# Surface waves

- Rayleigh and Love waves
- Particle motion
- Phase and group velocity
- Dispersion

- Reading:
  - Shearer, chapter 8
  - Telford *et al.*, 4.2.4, 4.2.6

# Mechanism

- Surface waves are always associated with a boundary.
- The (*e.g*., horizontal) boundary disrupts vertical wave propagation *but* provides for special wave modes propagating along it.
- Because there are 2 or 4 boundary conditions to satisfy (e.g., displacement and stress continuity), surface waves always consist of 2 or 4 interacting wave modes:
  - *P* and *SV* wave modes (*Rayleigh* or *Stoneley* waves);
  - Two *SH* modes (*Love* waves).
- Such waves are “tied” to the surface and exponentially decrease away from it.

# Surface wave potentials

- General wave equations for potentials:

$$\nabla^2 \phi = \frac{1}{V_P^2} \frac{\partial^2 \phi}{\partial t^2}, \qquad P\text{-wave}$$

$$\nabla^2 \psi_V = \frac{1}{V_S^2} \frac{\partial^2 \psi_V}{\partial t^2}, \qquad SV\text{-wave}$$

$$\nabla^2 \psi_H = \frac{1}{V_S^2} \frac{\partial^2 \psi_H}{\partial t^2}. \qquad SH\text{-wave}$$

- Surface waves are combinations of solutions with *complex* (*e.g.*, pure imaginary) wavenumbers along *z.*
  - *e.g.*, for Rayleigh wave:

$$\phi = Ae^{-mz} e^{i(kx - \omega t)},$$

$$\psi_V = Be^{-nz} e^{i(kx - \omega t)}.$$

  - Question: why are such solutions not allowed without a boundary?

# Depth dependence

- To satisfy the wave equations for any $k$ and $\omega$, $m$ and $n$ must equal (*show this*):

$$m=\sqrt{k^2-\frac{\omega^2}{V_P^2}},$$

*P*-wave component in Rayleigh wave

$$n=\sqrt{k^2-\frac{\omega^2}{V_S^2}}.$$

*SV*-wave component

  - note that therefore, for *any* surface wave:

$$k>\frac{\omega}{V_S}>\frac{\omega}{V_P},$$

and so $V_{Surface}=\frac{\omega}{k}<V_S.$

- To further describe the solution, we need to:
  1) consider $\omega$ and $A$ as free variables;
  2) determine B and $k(\omega)$ from the *boundary conditions*.

"Dispersion relation"

# Rayleigh waves
## ("ground roll")

- *Rayleigh waves* propagate along the *free surface*
- The displacements are as usual:

$$\vec{u}_P(\vec{x},\vec{z}) = \left(\frac{\partial\varphi}{\partial x}, 0, \frac{\partial\varphi}{\partial z}\right)$$ *P*-wave

$$\vec{u}_S(\vec{x},\vec{z}) = \left(\frac{-\partial\psi}{\partial z}, 0, \frac{\partial\psi}{\partial x}\right)$$ *SV*-wave

- and traction:

$$\vec{F}_P(\vec{x},\vec{z}) = \left(2\mu\frac{\partial^2\varphi}{\partial x\partial z}, 0, \lambda\nabla^2\varphi + 2\mu\frac{\partial^2\varphi}{\partial z^2}\right)$$

$$\vec{F}_S(\vec{x},\vec{z}) = \left(\mu\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right), 0, 2\mu\frac{\partial^2\psi}{\partial x\partial z}\right)$$

- For a free surface, the boundary conditions read: $\sigma_{xz} = \sigma_{zz} = 0$,
- Solution:

$$\phi = e^{-mz}e^{i(kx-\omega t)},$$

$$\psi_V = Be^{-nz}e^{i(kx-\omega t)}.$$

we can set $A = 1$ and seek $B$ and $k(\omega)$

# Rayleigh waves
## ("ground roll")

- Result (for $\sigma$=0.25): relative *P*- and *S*-wave amplitudes:

$$\phi = e^{-0.848kz} e^{i(kx-\omega t)},$$ *P*-wave

$$\psi_V = 1.468ie^{-0.393z} e^{i(kx-\omega t)}.$$ *SV*-wave

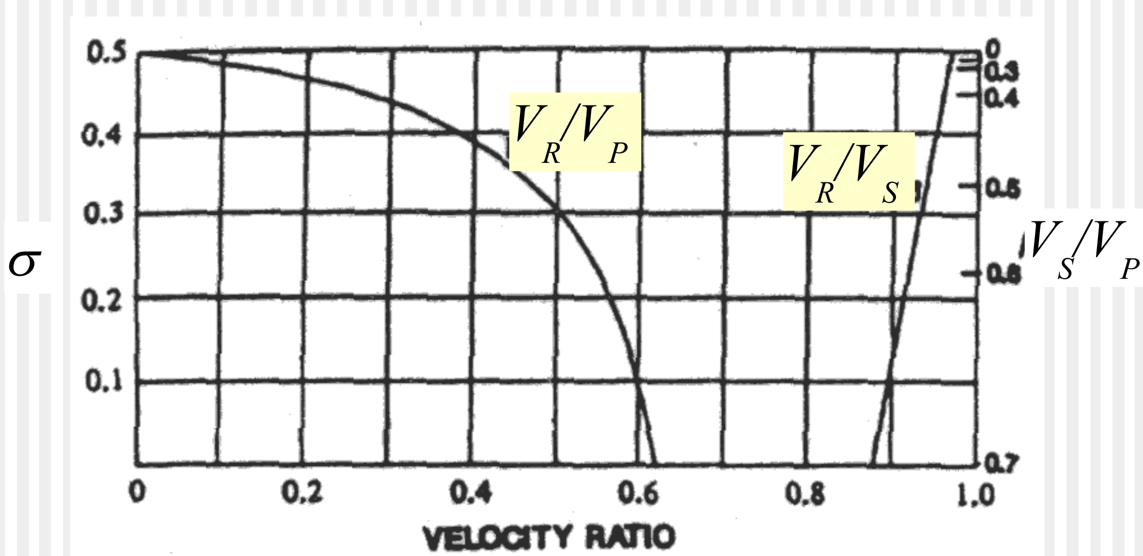
- ...and dispersion relation:

$$k = V_R \omega.$$ **(This means *no* dispersion!)**

- Rayleigh wave velocity:

$$V_R = 0.919 V_S$$

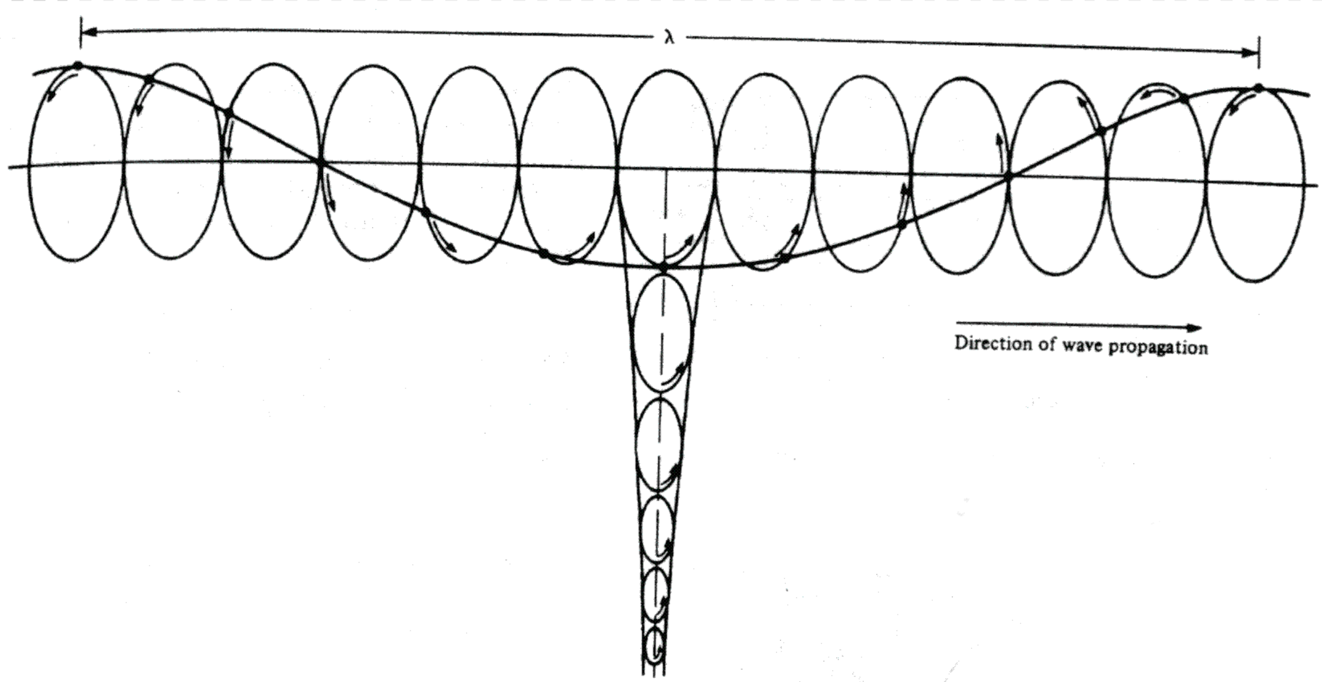
- For varying $\sigma$:



# Rayleigh waves
## ("ground roll")

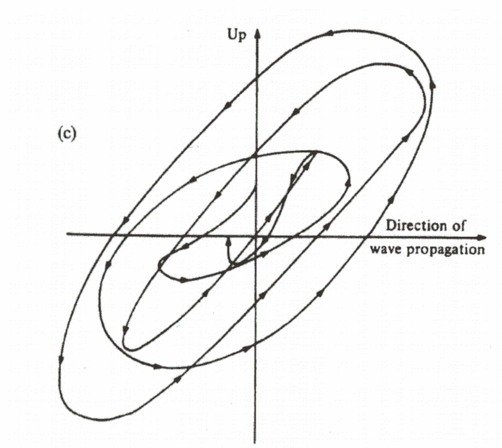
How does it follow from the equations for potentials and displacements?

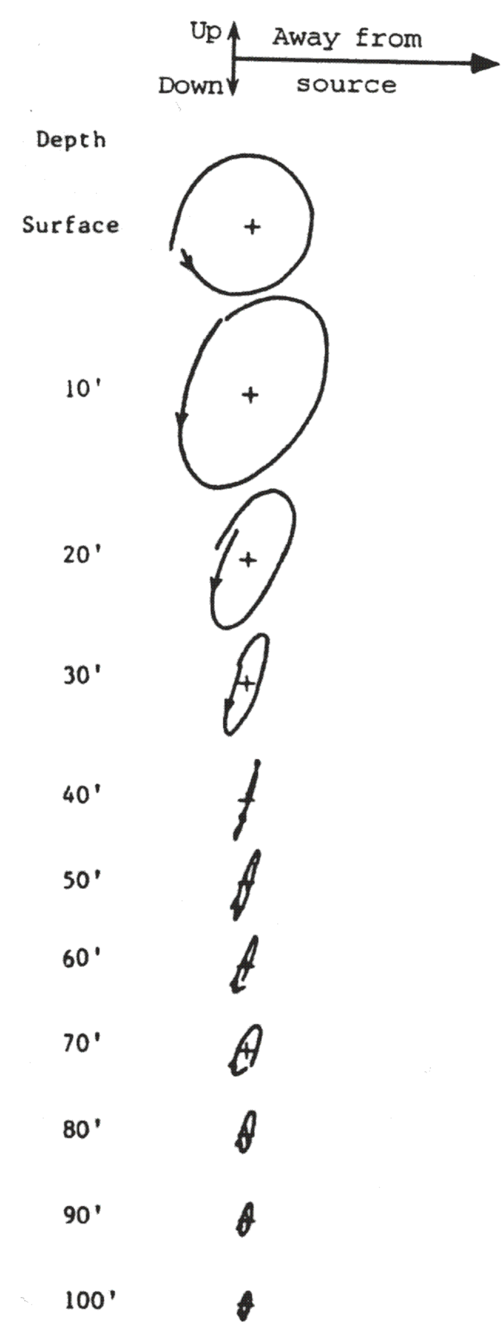
- Particle motion is elliptical and *retrograde* (counter-clockwise when the wave is moving left to right):



# Rayleigh waves
## ("ground roll")

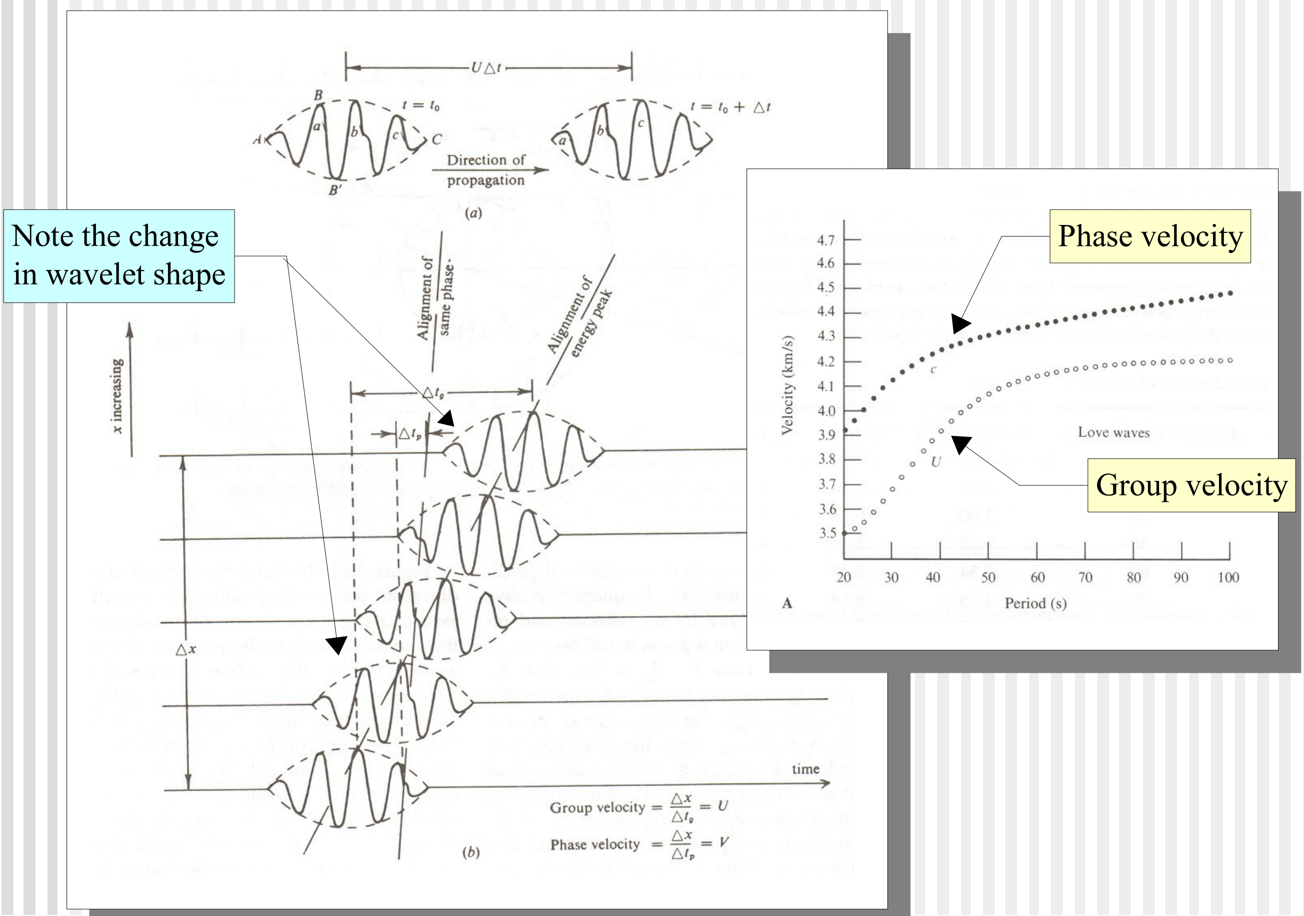
- Real Earth is never a *uniform* half-space, and thus in Rayleigh waves:
  - Particle motion paths are tilted and complex;
  - Retrograde motion may change into prograde at some depth;
  - Normal dispersion *is present*.



# Rayleigh-wave dispersion

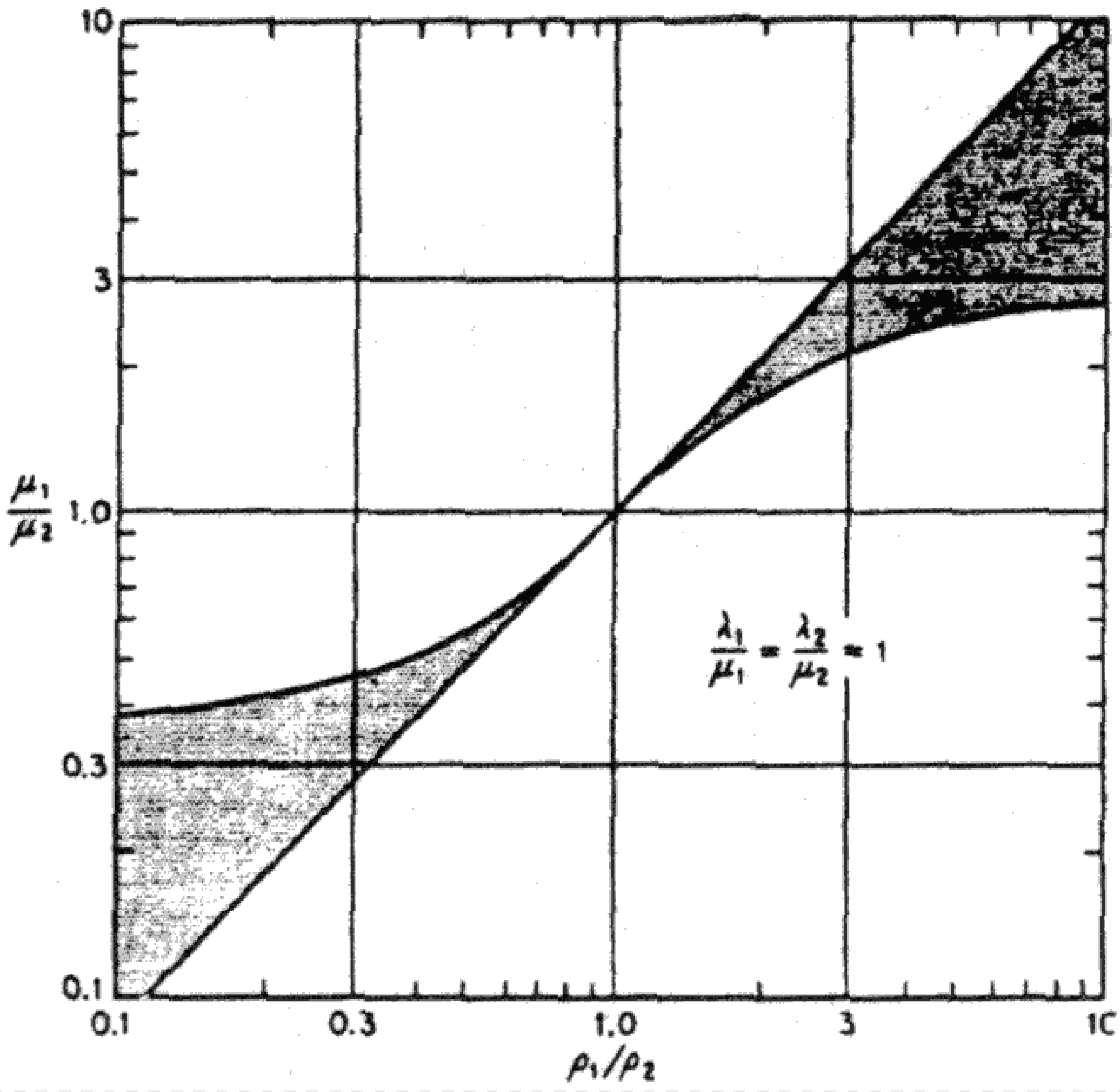
- Ideal Rayleigh wave (in a uniform half-space) is non-dispersive
- However, all *real* surface waves exhibit dispersion
  - It is because the subsurface is always layered



# Stoneley waves

- These waves propagate along the contact of *two* semi-infinite media
  - They are *P*/*SV* in nature, like Rayleigh waves;
  - They always exist when one of the media is a fluid;
    - An important example is *tube wave* propagating along a fluid-filled borehole.
  - If both media are solids, Stoneley waves exist only when $V_{S1} \approx V_{S2}$ and $\rho$ and $\mu$ lie within narrow limits *(plot on next page)*

# Stoneley waves



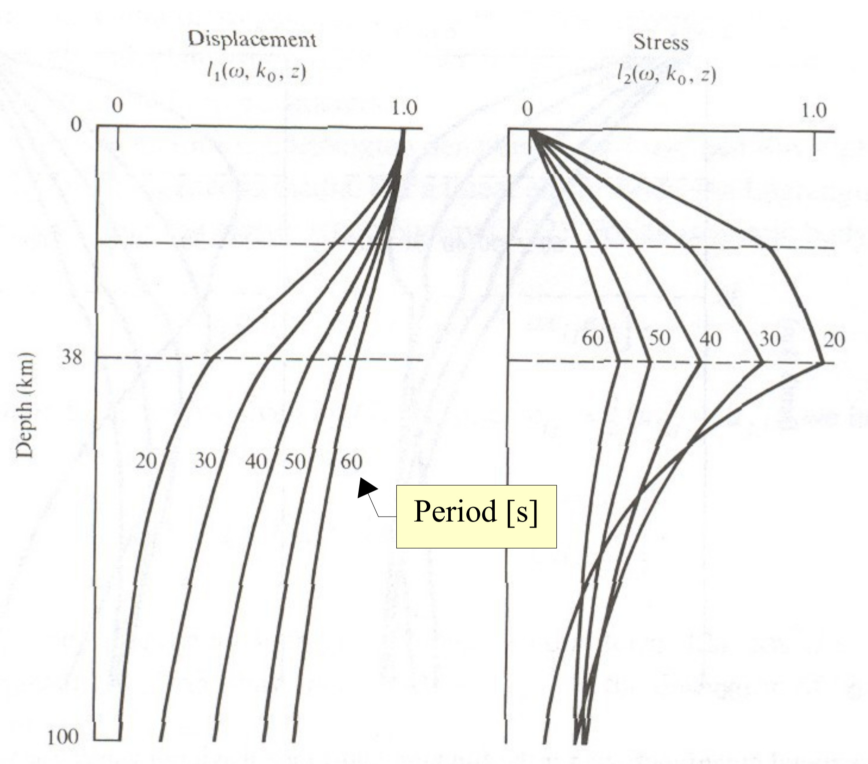
# Love waves

- These are *SH*-type waves propagating along the free surface
  - particle motion is transverse and parallel to the surface;
- Because there is just one *SH* potential, *two* modes are required to satisfy the two boundary conditions ($\sigma_{xz}=\sigma_{zz}=0$ on the free surface)
- Thus, Love waves exist when the semi-infinite medium is overlain with a layer with different elastic properties.
  - ... this situation is quite common.

# Love-wave dispersion

- Love waves are dispersive:
- At high frequencies, its velocity approaches the S-wave velocity in the surface layer
- At low frequencies, velocity is close to that of the lower layer.

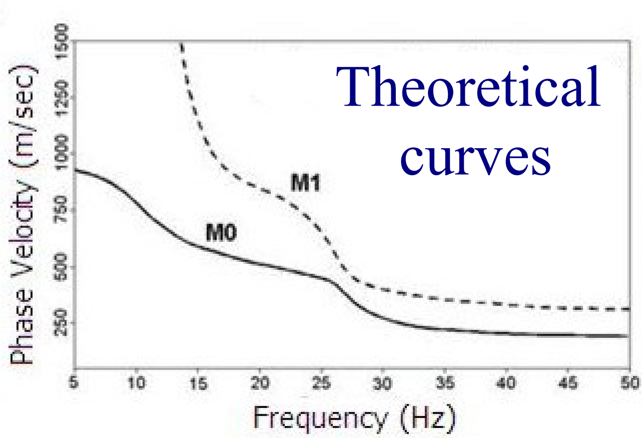
Displacement and stress in Love wave

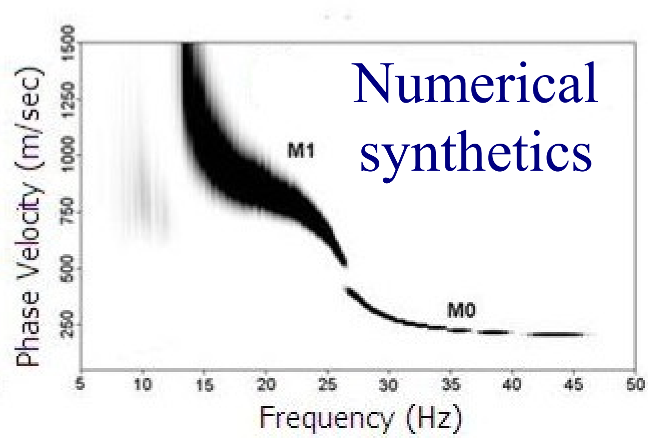


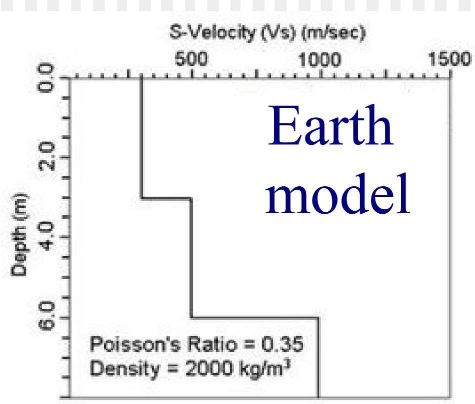

Depth of sampling increases with period. This is common to all surface waves.

# Surface-wave branches

- In layered structures, multiple surface-wave branches, or "modes" exist for the same frequency
- The branch with the lowest phase velocity (longest wavelength) is called the fundamental mode



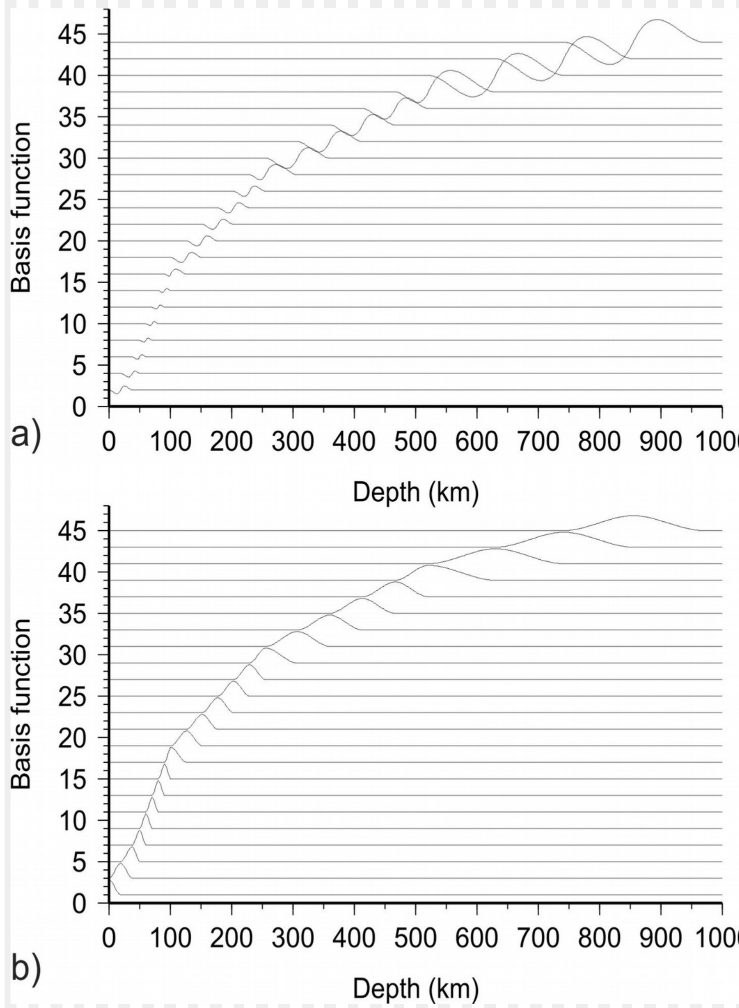



# Surface-wave branches (theory)

- To see that multiple branches exist and determine their parameters, the following matrix method can be used:

1) Chose a set of $N$ basis functions in depth, $f_i(z)$, and express the potential (or displacement) through them:

Example of basis functions for global Love waves




$$\psi(x,z,t)=e^{i(\omega t-kx)}\sum_{i=1}^{N}c_i f_i(z)$$

# Surface-wave branches (theory, cont.)

2) For a given *k*, express the total kinetic and potential energies; they will be quadratic matrix products of $c_i$:

Note that $E_{kin}$ is always proportional to $\omega^2$

$$E_{kin} = \int \left( \frac{\rho}{2} \dot{u}_i \dot{u}_i \right) dz = \omega^2 c_i A_{ij} c_j$$

$$E_{el} = \int \left( \frac{\lambda}{2} \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} \right) dz = c_i B_{ij} c_j$$

3) Note that in a wave, $E_{kin} = E_{el}$, and therefore:

$$\omega^2 c_i A_{ij} c_j = c_i B_{ij} c_j$$

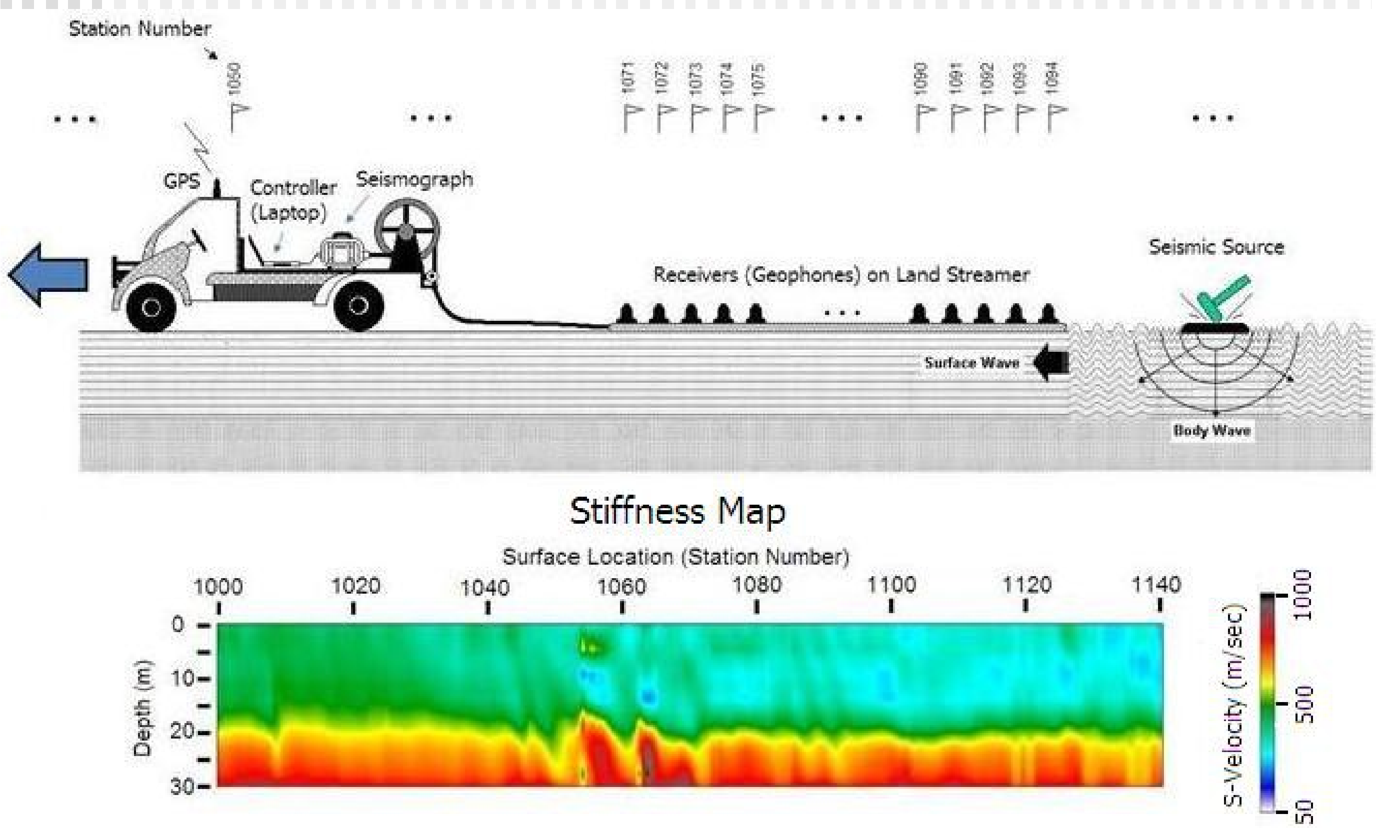
4) This means that $c_i$ is an eigenvector of matrix $\mathbf{A}^{-1}\mathbf{B}$, and $\omega^2$ is the corresponding eigenvalue:

There may be up to *N* positive eigenvalues. They give frequencies of up to *N* modes.

$$\left( A^{-1} B - \omega^2 I \right) c = 0$$

# MASW (SASW) method

- Multichannel (or Spectral) Analysis of Surface Waves
- Uses dispersion $V(f)$ measurements to invert for $V_s(z)$ and $\mu(z)$
    - Geotechnical applications

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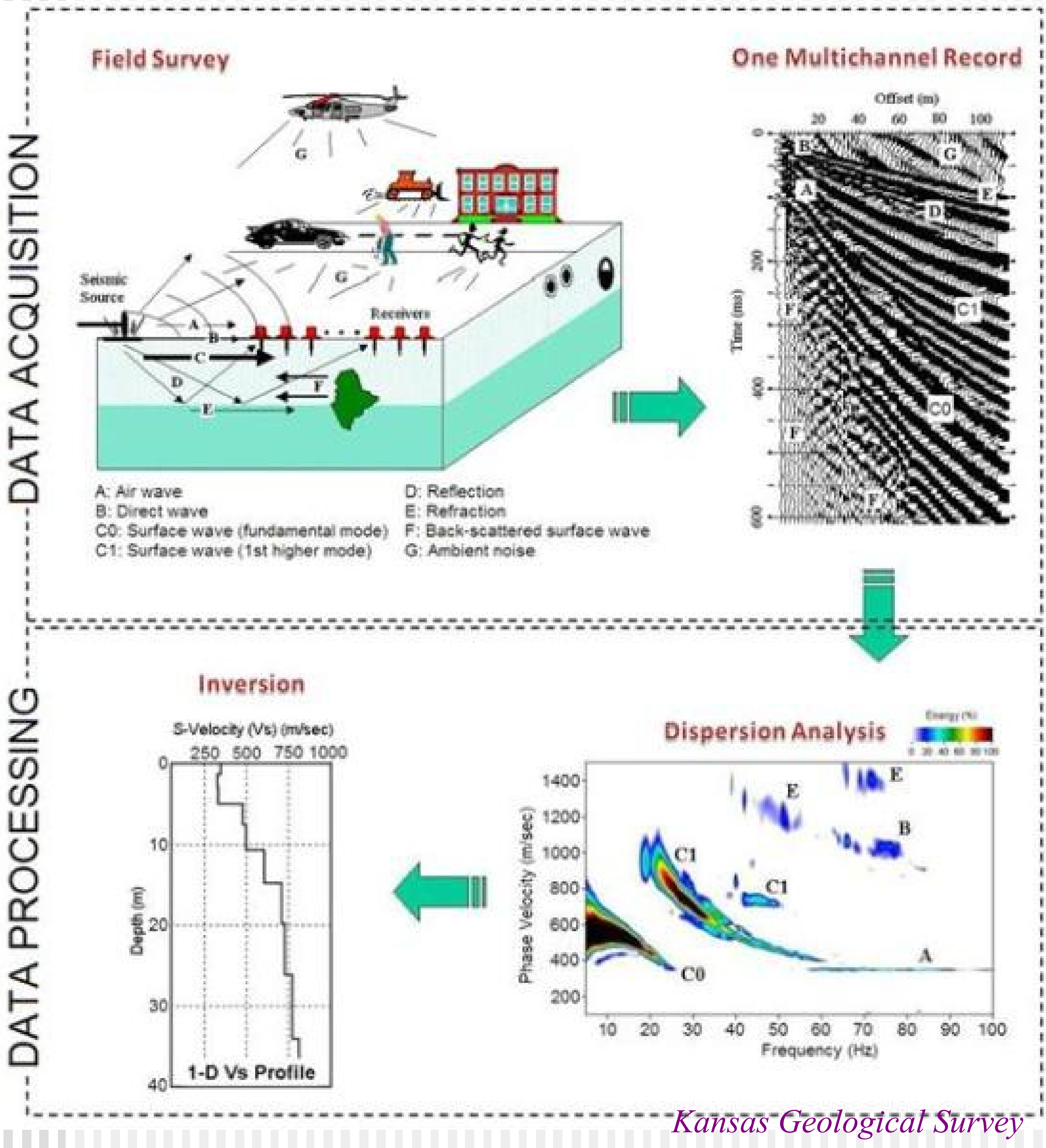
# MASW method


DATA ACQUISITION
DATA PROCESSING
Field Survey
One Multichannel Record
Seismic Source
Receivers
A: Air wave
B: Direct wave
C0: Surface wave (fundamental mode)
C1: Surface wave (1st higher mode)
D: Reflection
E: Refraction
F: Back-scattered surface wave
G: Ambient noise
Offset (m)
Time (ms)
Inversion
S-Velocity (Vs) (m/sec)
Depth (m)
1-D Vs Profile
Dispersion Analysis
Phase Velocity (m/sec)
Frequency (Hz)
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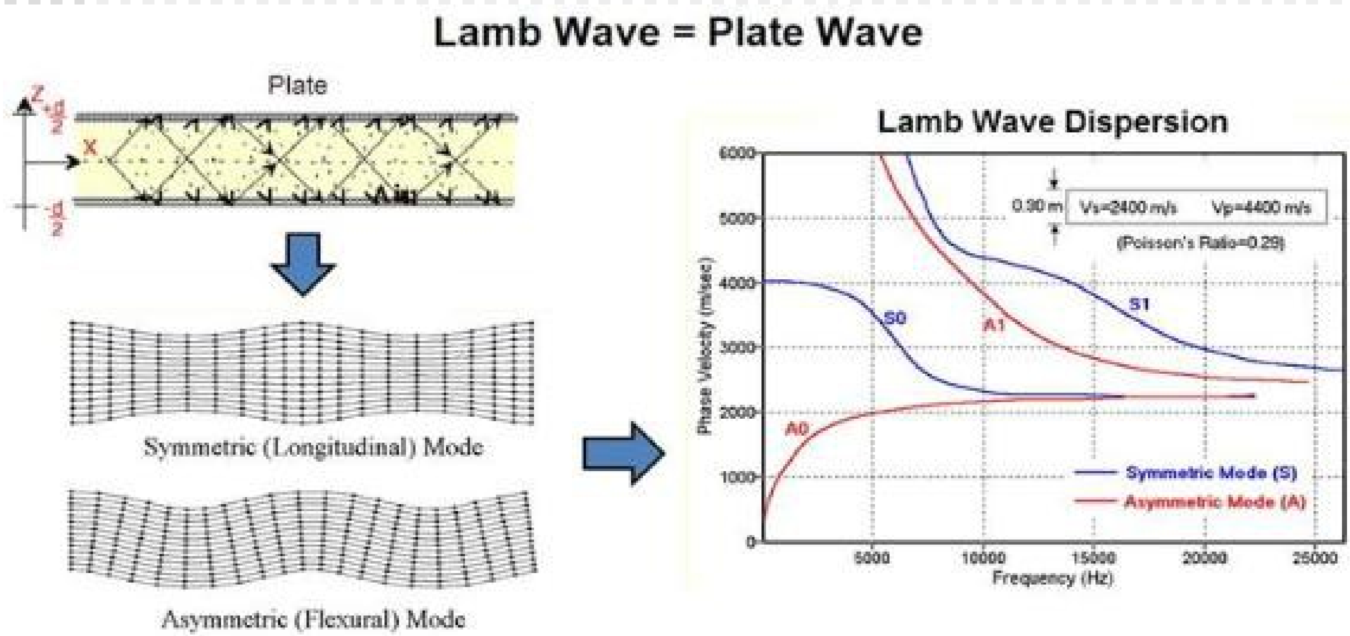
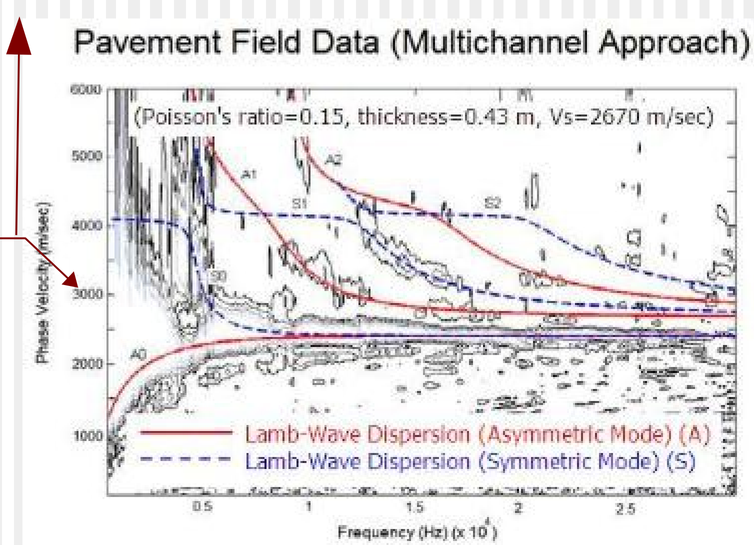
# Lamb (plate) wave

## (*e.g.*, in road pavement)

- The case of surface waves in a thin elastic plate is called the *Lamb's problem*




Note the difference In dispersion curves for symmetric and asymmetric deformations of the plate

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