GRAVITY ABOVE A THIN SHEET

$$\Delta g(h) = G \int_0^{\infty} \int_o^{2\pi} \frac{\sigma(r,\theta)}{(r^2+h^2)} cos\phi r dr d\theta$$

Splitting this into a contribution form mass in a circle right under the observation point and mass outside that circle

$$\begin{aligned}\Delta g(h) =& G \int_0^{\epsilon} \int_o^{2\pi} \frac{\sigma(r,\theta)}{(r^2+h^2)} cos\phi r dr d\theta \\ &+ G \int_{\epsilon}^{\infty} \int_o^{2\pi} \frac{\sigma(r,\theta)}{(r^2+h^2)} cos\phi r dr d\theta\end{aligned}$$

if $h << \epsilon$ then $\phi \approx \pi/2$ and $cos\phi \approx 0$, so in the limit as $h \rightarrow 0$, ie as we approach the plane, the second integral vanishes. This is true for any ϵ as long as $h << \epsilon$

As $\epsilon \to 0\, \sigma(r, \theta) \to \sigma(0, 0)$, call this $\sigma(0)$, so

$$\begin{aligned}\Delta g(h) =& G\sigma(0)2\pi \int_0^{\epsilon} \frac{cos\phi}{(r^2+h^2)} rdr \\ & G\sigma(0)2\pi h \int_0^{\epsilon} \frac{1}{(r^2+h^2)^{3/2}} rdr \\ & 2\pi G\sigma(0)h[\frac{1}{h} - \frac{1}{(\epsilon^2+h^2)^{1/2}}]\end{aligned}$$

and if $h << \epsilon$ as we have assumed

$$\Delta g(h) = 2\pi G\sigma(0)(1 - \frac{h}{\epsilon})$$

and in the limit as $h \to 0$

$$\Delta g(h) = 2\pi G\sigma(0)$$

or, if the origin in the plane has some arbitrary coordinates

$$\begin{aligned}\Delta g(x, y, 0) =& 2\pi G\sigma(x, y) \\ \Delta g(x, y) =& 2\pi G\sigma(x, y)\end{aligned}$$

S on a plane which has a surface density σ, gravity is everywhere equal to $2\pi G\times$ the local surface density.

This is a sort of 2-D Poisson's Eqtn, but unlike the 3-D case $\nabla^2 U = -4\pi G\rho$, we have gravity directly related to surface density.

The subsurface sources have gravity anomalies $\Delta g(x, y)$ on the survey plane. By the above Eqtn, we can define a fictitious surface density on that plane.

$$\sigma(x, y) = \frac{\Delta g(x, y)}{2\pi G}$$

Now, if the buried sources and the surface density have the same anomaly on the plane, then by the uniqueness th^{rm} (next page) they have the same anomaly everywhere above the plane as well. The surface density defined above is then the

EQUIVALENT STRATUM

for the buried sources.

THE UNIQUENESS THEOREM

If two potential fields have the same normal gradient everywhere on a closed surface are they the same everywhere outside the surface?

$U = U_1 - U_2$ The difference between two such fields

$$(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2 + (\frac{\partial U}{\partial z})^2 = \vec{\nabla} \cdot U\vec{\nabla}U - U\nabla^2 U$$

This is an identity for any scalar function.

Integrating over the volume outside the surface

$$\int_V (\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2 + (\frac{\partial U}{\partial z})^2 dV = \int_V \vec{\nabla} \cdot U\vec{\nabla}U - U\nabla^2 U dV$$

$$\int U\vec{\nabla} \cdot \hat{n} dS$$

The second line follows because U is a potential field $\nabla^2 U = 0$ and using the divergence theorem.

but $\vec{\nabla} U \cdot \hat{n} = 0$ on the surface because we said the normal gradients of U_1 and U_2 were equal on the surface.

so

$$\int_V (\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2 + (\frac{\partial U}{\partial z})^2 dV = 0$$

and since each element in the integrand is non-negative

$$(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2 + (\frac{\partial U}{\partial z})^2 = 0$$

and since each term in the sum is non-negative

$$\frac{\partial U}{\partial x} = 0 \quad \frac{\partial U}{\partial y} = 0 \quad \frac{\partial U}{\partial z} = 0$$

so

$$\frac{\partial U_1}{\partial x_i} = \frac{\partial U_2}{\partial x_i}$$

and

$$g_{x1} = g_{x2}$$

$$g_{y1} = g_{y2}$$

$$g_{z1} = g_{z2}$$

IF TWO POTENTIAL FIELDS HAVE EQUAL NORMAL DERIVATIVES (ie EQUAL GRAVITY OR MAGNETIC ANOMALIES) EVERYWHERE ON A CLOSED SURFACE (SURFACE OF THE EARTH) THEN THEY HAVE EQUAL ANOMALIES EVERYWHERE OUTSIDE (AND INSIDE) THE SURFACE

This is a general result for potential fields and follows from the properties of $\nabla^2 U = 0$

This last result suggests that if anomalies (gradients normal to the surface) are known everywhere on a closed surface they uniquely define the field inside and outside that surface (if Laplace's equation is satisfied inside and outside) as well and there is no further information to be had by measuring above the surface.

Here is the situation for the global case.

What about the exploration case, where we do not have gravity measurements everywhere on a closed surface?

We do not know anomalies everywhere on the surface enclosing this volume, so we cannot use the uniqueness theorem to claim there is only one solution compatible with the observed anomalies on the survey plane.

. . . .

An equivalent stratum can be defined for Mag as well.

$$Z(x,y) = \frac{M}{G\rho}\frac{\partial}{\partial z}\Delta g(x,y)$$

$$\frac{\partial A}{\partial z} = \frac{M}{G\rho}\frac{\partial}{\partial z}\Delta g(x,y)$$

and so

$$A = \frac{M}{G\rho}\Delta g + CNST$$

and Δg is the gravity anomaly due to the thin layer.

$$\Delta g = 2\pi G h(x,y) * \rho$$

$$A(x,y) = \frac{M}{g\rho}2\pi G\rho h(x,y)$$

$$A(x,y) = 2\pi M h(x,y)$$

$$A(x,y) = 2\pi M_s(x,y)$$

UPWARD CONTINUATION
(INTEGRAL FORM)

Instead of

$$\Delta g(x, y, -h) = \int_v G\rho(x', y', z') cos\phi' dv$$

Note: division by r squared missing here.

we want to calculate gravity anomalies at height $-h$ above survey plane, but we do not know $\rho(x', y', z')$. We can take advantage of the equivalent stratum to calculate

$$\Delta g(x, y, -h) = G \int \eta\xi \frac{\sigma(\xi, \eta)}{l^2} cos\phi d\eta d\xi$$

and

$$\sigma(\xi, \eta) = \frac{\Delta g(\xi, \eta, -h)}{2\pi G}$$

so

$$\Delta g(x, y, -h) = \frac{h}{2\pi} \int \int_{-\infty}^{\infty} \frac{\Delta g(\xi, \eta)}{((x - \xi)^2 + (y - \eta)^2 + h^2)^{3/2}} d\xi d\eta$$

This is the INTEGRAL FORM OF UPWARD CONTINUATION

Note that the integration limits are from $-\infty$ to ∞. What if you do not have data to $\pm\infty$?

THE EFFECT OF WHAT YOU DID NOT SURVEY

$$\Delta g(x, y, -h) = \frac{h}{2\pi} \int_{-\infty}^{\infty} \int \frac{\Delta g(\xi, \eta)}{((x-\xi)^2 + (y-\eta)^2 + h^2)^{3/2}} d\xi d\eta$$

OR

$$\Delta g(x, y, -h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int \frac{\Delta g(\xi, \eta)}{((x-\xi)^2 + (y-\eta)^2 + h^2)^{3/2}} d\xi d\eta$$

or $(0, 0, -h)$ or $(\xi^2 + \eta^2 + h^2)$

If the anomalies on the survey plane, $z = 0$, are such that the average anomaly outside $[-\eta, \eta], [-\xi, \xi]$ is no bigger than the average anomaly inside the survey area then the error caused by not doing the integration outside the survey area is

$$E(\Delta g(x, y, -h)) \approx \frac{\overline{\Delta g(\xi, \eta)}}{2\pi} \int_{\phi_o}^{\pi/2} \frac{cos\phi}{l^2} 2\pi l^2 \sin\phi cos\phi d\phi$$

where $\overline{\Delta g(\xi, \eta)}$ is the average value of unsurveyed anomalies which we take to be the same as the average surveyed gravity.

$$
\begin{aligned}
E(\Delta g(x,y,-h)) &\approx \overline{\Delta g(\xi,\eta)} \int_{\phi_o}^{\pi/2} cos^2\phi sin\phi d\phi \\
&\approx -\overline{\Delta g(\xi,\eta)} \frac{cos^3\phi}{3} |_{\phi_o}^{\pi/2} \\
&\approx \overline{\Delta g(\xi,\eta)} \frac{cos^3\phi_o}{3}
\end{aligned}
$$

So, if we want the error in the continued anomaly to be less than say 10%, of the unsurveyed anomaly then

$$
\frac{E(\Delta g(0,0,-h))}{\overline{\Delta g(\xi,\eta)}} < 0.1
$$

which means

$$
cos^3\phi_o < 0.3 \rightarrow \phi_o > 49^o
$$

So we would need to come in $h/tan(41)$ deg $\approx 1.1h$ from the edges, or if we want the error to be less than 1% of the mean of unsurveyed values,

$$
cos^3\phi_o < 0.03 \rightarrow \phi_o > 72^o
$$

So we would need to come in $h/tan(18)$ deg $\approx 3h$ from the edges

Another way of looking at this...

We have data on the survey plane $z = 0$ from $-\xi$ to ξ and $-\eta$ to η

We can compute accurate anomalies on the plane at $z = -h$ from $-\overline{X}$ to $\overline{X}$ and $-\overline{Y}$ to $\overline{Y}$ where

$$tan\phi_o = \frac{\overline{\xi} - \overline{X}}{h}$$

$$\texttt{tan(phi0)}\ h = \overline{\xi} - \overline{X}$$

$$\overline{X} = \overline{\xi} - h\texttt{tan(phi0)}$$

So, if we demand 10% accuracy in the upward continued values, we must be content with an array size on the plane $z = -h$ which is smaller than the array size on the survey plane. In this case the array on the $z = -h$ plane would have to be $h/.6$ in on each edge.

If we go for 1% accuracy then we would need to come in 7h on each side!

THE LESSON IS YOU CANNOT UPWARD CONTINUE TO THE MOON BECAUSE THE AREA OVER WHICH YOU CAN CALCULATE ACCURATE VALUES SHRINKS AS THE CONTINUED HIGHT INCREASES

In the above, we assumed that unsurveyed anomalies were on average equal to or less than of the survey anomaly. Is this always true?

If you do the regional residual separation properly, and isolate the anomaly, then the average anomaly outside the survey area is very much smaller than the anomaly inside the survey area. I am not sure this is always true: there may exist similar anomalies outside the survey area.

What happens if there is a large anomaly outside the survey area?

What about the spatial dimensions of the anomaly after upward continuation?

Your upward continued result is only due to the separated anomaly, and cannot reflect the unmeasured anomaly. That is, your upward continued result will not agree with an actual measurement.