DERIVATIVES OF POTENTIAL FIELDS

Potential fields satisfy

$$\frac{\partial^2 \Delta g}{\partial x^2} + \frac{\partial^2 \Delta g}{\partial y^2} + \frac{\partial^2 \Delta g}{\partial z^2} = 0$$

and of course we could substitute the mag total field or vertical field for Δg

The second vertical derivative is then

$$\frac{\partial^2 \Delta g}{\partial z^2} = -(\frac{\partial^2 \Delta g}{\partial x^2} + \frac{\partial^2 \Delta g}{\partial y^2})$$

The second vertical derivative is used as a high pass filter to enhance short horizontal scale features (short wavelength) at the expense of long horizontal scale features (long wavelength).

SECOND VERTICAL DERIVATIVE CONVOLUTION OPERATOR

Suppose we take three elements of a row or column of a gridded data set.

$$g_{i,k-1}\, g_{j,k}\, g_{j,k+1}$$

Then, we can estimate the first horizontal derivative at two points

$$\frac{\partial g}{\partial x}|_{j,k-1/2} \approx \frac{g_{j,k} - g_{j,k-1}}{1}$$
$$\frac{\partial g}{\partial x}|_{j,k+1/2} \approx \frac{g_{j,k} - g_{j,k+1}}{1}$$

where $j, k \pm 1/2$ means the point half way between k and $k \pm 1$. An estimate of the second horizontal derivative is then

$$\frac{\partial^2 g}{\partial x^2} \approx \frac{\frac{\partial g}{\partial x}|_{j,k-1/2} - \frac{\partial g}{\partial x}|_{j,k-1/2}}{1}$$
$$\approx g_{j,k+1} - g_{j,k} - g_{j,k} + g_{j,k-1}$$
$$\approx g_{j,k+1} + g_{jk-1} - 2gj, k$$

Here we are differentiating along the column index, so in the x direction, we could do the same thing in the y direction by working on the first index. So, a three element convolution filter to calculate the second vertical derivative would be.

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

SECOND VERTICAL DERIVATIVE (FOURIER)

$$\Delta g(x,y,0) = \int\int G(\omega_x,\omega_y)e^{+i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$
$$\frac{\partial^2 \Delta g(x,y,0)}{\partial x^2} = -\int\int \omega_x^2 G(\omega_x,\omega_y)e^{+i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$
$$\frac{\partial^2 \Delta g(x,y,0)}{\partial y^2} = -\int\int \omega_y^2 G(\omega_x,\omega_y)e^{+i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$
$$\frac{\partial^2 \Delta g(x,y,0)}{\partial z^2} = \int\int (\omega_x^2+\omega_y^2) G(\omega_x,\omega_y)e^{+i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$

Once we have $G(\omega_x, \omega_y$ ie we have done a Fourier transform for any reason, we can easily calculate the second vertical derivative in wavenumber space and then inverse transform.

The preceding should make it clear that the gain of a second vertical derivative operator increases as the square of frequency.

So we could potentially have a serious problem with Gibbs' phenomenon at high frequencies. To avoid this, a high cut (high frequencies are attenuated) operator could be applied.

$$2^{nd} V.D. = V * S$$

The resulting filter would have a gain that initially increases as the square of frequency, but at high frequencies the high cut filter limits the gain.

Or in the Fourier context, a low pass or high cut filter could be applied. Remember that it does not matter which order these operations are done in. We could apply the high cut filter first and then the second vertical derivative, or the other way around.

FIRST VERTICAL DERIVATIVE (FOURIER)

See comments file (note a couple typos in formulas below)

The gravity anomaly at depth z is

$$\Delta g(x, y, z) = \int\int G_o(\omega_x, \omega_y) e^{(\omega_x^2+\omega_y^2)^{1/2} z + i(\omega_x x + \omega_y y)} d\omega_x d\omega_y$$

and we can differentiate this wrt z

$$\frac{\partial \Delta g(x, y, z)}{z} = \int\int \frac{|z|}{z} (\omega_x^2+\omega_y^2)^{1/2} G_o(\omega_x, \omega_y) e^{(\omega_x^2+\omega_y^2)^{1/2} z + i(\omega_x x + \omega_y y)} d\omega_x d\omega_y$$

and at z=0, but in the limit from below

$$\frac{\partial \Delta g(x, y, 0)}{z} = \int\int \frac{|z|}{z} (\omega_x^2 + \omega_y^2)^{1/2} (\omega_x^2 + \omega_y^2) \times G_o(\omega_x, \omega_y) e^{(\omega_x^2+\omega_y^2)^{1/2} z + i(\omega_x x + \omega_y y)} d\omega_x d\omega_y$$

When we did continuation, our result worked from level surface to level surface. We could think about first and second derivatives a s a way to go from a curved surface to a level surface or level to curved or curved to curved.

Drape to level?

Level to drape?

THE ANALYTIC SIGNAL

The analytic signal is a relatively new derivative tool for mag.

$$AS = |\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}|$$

it can either be measured directly, which has the major advantage that a base station is not required, or it can be computed form total field measurements.

- TENDS TO SHOW A SINGLE PEAK CENTERED OVER SOURCE
- GRADIENT MEASURED OVER SMALL DISTANCES IS NOISY
- LONG λ's HAVE LOW S/N
- NOT MUCH SOFTWARE FOR INTERPRETATION

THE GRADIENT TENSOR

$$\begin{pmatrix} \frac{\partial T_x}{\partial x} & \frac{\partial T_x}{\partial y} & \frac{\partial T_x}{\partial z} \\ \frac{\partial T_y}{\partial x} & \frac{\partial T_y}{\partial y} & \frac{\partial T_y}{\partial z} \\ \frac{\partial T_z}{\partial x} & \frac{\partial T_z}{\partial y} & \frac{\partial T_z}{\partial z} \end{pmatrix}$$

The gradient tensor is similar to analytic signal (centered over source) but a direct measurement requires at least four magnetometers.

EULER DECONVOLUTION

this is Euler's homogeneous equation of degree N

$$(x - x_o)\frac{\partial T}{\partial x} + (y - y_o)\frac{\partial T}{\partial y} + (z - z_o)\frac{\partial T}{\partial x} = -N(B - T)$$

Here, x_o, y_o, z_o is the source location, B is a regional field and N is a structural index, much like the structural indeces we used in depth estimates.

There are 5 unknowns x_o, y_o, z_o, N, B. If we guess N then there are four unknowns in a linear equation, so we need at least four equations.