Time and Spatial Series and Transforms

- Z- and Fourier transforms
- Gibbs' phenomenon
- Transforms and linear algebra
- Wavelet transforms

- Reading:
  - Sheriff and Geldart, Chapter 15

Z-Transform

- Consider a digitized record of *N* readings: U={u_0, u_1, u_2, …, u_{N-1}}. How can we represent this series differently?
- The *Z* transform simply associates with this time series a *polynomial*:

$$Z(U) = u_0 + u_1 z + u_2 z^2 + u_3 z^3 + \ldots$$

  - For example, a 3-sample record of {1,2,5} is represented by a quadratic polynomial:
    $1 + 2z + 5z^2$.
- In *Z*-domain, the all-important operation of *convolution* of time series becomes simple multiplication of their Z-transforms:

$$U_1 * U_2 \leftrightarrow Z(U_1)Z(U_2)$$

Fourier Transform

- To describe a polynomial of order *N*-1, it is sufficient to specify its values at *N* points in the plane of "*z*".
- The *Discrete Fourier transform* is obtained by taking the *Z*-transform at *N* points uniformly distributed around a unit circle on the complex plane of *z*:

$$U(k) = \sum_{m=1}^{N-1} e^{i\frac{2\pi k}{N}m} u(t_m) \quad k = 0, 1, 2, \ldots, N-1$$

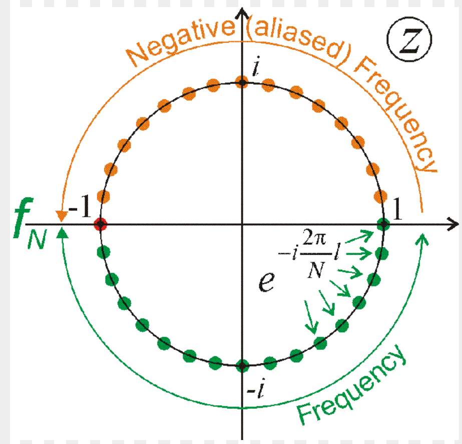
- Each term (*k*>0) in the sum above is a *periodic function* (a combination of *sin* and *cos*), with a period of *N*/*k* sampling intervals:

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

- Thus, the Fourier transform expresses the signal in terms of its frequency components,
  - and also has the nice property of the *Z*-transform regarding convolution

Relation of Fourier to Z-transform

- z-points used to construct the Fourier transform:



- **Aliasing**: for a real-valued signal, the values of FT at frequencies below and above the Nyquist (orange and green dots) are complex conjugate. Thus, only a half of the frequency band describes the process uniquely.
- This ambiguity is the source of aliasing.
- For this reason, frequencies above f_N should not be used.
- **Note**: forward and inverse FT result in a signal whose *N* samples are repeated periodically in time.

Matrix form of Fourier Transform

- Note that the Fourier transform can be shown as a matrix operation:

$$\begin{pmatrix} U(\omega_1) \\ U(\omega_2) \\ U(\omega_3) \\ \dots \end{pmatrix} = L \begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \\ \dots \end{pmatrix}$$

$$F = \begin{pmatrix} e^{i\omega_1 t_1} & e^{i\omega_1 t_2} & e^{i\omega_1 t_3} & \dots \\ e^{i\omega_2 t_1} & e^{i\omega_2 t_2} & e^{i\omega_2 t_3} & \dots \\ e^{i\omega_3 t_1} & e^{i\omega_3 t_2} & e^{i\omega_3 t_3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

- Inverse:

$$F^{-1} = \frac{1}{N} \begin{pmatrix} e^{-i\omega_1 t_1} & e^{-i\omega_2 t_1} & e^{-i\omega_3 t_1} & \dots \\ e^{-i\omega_1 t_2} & e^{-i\omega_2 t_2} & e^{-i\omega_3 t_2} & \dots \\ e^{-i\omega_1 t_3} & e^{-i\omega_2 t_3} & e^{-i\omega_3 t_3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Resolution of Fourier Transform

- Resolution matrix:

$$R_F = F^{-1} F$$

- If all *N* frequencies are used:

$$R_F = I$$

- If fewer than *N* frequencies are used (Gibbs phenomenon):

$$R_F \neq I$$

Integral Fourier Transform

- For continuous time and frequency (infinitesimal sampling interval and infinite recording time), Fourier transform reads:
  - Forward: $U(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, u(t) e^{i\omega t}.$
  - Inverse: $u(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, U(\omega) e^{-i\omega t}.$
- In practice, the bandwidth (and time) is always limited, and so the actual combination of the forward and inverse transforms is rather:

$$u(t)=\frac{1}{2\pi}\int_{-\omega_{max}}^{\omega_{max}} d\omega \left[\int_{-\infty}^{\infty} d\tau\, u(\tau) e^{i\omega\tau}\right] e^{-i\omega t}.$$

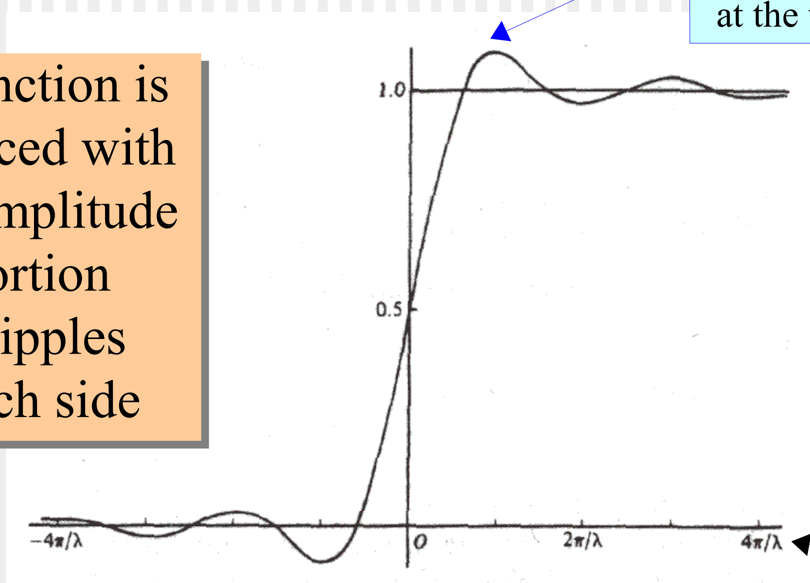
$$u(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, u(\tau) \left[\int_{-\omega_{max}}^{\omega_{max}} d\omega\, e^{i\omega(t-\tau)}\right].$$

Gibbs' phenomenon

- At a discontinuity, application of the Fourier forward and inverse transform (with a limited bandwidth), results in ringing.

Note the ~9% "overshoot" at the top and the bottom

Step function is reproduced with ~18% amplitude distortion and ripples on each side



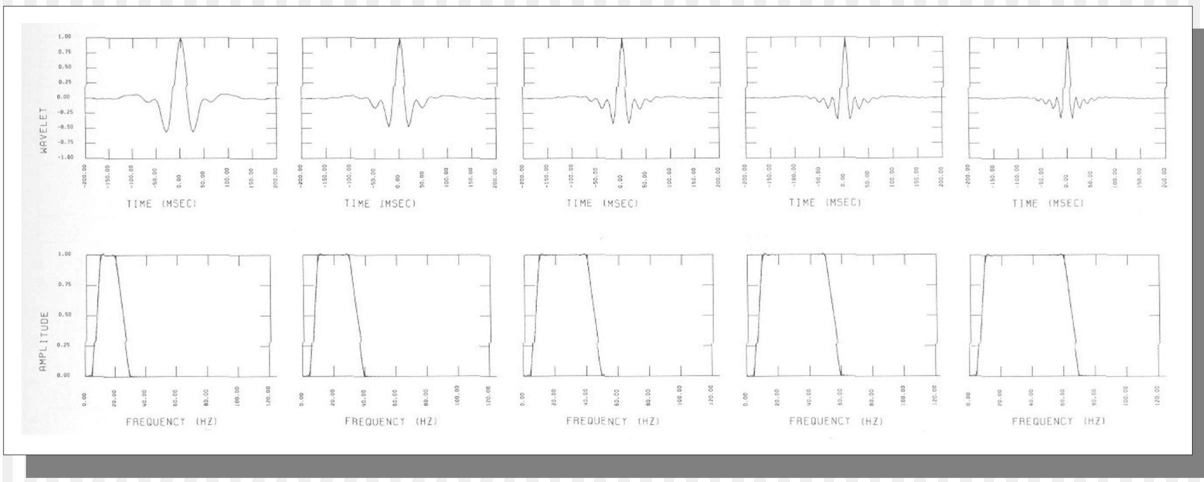
λ is our ω_{max}

- This is important for constructing time and frequency windows
  - Boxcar windows create ringing at their edges.
  - "Hanning" (cosine) windows are often used to reduce ringing:

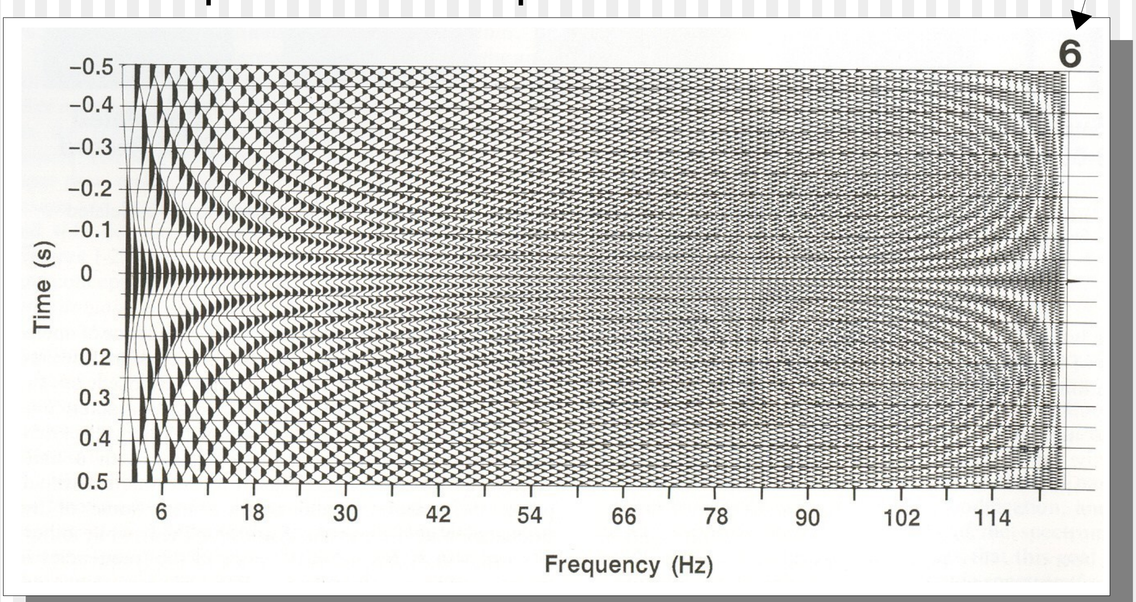
$$H_{\Delta t}(t) = \frac{1}{2}\left(1 - \cos\frac{\pi t}{\Delta t}\right).$$

Spectra of Pulses

- For a pulse of width *T* s, its spectrum is about 1/*T* Hz in width:

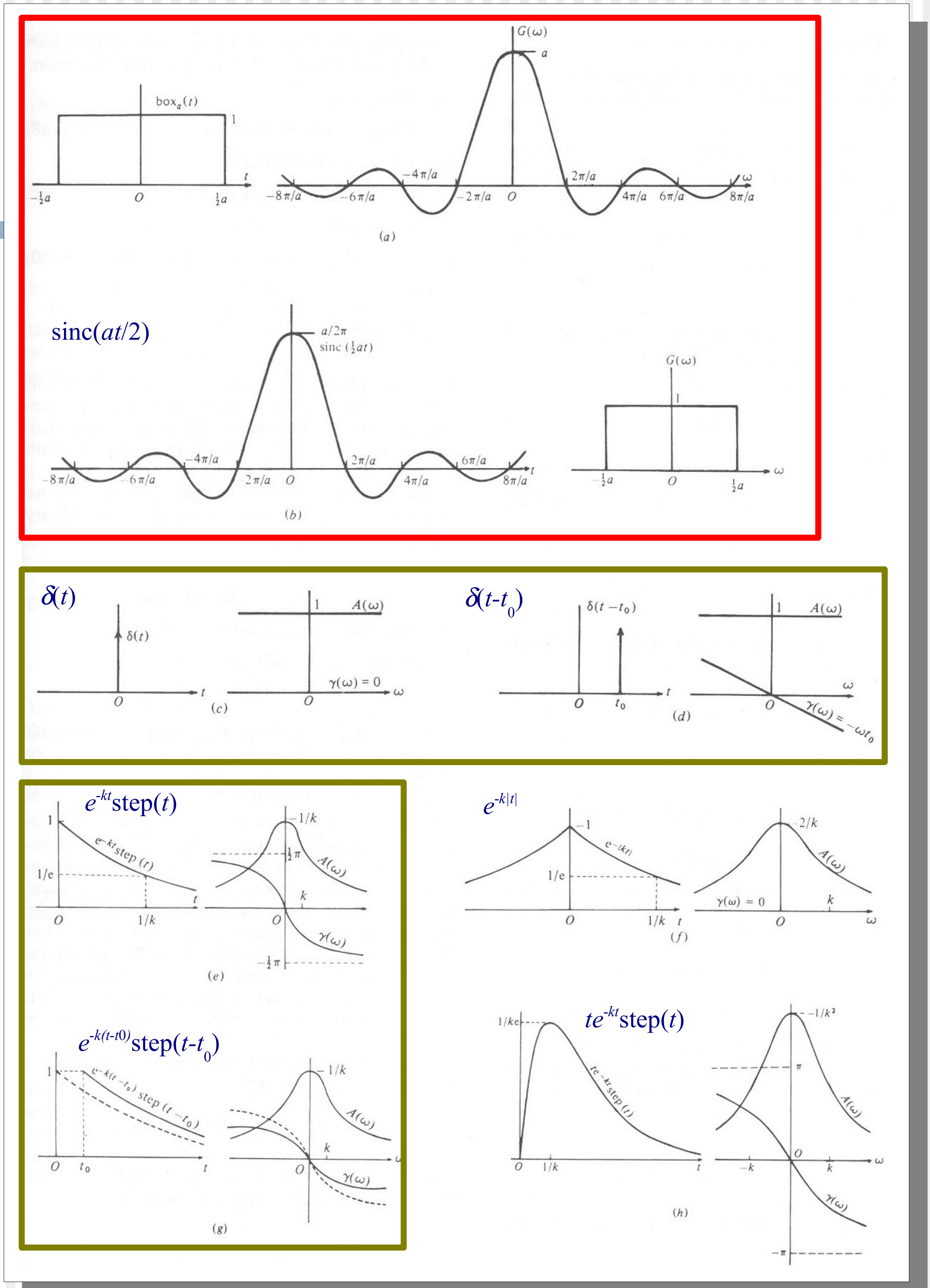


- Equal-amplitude (co)sinusoids from 0 to f_N add up to form a spike:



From Yilmaz, 1987

Sample Fourier Transforms



From Sheriff, Geldart, 1995

- Compare the transforms within the boxes...

Wavelet transforms

- Like the inverse Fourier transform, *wavelet decomposition* represents the time-domain signal as a combination of *wavelets* of some desired shapes:

$$\begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \\ \dots \end{pmatrix} = \begin{pmatrix} f_1(t_1) & f_2(t_1) & f_3(t_1) & \dots \\ f_1(t_2) & f_2(t_2) & f_3(t_2) & \dots \\ f_1(t_3) & f_2(t_3) & f_3(t_3) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \end{pmatrix}$$

Wavelet shapes

Wavelet amplitudes

- Ideally, wavelets should form a *complete orthonormal basis*:

$$\sum_{k=0}^{N-1} f_i(t_k) f_j(t_k) = \delta_{ij}$$

exp(...) functions used in Fourier transforms satisfy this property

although this is not really necessary

- Usually, functions *f*(*t*) represent time-scaled and shifted versions of some "wavelet" *W*(*t*)