THE SAMPLING THEOREM

How often do we have to sample a continuous function in order to adequately describe that function with discrete samples?

Intuitively, ΔT must be smaller than the smallest wiggle in the continuous signal. The sampling theorem is a precise mathematical expression of that intuitive idea.

THE SAMPLING THEOREM

IF A FUNCTION $f(t)$ HAS A FOURIER TRANSFORM THAT IS ZERO FOR ALL FREQUENCIES GREATER THAN SOME FREQUENCY σ_c THEN $f(t)$ CAN BE UNIQUELY DETERMINED FROM A KNOWLEDGE OF ITS SAMPLED VALUES

This is an important theorem to consider before we move on to discrete Fourier transforms, because it says that when we sample a continuous function like gravity or mag discretely we can actually know all there is to know about that continuous function (in the sense that we can re-create the continuous function) if we follow a few simple rules.

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You need at least two samples per shortest period
you need to reproduce.
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A function whose transform is zero for all values above a certain frequency is called **band-limited.**

In particular, the maximum size of the sampling interval needed to re-create the continuous function is

$$\Delta T = \frac{1}{2\sigma_c} \quad \text{or} \quad \sigma_c = \frac{1}{2\Delta T}$$

That is, if the sampling interval is one over twice the cutoff frequency, or smaller, then the sampling theorem applies

Are gravity and mag band-limited?

For example, the human ear can hear frequencies from say ten Hz to 20 kHz, This means in order to record sound digitally onto a CD we would need to sample the waveform at 40 kHz, or 40,000 samples a sec. I think the conventional rate is 44kHz ~~to accommodate those who can hear to 22kHz.~~ Of course mathematically reproducing a analogue signal from a digital record is different from making a device that does that.

In seismic recording we would set the recording interval at a millisecond or ten ms depending on the highest frequency we expected to receive in the data. In shallow work you need a higher sampling frequency because the highest frequencies generated may still be in the reflections.

In potential fields the station spacing (the sampling interval) determines the shallowest depth of investigation.

Once you pick the sampling interval, you have defined the **NYQUIST** frequency,

$$f_N = \frac{1}{2\Delta T}$$

The Nyquist frequency is sometimes also called the FOLDING frequency for reasons that will become apparent and also the ALIASING FREQUENCY.

HOW DOES ALIASING AFFECT GRAVITY AND MAG SURVEYS?

The sampling interval determines the shallowest depths that can be accurately imaged.

Anomalies from shallow sources shorter than Nyquist period will appear as long anomalies from deeper sources.

We cannot filter such short anomalies away, so we have to hope that they do not exist!

Using the full width at half height rule for depth, $depth = 0.65w_{1/2}$, so let's say the sampling interval $\Delta x = w_{1/2}$ then sources shallower than $0.65\Delta x$ will be aliased. The field school gravity is mostly at 50 m spacing, so we can get an unaliased view of structure as shallow as 33 m. Of course the full width at half height rule is for a point source, so more distributed structures shallower than 33 m can be imaged properly, but not edges on those structures.

DISCRETE FOURIER TRANSFORMS

By a simple minded conversion of the integrals in the integral transform to sums we have

$$F(\frac{n}{NT}) = \sum_{k=0}^{N-1} f(kT)e^{-i2\pi nk/N} \quad n = 0, 1, 2, \cdots N-1$$

$$f(kT) = \frac{1}{N}\sum_{n=0}^{N-1} F(\frac{n}{NT})e^{i2\pi nk/N} \quad k = 0, 1, 2 \cdots N-1$$

as a discrete Fourier transform pair with a sampling interval T. We can check that they do what a transform is supposed to do by substituting one into the other and verifying that we get back to the original.

Also,

$$F(\frac{n}{NT}) = F(\frac{(rN+n)}{NT})$$

$$f(kT) = f((rN+k)T)$$

where r is any integer, so that both the transform and the function are periodic outside the interval in which they are defined (just like Fourier series and for the same reason)

But note that under this definition the highest frequency is $\frac{N-1}{NT} > \frac{1}{2T} = f_N$, and we do not want any frequency higher than the Nyquist.

NEW DEFINITION

$$F(\frac{n}{NT}) = \sum_{-N/2}^{N/2-1} f(kT)e^{-i2\pi nk/N} \quad n = -N/2+1, -N/2+2, \cdots 0 \cdots N/2$$

$$f(kT) = \frac{1}{N} \sum_{n=-N/2+1}^{N/2} F((\frac{n}{NT})e^{i2\pi nk/N} \quad k = -N/2, -N/2+1 \cdots N/1-1$$

so the frequencies are now

$$f = \frac{n}{NT} = \frac{nf_n}{N/2} = n/L$$

where L is the span of the data set. The frequencies are equispaced at

$$\Delta f = \frac{f_N}{N/2} = \frac{1}{L}$$

and they are specifically

$$f = o, \pm\frac{f_n}{N/2}, \pm\frac{2f_N}{N/2}, \pm\frac{3f_N}{N/2} \cdots f_N$$

so under this definition the highest frequency is f_N. The periods are

$$P = \infty, \pm N/2P_N, \pm\frac{N/2P_N}{2}, \pm\frac{N/2P_n}{3} \cdots P_N$$

where P_N is the Nyquist period.

DFT OF a 10 Hz SINE at 0.015873 s for 1 sec

$$\text{REAL PART} = -.478 at f = \pm 9.84 Hz$$

$$\text{IMAGINARY PART} = -30.5 \text{ at } f = 9.84 Hz$$

$$30.5 \text{ at } f = -9.84 Hz$$

From the equation for the reverse transform we have

$$\begin{aligned} S =& \frac{1}{N} \sum_{\pm\omega} F(\omega) e^{i\omega t} \\ =& \frac{1}{N} \sum_{\pm\omega} F_r(\omega) = iF_i(\omega)(cos(\omega t) + isin(\omega t)) \\ =& \frac{1}{64}[(-.478 - i30.5)(cos(9.84t) + isin(9.84t) \\ & - .478 + i30.5)(cos(-9.84t) + isin(-9.84t)] \\ =& \frac{1}{64}[61 sin(9.84t) - .96 cos(9.84t)] \\ =& 0.953 sin(9.84t - 0.0025cy) \end{aligned}$$

The digitized signal was

$$S = 1.0 sin 10t$$

so the signal we have recovered by inverting the values at the peak of the transform are not quite the right amplitude, frequency or phase!! **Why?**

The reason the frequency is not right is that once the sampling interval was chosen the available frequencies were set. 9.84 Hz happened to be the nearest Fourier frequency with the sampling interval chosen, so the spectrum peaked there.

Because the 10 Hz wave could not be perfectly described by one of the Fourier frequencies, most of the power went to 9.84 Hz and the rest was **leaked** to nearby frequencies. If I had chosen the sampling interval so that 10 Hz was exactly one of the Fourier frequencies then we would have seen amplitude in the spectrum at only that frequency. This of course means that you cannot in general know the frequency of a wave to better than plus or minus a Fourier frequency interval $\pm \frac{f_N}{N/2}$.

The phase and amplitude are not exactly correct for a similar reason.

What do you think would be the result of re-synthesizing the wave by summing over two frequencies in the spectrum? What about all (64) of them?

Can a signal be missed by falling in between two Fourier frequencies?

A DFT has a lowest frequency as well as a highest frequency. The lowest available frequency is one over the length of the profile and the highest is the Nyquist. Thus the lowest frequency is determined by how wide the data set is and the highest by the station spacing.

Nyquist

Just as we want to avoid having frequencies higher than the highest available Fourier frequency - to avoid aliasing - we want to make sure there are no frequencies lower than the lowest available frequency.

frequency increment

We want to ensure that the Fourier-transformed function is indeed periodic with period L - want to remove the "DRIFT"

REMOVE THE DRIFT

The drift in this context is the trend in the data along the profile. You must remove everything with longer horizontal scale than the width of the data before attempting a Fourier Transform.

TAPER THE START AND END

"Gibbs' phenomenon"

To avoid Gibbs' you must remove the discontinuities implied by the periodicity requirement if the first and last points are not at the same level. Taper the first and last ∼ twenty percent of the data to the mid-point of the first and last samples.

and derivatives

10% in program QFT.m in lab 5

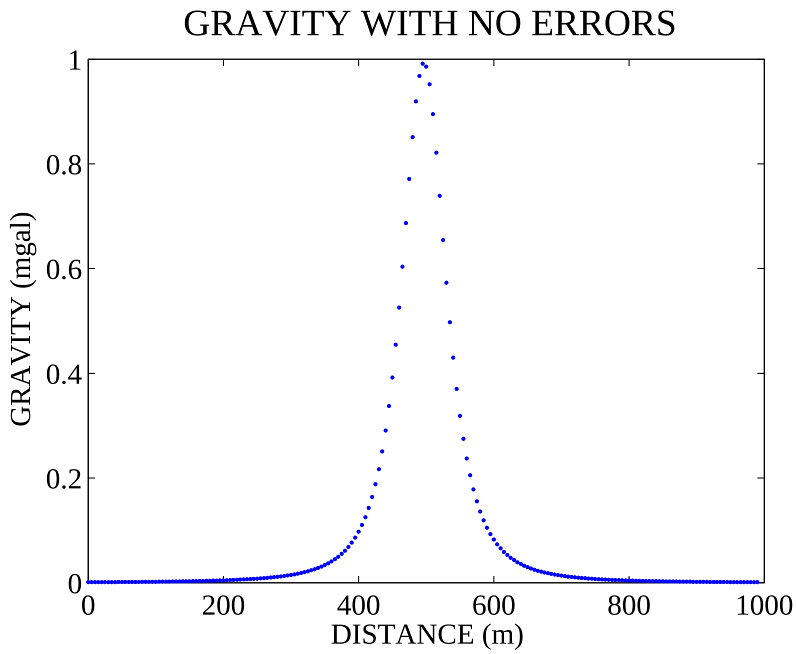
REMOVE THE DC

In this context the DC is the mean value of the data set. It is the a_o term in the Fourier series and the zero frequency term in the Fourier transform.

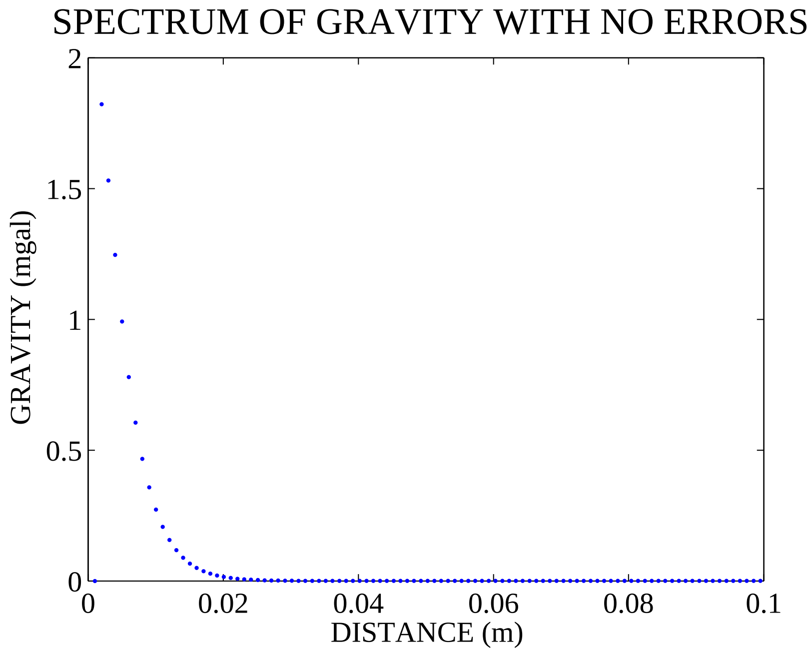
The above order is important. Why?

Removal of the DC should not be really important

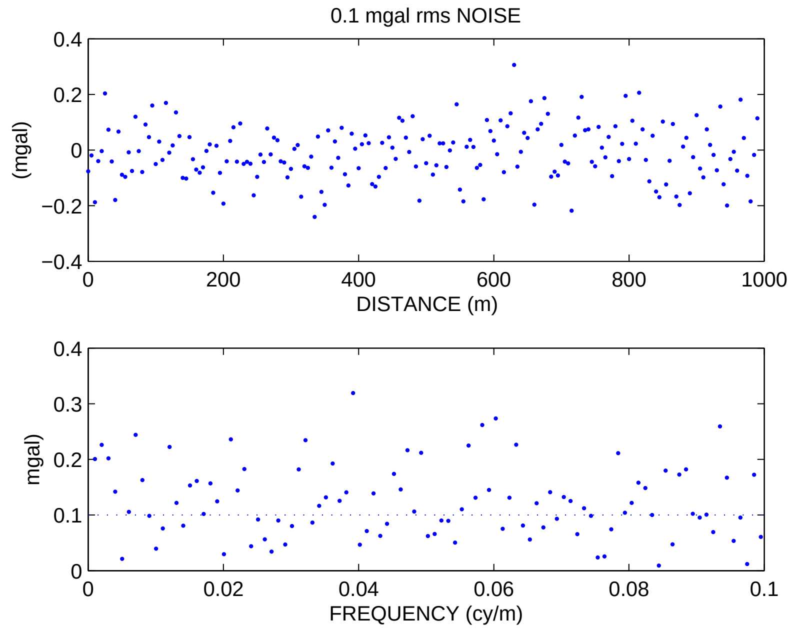
Here is an error free gravity anomaly



and here is its spectrum.

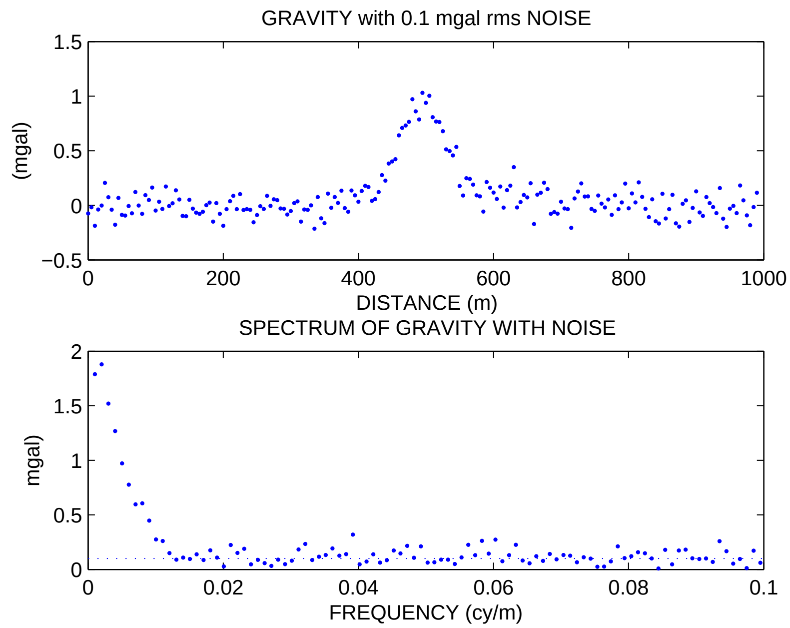


Here is rms (0.1 mgal) noise and its spectrum.



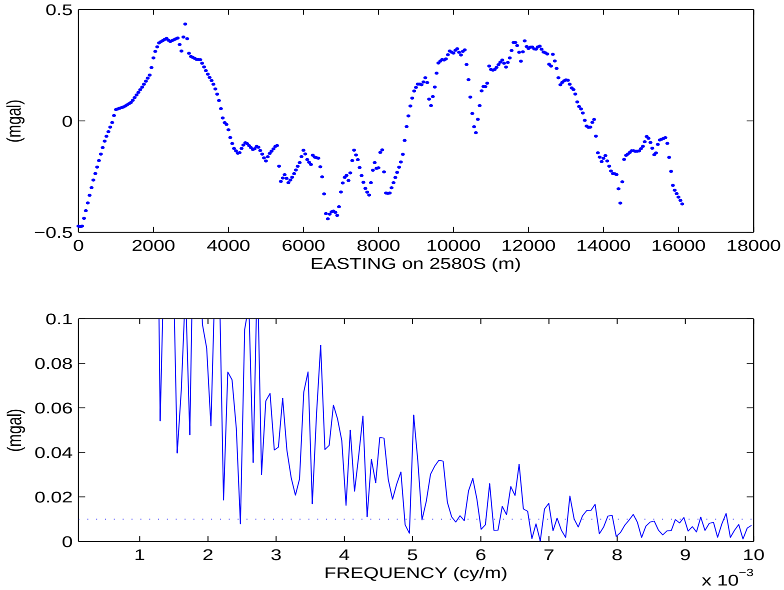
Random noise with some standard has a flat spectrum - there is no variation in the level with frequency. Most sources of noise in a gravity or mag survey will be like this.

Here is gravity with noise and its spectrum of gravity. Notice that because the Fourier transform is linear, if we add noise to data we add the spectrum of noise to the spectrum of gravity.



The noise free spectrum flattened out at 0. The noisey spectrum flattens out at the noise level, so we can tell what the noise in the data is.

Finally here is the gravity on 2580 South (trend removed) and its spectrum. The noise is about .01 mgal, which is an underestimate, probably because the raw data was smoothed by interpolation to 50 m spacing.



2-D FOURIER TRANSFORMS

USES

1) EXAMINE THE SPECTRUM
2) FILTER FOR SMOOTHING ETC
3) SEPARATE REGIONAL/RESIDUAL
4) TRANSFORM MAGNETIC FIELDS TO POLE
5) UPWARD AND DOWNWARD CONTINUATION
6) STRIKE FILTER
7) HORIZONTAL DERIVATIVES

HOW TO DO A 2-D TRANSFORM

1-D transform every column

$$g(kx, \frac{nf_{Ny}}{N/2}) = \sum_{l=-N/2}^{N/2-1} f(kx, ly) e^{-2\pi i \frac{nl}{N}} \quad n = -N/2+1 \rightarrow N/2$$

$$k = -M/2 \rightarrow M/2-1$$

then 1-D transform every row of the above result

$$F(\frac{mf_{Nx}}{M/2}, \frac{nf_{Ny}}{N/2}) = \sum_{k=-M/2}^{M/2-1} g(kx, \frac{nf_{N}y}{N/2}) e^{-2\pi i \frac{mk}{M}} \quad n = -N/2+1 \rightarrow N/2$$

$$m = -M/2+1 \rightarrow M/2$$

$$= \sum_{l=-N/2}^{N/2-1} \sum_{k=-M/2}^{M/2-1} f(kx, ly) e^{-2\pi i (\frac{mk}{M} + \frac{nl}{N})}$$

So a 2-D transform is simply a series of 1-D transforms

FFT FAST FOURIER TRANSFORM

A discrete Fourier transform requires the addition of N multiplications or N^2 operations. If the number of points is $N = 2^M$ or $N = 3^M$ etc the total number of operations can be reduced through the application of the Fast Fourier Transform resulting in NlogN operations where the number of samples is N. MATLAB uses a mixed radix formula that is, it factors N into $2^M \cdot 3^L \cdots$.

What if the number of points is prime?

at the matlab prompt

>>tic,fft(ones(1,1024));toc

tic starts mat lab's clock running and toc stops it and displays the elapsed time, which in this case is the time to do a $1024 = 2^{10}$ point Fourier transform. Now do

>>tic,fft(ones(1,1031));toc

1031 is a prime number. You should find that it takes nearly twice as long to do the 1031 point transform as the 1024 point transform. The elapsed time is short in either case, but what if you hit on a prime in the millions? Try executing either of the above a number of times. Try 2048 points and 4096 and 8192 and 16384 and 32768 etc to see how the time increases with the number of points.

Before the mixed radix algorithm people would either pad their data with zeros to get 2^N, or truncate.