FOURIER SERIES

PROPOSED BY FOURIER in 1807 from a military contract to study the heating of cannon barrels. They are applied on periodic functions with some limitations. The original paper was rejected (Laplace was one of the reviewers!)

FOURIER TRANSFORM

Extension of Fourier series to continuous aperiodic functions.

DISCRETE FOURIER TRANSFORM

Extension of Fourier transform to discretely sampled functions.

FAST FOURIER TRANSFORM

Extension of discrete Fourier transforms for speed.

Many uses for all of these in Seismology, electromagnetics and potential fields

FOURIER SERIES

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n cos(nx) + b_n sin(nx) \right]$$

a_n, b_n are Fourier coefficients

DIRICHLET CONDITIONS

1) $f(x)$ is periodic or defined over a finite range

2) $f(x)$ is piecewise continuous

3) $f(x)$ has a finite number of maxima and minima

3) $\int_{-\pi}^{\pi} f(x)dx$ exists

Critical to all the Fourier series and of wider applicability is the idea of orthogonality. Sine and cosine functions are orthogonal.

$$\int_o^{2\pi} sin(mx)cos(nx)dx = 0 \quad \text{all m,n}$$

$$\int_o^{2\pi} sin(mx)cos(nx)dx = \frac{1}{2}\int_0^{2\pi} sin((m+n)x) + sin((m-n)x)dx$$

and

$$\int_0^{2\pi} sin((m+n)x)dx = 0 \quad \text{ if n,m are integers}$$

and

$$\int_0^{2\pi} sin((m-n)x)dx = 0 \quad \text{ if n,m are integers}$$

So, the integral of the product of a cos and a sine is **zero** as long as the integral is over an integral number of wavelengths.

Also,

$$\int_o^{2\pi} sin(mx)sin(nx)dx = \frac{1}{2}\int_0^{2\pi} cos((m+n)x) + cos((m-n)x)dx$$

$$= \frac{1}{2}[\frac{sin((n+m)x)}{(n+m)} + \frac{sin((n-m)x)}{(n-m)}]_0^{2\pi}$$

$$= 0 \quad \text{if } n \neq m \text{ because } sin(0) = sin(2\pi(n+m) = 0$$

If $n = m$ the $\frac{1}{n-m}$ is undetermined, but then we can just integrate $\frac{1}{2}\int cos((n-m)x) = \pi$

So,

$$\int_o^{2\pi} sin(mx)sin(nx)dx = 0 \quad \text{if } n \neq m$$
$$= \pi \quad \text{if } n = m$$
$$= 0 \quad \text{if } n = m = 0$$

and

$$\int_o^{2\pi} cos(mx)cos(nx)dx = \pi \quad \text{if } n = m$$
$$= 0 \quad \text{if } n \neq m$$
$$= 2\pi \quad \text{if } n = m = 0$$

and

$$\int_0^{2\pi} sin(mx)cos(nx)dx = 0 \quad \text{all } n, m$$

These are the orthogonality conditions for sine and cosines.

Multiplying the Fourier expansion of $f(x)$ by $cos(mx)$ and integrating

$$\int_0^{2\pi} f(x)cos(mx)dx = \int_0^{2\pi} (\frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n cos(nx) + b_n sin(nx))cos(mx)]dx$$

and using the orthogonality conditions

$$= \int_0^{2\pi} (\frac{a_o}{2} cos(mx) + \sum_{n=1}^{\infty} a_n cos(nx)cos(mx) + b_n sin(nx)cos(mx)dx$$

$$= \cancel{a_o \pi} + \sum_{n=1}^{\infty} a_n \delta_n^m \pi$$

$$= a_m \pi$$

Because

$$\sum a_n \delta_n^m = a_1 \times 0 + a_2 \times 0 + ... + am \times 1 + a_{m+1} \times 0.... = a_m$$

So

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x)cos(mx)dxm = 0, 1, 2...$$

and

$$b_m = \frac{1}{\pi} \int_0^{2\pi} f(x)sin(mx)dxm = 0, 1, 2...$$

Alternatively,

$$f(x) = \frac{a_o}{2} + \sum_{0}^{\infty} P_n cos(nx - \phi_n)$$

$$= \frac{a_o}{2} \sum_{0}^{\infty} \lfloor P_n cos(\phi_n) cos(nx) + P_n sin(\phi_n) sin(nx) \rfloor$$

$P_n = (a_n^2 + b_n^2)^{1/2}$ AMPLITUDE SPECTRUM

$\phi_n = Tan^{-1}(b_n/a_n)$ PHASE SPECTRUM

and

$a_n =$ SPECTRAL COEFFICIENTS or

$b_n =$ SPECTRUM

Notice that Fourier series (and transforms) is linear - the spectrum of the sum of two functions is the sum of their spectra

$$f(x) + g(x) \rightarrow F_n + G_n$$

where an upper case letter denotes the spectrum of the lower case.

This is important because it allows us to operate in the frequency domain knowing the operation is linear in the observation domain.

regional + residual $\rightleftharpoons$ REGIONAL + RESIDUAL

Because we limited $f(x)$ to periodic functions we ended up with discrete frequencies

PERIODIC FUNCTIONS $\rightleftharpoons$ DISCRETE FREQUENCIES

what about aperiodic functions defined on $[-\infty, \infty]$ instead of $[0, 2\pi]$?

$$sin(x) \text{ on } -\pi \leq x \leq \pi \quad \equiv sin(\frac{\pi x}{L}) \text{ on } -L \leq x \leq L$$

So,

$$\begin{aligned} f(x) =& \frac{1}{2L}\int_{-L}^{L} f(t)dt \\ &+ \frac{1}{L}\sum_{n=1}^{\infty} cos(\frac{n\pi x}{L})\int_{-L}^{L} f(t)cos(\frac{n\pi t}{L})dt + \frac{1}{L}\sum_{n=1}^{\infty} sin(\frac{n\pi x}{L})\int_{-L}^{L} f(t)sin(\frac{n\pi t}{L})dt \\ =& \frac{1}{2L}\int_{-L}^{L} f(t)dt - \frac{1}{L}\sum cos(\frac{n\pi(x-t)}{L})f(t)dt \end{aligned}$$

integral here
as well

and now

$$\omega_n = \frac{n\pi}{L} \quad n = 0, 1, 2, ... \quad \text{is a frequency}$$

and we can write

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt - \frac{1}{\pi}\int_{-L}^{L}\sum_{n=1}^{\infty} cos(\omega_n(x-t))f(t)\Delta\omega dt$$

and then as $L \to \infty$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} cos(\omega(x-t))f(t)d\omega dt$$

and since cos is an even function $\int_{-a}^{a} cos(\omega)d\omega = 2\int_{0}^{a} cos(\omega)d\omega$ and we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cos(\omega(x-t))f(t)d\omega dt$$

Also, since sine is an odd function $\int_{-a}^{a} sin(\omega)d\omega = 0$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} isin(\omega(x-t))f(t)d\omega dt = 0$$

and we can add this to the previous equation without changing anything

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i(\omega(x-t)}d\omega dt \quad (e^{i\theta} = cos\theta + isin\theta)$$

and switching the order of integration

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dtd\omega$$

The inner integral

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad \text{FORWARD TRANSFORM}$$

and the outer integral

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \qquad \text{INVERSE TRANSFORM}$$

Also,

$$2\pi\sigma = \omega$$

circular frequency   radial frequency

cy/whatever   radian/whatever

FOURIER SERIES

Represents a periodic function $f(x)$ as an infinite sum over discrete frequencies

FOURIER TRANSFORM

Represents an aperiodic function as an integral over a continuum of frequencies

f and its transform F should both be regarded as complex, although if f is physical it will be real, but F is in general complex.

There are some important properties of Fourier transforms

SYMMETRY OR RECIPROCITY

note: $\widetilde{F}[f(t)]$ means take the Fourier transform of $f(t)$

$$\widetilde{F}[f(t)] = F(\omega)$$

and

$$\widetilde{F}[F(t)] = f(-\omega)$$

If the shape f in time or space transforms to the shape F in frequency, then the shape F in time or space transforms to the shape f in frequency

SCALING (in TIME OR SPACE)

$$\widetilde{F}[f(at)] = \frac{1}{|a|}F(\omega/a)$$

That is, if the space or time axis is scaled by a then the frequency axis is scaled by $1/a$ and the amplitude is scaled by $1/|a|$.

$$\begin{aligned}
F(\omega) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
=& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)e^{-i\omega at}dat \\
\text{and}& \\
F(\omega/a) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)e^{-i\omega t}dat \\
\text{so}& \\
\frac{F(\omega/a)}{|a|} =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)e^{-i\omega t}dt \\
\frac{F(\omega/a)}{|a|} =& \widetilde{F}[f(at)]
\end{aligned}$$

A corollary is

FREQUENCY SCALING

$$\widetilde{F}[\frac{f(t/a)}{|a|}] = F(a\omega)$$

or

$$\widetilde{F}[F(a\omega)] = \frac{F(t/a)}{|a|}$$

SHIFTING (TIME OR DISTANCE)

$$\widetilde{F}[f(t \pm a)] = F(\omega)e^{\pm i\omega a}$$

DERIVATIVES OF FUNCTIONS

$$\frac{df}{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega F(\omega) e^{i\omega t} d\omega$$

so that differentiating in the observation domain (space or time) is equivalent to multiplication by $i\omega$ in the frequency domain.

$$iw = we^{i\pi/2}$$

so differentiation can also be seen as a phase shift of $\pi/2$

For example,

$$\begin{aligned} f(t) =& cos 2\pi t + cos 20\pi t \\ \frac{df(t)}{dt} =& -2\pi(sin 2\pi t + 10 sin 20\pi t)) \\ \frac{d^2 f(t)}{d^2 t} =& -4\pi^2(cos 2\pi t + 100 cos 20 * \pi t) \end{aligned}$$

The second derivative is phase shifted by 180 degrees and the high frequency wave is much larger in amplitude than the low frequency wave.

In 2D this will turn out to be a useful operation.

There are some functions whose Fourier transforms are especially important

THE DIRAC DELTA FUNCTION

$$y(t) = \int_{-\infty}^{\infty} y(x)\delta(t-x)dx$$

So

$$\int_{-\infty}^{\infty} 1\delta(t-x)dx = 1$$

Fourier transforming the Dirac delta

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\delta(t)dt$$
$$\frac{1}{\sqrt{2\pi}}e^{0} = \frac{1}{\sqrt{2\pi}}$$

So a delta function, or a spike, transforms to a constant function at all frequencies.

That is, a spike contains all frequencies!

It follows from the symmetry property that a constant function transforms to a spike.

If a spike - a function with no width - has an infinitely wide transform it makes intuitive sense that a narrow function has a broader spectrum than a wider function. This is true.

THE SHARPER A FUNCTION IS IN THE OBSERVATION DOMAIN THE BROADER IT IS IN THE FREQUENCY DOMAIN

and

THE LONGER A FUNCTION IS OBSERVED, THE MORE PRECISELY ITS FREQUENCY SPECTRUM CAN BE DEFINED.

The boxcar function is defined by

$$\begin{aligned} b(t) =& 1 \quad -T/2 \leq t \leq T/2 \\ =& 0 \quad \text{all other t} \end{aligned}$$

and its transform is the Sinc function

$$B(\omega) = T\frac{sin\omega T/2}{\omega T/2}$$

This peaks at 0 where the amplitude is T and has zero crossings at $\pm 2n\pi/T$. The first side lobe is much smaller than the central peak $\approx T/3\pi \approx .1T$ and of course the side lobes get progressively smaller.

FOURIER TRANSFORM OF $e^{i\omega_o t}$ (a sinusoid with a single frequency)

$$
\begin{aligned}
\widetilde{F}[e^{+i\omega_o t}] = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(\omega-\omega_o)t} dt \\
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} cos(\omega-\omega_o)t + isin(\omega-\omega_o)tdt \\
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} cos\omega t cos\omega_o t + sin\omega t sin\omega_o t \\
& \qquad + i(sin\omega t cos\omega_o t - cos\omega t sin\omega_o t)dt
\end{aligned}
$$

and from the orthogonality properties this is equal to zero if $\omega \neq \omega_o$ and

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\pi \quad \text{per wavelength times } \infty \text{ wavelengths} \\
&= \infty
\end{aligned}
$$

or an infinite amplitude spike with zero width

FOURIER TRANSFORM OF $a sin\omega_o t$

$$F(\omega) = \frac{1a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{[e^{i\omega_o t} - e^{-i\omega_o t}]}{2i} dt$$

$$= \frac{1a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{[e^{-i(\omega-\omega_o)t} - e^{i(\omega+\omega_o)t}]}{2i} dt$$

$$= \frac{-ai}{2\sqrt{2\pi}} [\delta(\omega - \omega_o) - \delta(\omega + \omega_o)]$$

which is two spikes of amplitude $\pm i$ at $\pm\omega_o$. So a sine of a single frequency has a purely imaginary transform with power only at $\pm$ whatever the frequency is.

We can do the transform of $cos\omega_o t$ using similar mathematics

so cos transforms to two spikes again at $\pm\omega_o$ but now with real and equal amplitudes.