as the frequency counter increases, or, as the wavelength decreases, the continuation factor increases dramatically. The continuation factor may increase faster with frequency than the spectrum decreases in amplitude with frequency, so downward continuation is inherently unstable.

CUTOFFS FOR DOWNWARD CONTINUATION

1) RADIAL FREQUENCY SPECTRUM

2) CHARACTERISTIC SOURCE DEPTH

From the full width at half height, the source depth is

$$h \approx 0.65 \times w_{1/2} \sim \frac{0.65\lambda_D}{2\,\text{to}\,3} \sim \frac{\lambda}{3\,\text{to}\,4}$$

Lets call a source depth calculated from the dominant wavelength the CHARACTERISTIC SOURCE DEPTH

So

$$\lambda_D \sim 4h \quad \text{or} \quad h_{CSD} \sim \frac{\lambda_d}{4}$$

DOMINANT WAVELENGTH $\sim 4\times$ CHARACTERISTIC SOURCE DEPTH

This means that

SOURCES BURIED AT $\lambda/4 = 2\Delta x/4 = \Delta x/2$ WILL HAVE A DOMINANT WAVELENGTH AT THE NYQUIST AND WILL CERTAINLY BE ALIASED, SO WE CANNOT HAVE SOURCES SHALLOWER THAN $\Delta x/2$

and that

THE LONGEST FOURIER WAVELENGTH IS $N\Delta x$, WHICH WOULD HAVE A CHARACTERISTIC SOURCE DEPTH OF $N\Delta x/4$ WHICH IS 1/4 OF THE SURVEY WIDTH

THE SHORTEST FOURIER WAVELENGTH IS $2\Delta x$ WHICH HAS A CHARACTERISTIC SOURCE DEPTH OF $\Delta x/2$

RULE OF THUMB FOR CUTOFF IN DOWNWARD CONTINUATION

Cut at these frequencies...

When the source is at these depths...

HARMONIC n	FREQUENCY	CSD
1	$f_N/(N/2)$	$N/4$
2	$2f_N/(N/2)$	$N/8$
3	$3f_N/(N/2)$	$N/12$
4	$4f_N/(N/2)$	$N/16$
.	.	.
.	.	.
.	.	.
$N/2-1$	$(N/2-1)f_n/(N/2)$	$N/4/(N/2-1)$
$N/2$	f_N	$1/2$

CONTINUATION FILTERS

You can view the downward- and upward continuation as depth-dependent linear filtering (convolution).

$$\Delta g(x,y,z) = \frac{1}{N^2}\Sigma_n\Sigma_m e^{2\pi\sqrt{\frac{(n^2+m^2)}{N/2}}z/2} G(\frac{nf_{nx}}{N/2}, \frac{nf_{my}}{N/2}) e^{2\pi i(\frac{nx}{N}+\frac{my}{N})}$$

where z is the number of grid increments to continue $N = M$ and n, m are the counters for N and M

Or, rearranging

$$\Delta g(x,y,z) = \frac{1}{N^2}\Sigma_n\Sigma_m e^{2\pi\sqrt{\frac{(n^2+m^2)}{N/2}}z/2} \Sigma_\xi\Sigma_\eta \Delta g(\xi,\eta) e^{2\pi i(\frac{n(x-\xi)}{N}+\frac{m(y-\eta)}{N})}$$

$$\Delta g(0,0,z) = \frac{1}{N^2}\Sigma_n\Sigma_m e^{2\pi\sqrt{\frac{(n^2+m^2)}{N}}z} \Sigma_\xi\Sigma_\eta \Delta g(\xi,\eta) e^{-2\pi i(\frac{n(\xi)}{N}+\frac{m(\eta)}{N})}$$

$$\Delta g(0,0,z) = \Sigma_\xi\Sigma_\eta \Delta g(\xi,\eta) \frac{1}{N^2}\Sigma_n\Sigma_m e^{2\pi\sqrt{\frac{(n^2+m^2)}{N}}z} e^{-2\pi i(\frac{n(\xi)}{N}+\frac{m(\eta)}{N})}$$

or

$$\Delta g(0,0,z) = \Sigma_\xi\Sigma_\eta \Delta g(\xi,\eta) F(-\xi,-\eta) \quad \text{or more generally}$$

$$\Delta g(x,y,z) = \Sigma_\xi\Sigma_\eta \Delta g(\xi,\eta) F(x-\xi, y-\eta)$$

HOW FAR CAN YOU DOWNWARD CONTINUE?

IT DEPENDS ON Δx and λ_D

AS YOU DOWNWARD CONTINUE, λ_D WILL DECREASE. AT WHAT DEPTH WILL $\lambda_D = \lambda_N$?

$$\lambda_D = 4z_o = 2\Delta x \rightarrow z_o = \Delta x/2$$

SO IF YOU DOWNWARD CONTINUE TO WITHIN $\Delta x/2$ OF THE SOURCE, THE DOMINANT WAVELENGTH WILL BE THE NYQUIST.

IN PRACTICE YOU WILL NOT EVEN BE ABLE TO GET THIS CLOSE. $2\Delta x$ OR $4\Delta x$?

IN GENERAL CONTINUING UP IS MUCH SAFER THAN CONTINUING DOWN.

CONTINUATION SUMMARY

UPWARD CONTINUATION IS STABLE - NO FILTERING REQUIRED- BUT THE AREA IN WHICH CONTINUED VALUES ARE ACCURATE SHRINKS THE FURTHER UP YOU CONTINUE. ABOUT 1/12 th OF THE SURVEY WIDTH MAY BE THE MAXIMUM.

DOWNWARD CONTINUATION IS UNSTABLE AND SOME LOW PASS FILTERING IS REQUIRED.

ONE HALF THE DEPTH TO SOURCE IS A PRACTICAL MAXIMUM DOWNWARD CONTINUATION

YOU CANNOT CONTINUE TO WITHIN Δx or better $2\Delta X$ OF THE SOURCE

THE RADIAL FREQUENCY SPECTRUM AND THE RULE-OF-THUMB HELP TO DECIDE THE CUTOFF FOR THE LOW PASS FILTER.

Some problems with downward continuation

Continuation only works if the quantity being continued satisfies Laplace's eqtn, because ~~only then is there a uniqueness Th^{rm}~~ otherwise one needs to know the source density. The density in the ground is not zero, so we should not be able to downward continue, except that in the process of reduction, and in particular regional residual separation, we accounted for the attraction of the host rock, so that the only mass producing the anomaly is the mass excess or deficit with the host rock, that is, the anomaly looks like it is caused by a body with mass $(\rho - \rho_h)V$ and the host rock now has zero density.

We found that upward continuation worked in a smaller area the farther we upward continued. The situation is worse for downward continuation. The volume in which we can make the uniqueness Th^{rm} work for downward continuation is an inverted cone with its apex at the source and its base the survey plane.

Rather,not the uniqueness theorem but approximation of zero density anomaly within the zone of continuation.