

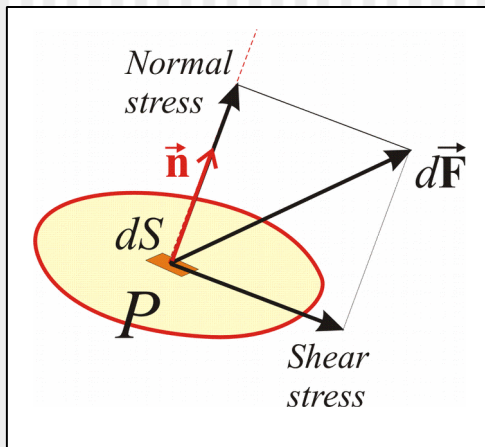
Elements of Rock Mechanics

- Stress and strain
- Creep
- Constitutive equation
 - ◆ Hooke's law
- ◆ Empirical relations
- Effects of porosity and fluids
- Anelasticity and viscoelasticity

- Reading:
 - › Shearer, 3

Stress

- Consider the interior of a deformed body:



At point P , force $d\mathbf{F}$ acts on any infinitesimal area dS . $d\mathbf{F}$ is a projection of *stress tensor*, σ , onto \mathbf{n} :

$$dF_i = \sigma_{ij} n_j dS$$

- Stress σ_{ij} is measured in [*Newton/m²*], or *Pascal* (unit of pressure).
- $d\mathbf{F}$ can be decomposed into two components relative to the orientation of the surface, \mathbf{n} :
 - Parallel (*normal stress*)

$$(dF_n)_i = n_i \cdot (\text{projection of } F \text{ onto } n) = n_i \sigma_{kj} n_k n_j dS$$

- Tangential (*shear stress, traction*)

$$d\vec{F}_\tau = d\vec{F} - d\vec{F}_n$$

Note summation over k and j

Forces acting on a small cube

- Consider a small parallelepiped ($dx \times dy \times dz = dV$) within the elastic body.
- **Exercise 1:** show that the *force* applied to the parallelepiped from the outside is:

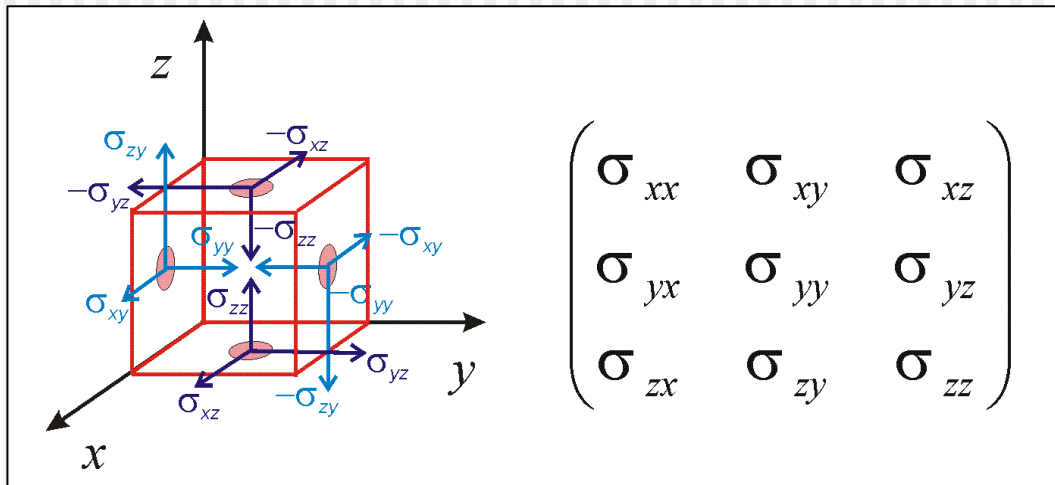
$$F_i = -\partial_j \sigma_{ij} dV$$

Keep in mind implied summations over repeated indices

(This is simply minus divergence ("convergence") of stress!)

- **Exercise 2:** Show that *torque* applied to the cube from the outside is:

$$L_i = -\epsilon_{ijk} \sigma_{jk} dV$$



Symmetry of stress tensor

- Thus, L is proportional to dV : $L = O(dV)$
- The *moment of inertia* for any of the axes is proportional to $dV \cdot \text{length}^2$:

$$I_x = \int_{dV} (y^2 + z^2) \rho dV$$

and so it tends to 0 faster than dV : $I = o(dV)$.

- Angular acceleration: $\theta = L/I$, must be *finite* as $dV \rightarrow 0$, and therefore:

$$L_i/dV = -\epsilon_{ijk} \sigma_{jk} = 0.$$

- Consequently, the stress tensor is *symmetric*:

$$\sigma_{ij} = \sigma_{ji}$$

- σ_{ji} has only 6 independent parameters out of 9:

$$\begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = dS \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

$$\sigma_{xy} = \sigma_{yx},$$

$$\sigma_{xz} = \sigma_{zx},$$

$$\sigma_{yz} = \sigma_{zy}$$

Shear stress components are symmetric

Normal stress components

Principal stresses

- The symmetric stress matrix can always be *diagonalized* by properly selecting the (X, Y, Z) directions (*principal axes*)
 - For these directions, the stress force **F** is orthogonal to dS (that is, parallel to directional vectors **n**)
 - With this choice of coordinate axes, the stress tensor is *diagonal*:

$$\boldsymbol{\sigma}_{\text{principal}} = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

Negative values mean *pressure*, positive - *tension*

- For a given σ , principal axes and stresses can be found by solving for *eigenvectors* of matrix σ :

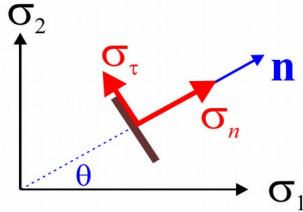
$$\boldsymbol{\sigma} \mathbf{e}_i = \lambda_i \mathbf{e}_i$$

Principal direction vector

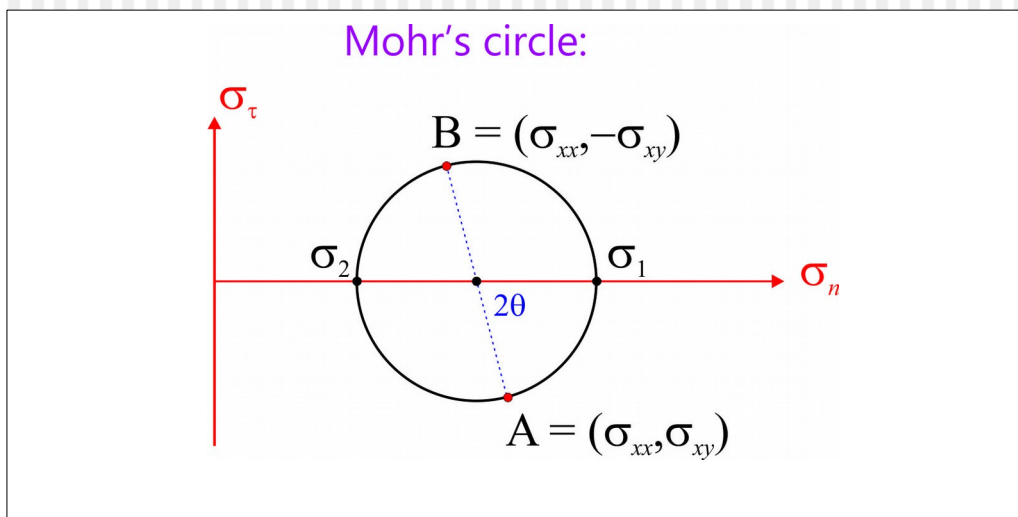
Principal stress

Mohr's circle

- It is easy to show that in 2D, when the two principal stresses equal σ_1 and σ_2 , the normal and tangential (shear) stresses on surface oriented at angle θ equal:

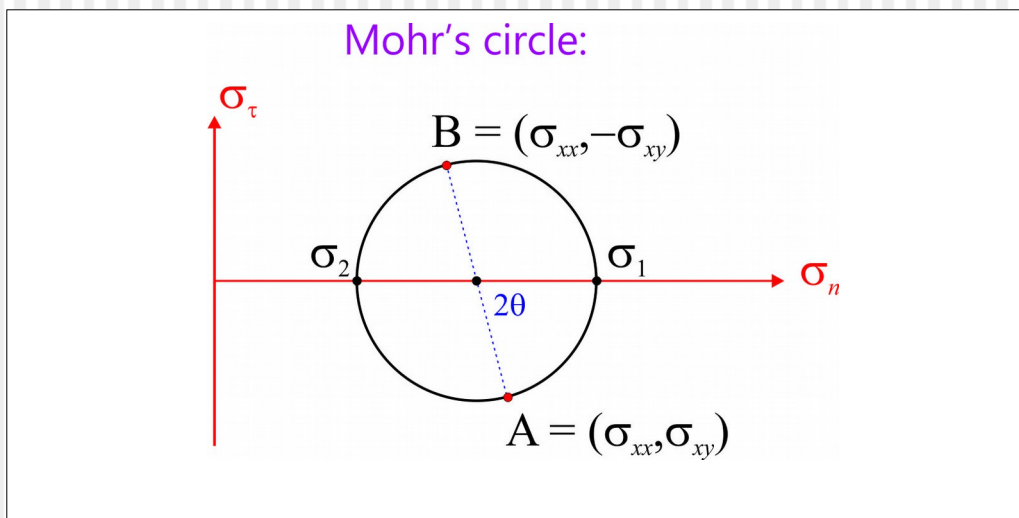
$$\begin{cases} \sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta, \\ \sigma_\tau = -\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta. \end{cases}$$


- Mohr (1914) illustrated these formulas with a diagram:



Mohr's circle (cont.)

- Two ways to use:
 - 1) When knowing the principal stresses and angle θ , start from points σ_1 , σ_2 , and find σ_n and σ_τ .
 - When knowing the stress tensor (σ_{xx} , σ_{xy} , and σ_{yy}), start from points A and B and find σ_1 , σ_2 , and the direction of principal direction σ_1 (θ).

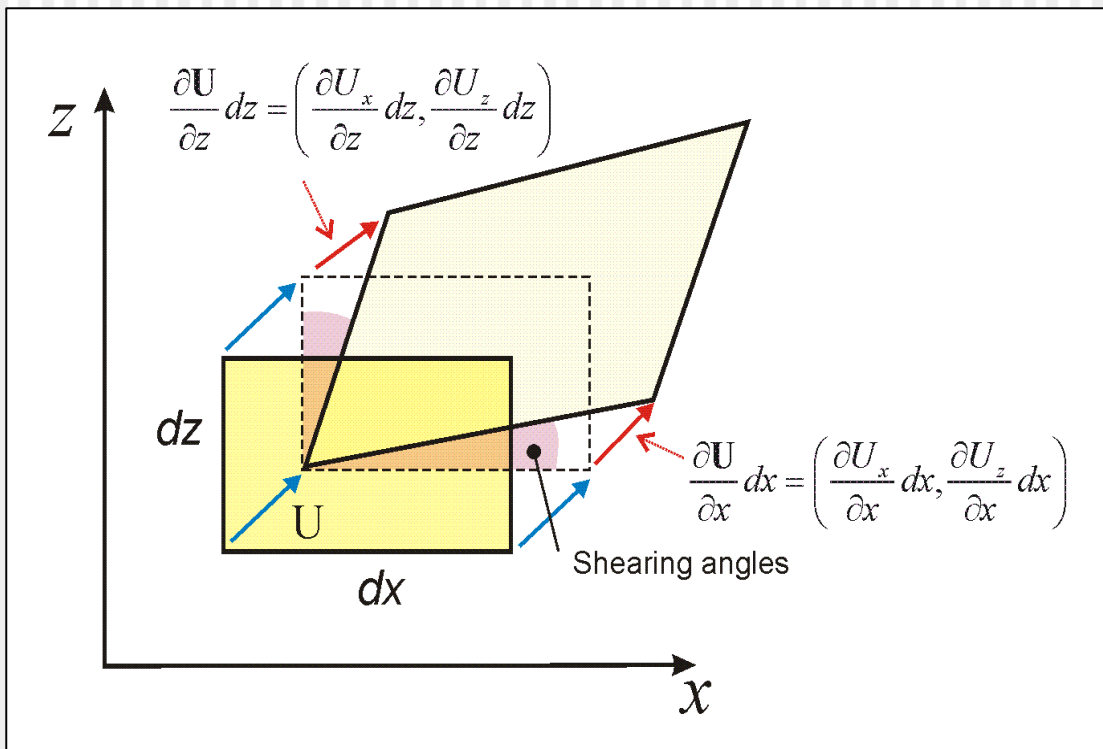


Strain

- Strain is a measure of deformation, i.e., *variation of relative displacement* as associated with a *particular direction* within the body
- It is, therefore, also a *tensor*
 - Represented by a matrix
 - Like stress, it is decomposed into *normal* and *shear* components
- Seismic waves yield strains of 10^{-10} - 10^{-6}
 - So we can rely on *infinitesimal* strain theory

Elementary Strain

- When a body is deformed, *displacements* (\mathbf{U}) of its points are dependent on (x,y,z) , and consist of:
 - Translation (**blue arrows** below)
 - Deformation (**red arrows**)
- Elementary strain is:
$$e_{ij} = \frac{\partial U_i}{\partial x_j}$$



Stretching and Rotation

- Exercise 1: Derive the elementary strain associated with a uniform stretching of the body:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1+\gamma & 0 \\ 0 & 1+\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Exercise 2: Derive the elementary strain associated with rotation by a small angle α :

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- What is characteristic about this strain matrix?

Strain Components

- Anti-symmetric combinations of e_{ij} yield rotations of the body without changing its shape:
 - ♦ e.g., $\frac{1}{2}\left(\frac{\partial U_z}{\partial x} - \frac{\partial U_x}{\partial z}\right)$ yields rotation about the 'y' axis.
 - ♦ So, the case of $\frac{\partial U_z}{\partial x} = \frac{\partial U_x}{\partial z}$ is called *pure shear* (no rotation).
- To characterize *deformation*, only the symmetric component of the elementary strain is used:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right),$$

$$\varepsilon_{ij} = \varepsilon_{ji}, \text{ where } i, j = x, y, \text{ or } z$$

$$\varepsilon = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{1}{2} \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y} \right) & \frac{\partial U_y}{\partial y} & \frac{1}{2} \left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial U_z}{\partial y} + \frac{\partial U_y}{\partial z} \right) & \frac{\partial U_z}{\partial z} \end{pmatrix}$$

Dilatational Strain (relative volume change during deformation)

- Original volume: $V = \delta x \delta y \delta z$
- Deformed volume: $V + \delta V = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz})\delta x \delta y \delta z$
- Dilatational strain:

$$\Delta = \frac{\delta V}{V} = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz}) - 1 \approx \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$\Delta = \varepsilon_{ii} = \partial_i U_i = \vec{\nabla} \cdot \vec{U} = \text{div } \vec{U}$$

- Note that *shearing (deviatoric) strain does not change the volume.*

Deviatoric Strain (pure shear)

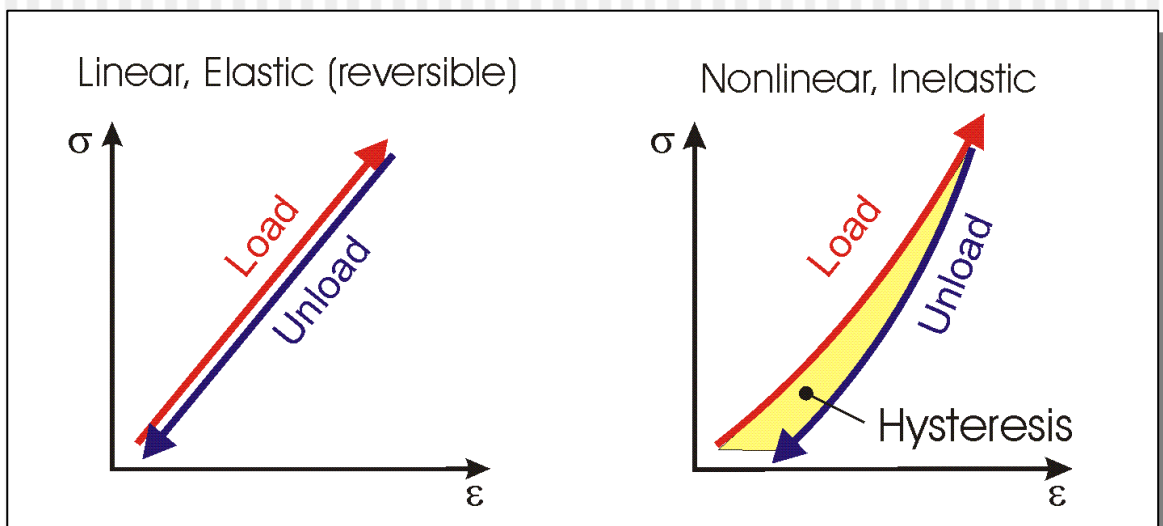
- Strain without change of volume:

$$\tilde{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{\Delta}{3} \delta_{ij}$$

$$\text{Trace}(\tilde{\varepsilon}_{ij}) \equiv \tilde{\varepsilon}_{ii} = 0$$

Constitutive equation

- The “constitutive equation” describes the **stress** developed in a **deformed body**:
 - $\mathbf{F} = -k\mathbf{x}$ for an ordinary spring (1-D)
 - $\sigma \sim \varepsilon$ (in some sense) for a '*linear*', '*elastic*' 3-D solid. This is what these terms mean:



- For a general (*anisotropic*) medium, there are 36 coefficients of proportionality between six independent σ_{ij} and six ε_{ij} :

$$\sigma_{ij} = \Lambda_{ij,kl} \varepsilon_{kl}.$$

Hooke's Law (isotropic medium)

- For isotropic medium, the instantaneous strain/stress relation is described by just 2 constants:


$$\sigma_{ij} = \lambda \Delta \delta_{ij} + 2\mu \varepsilon_{ij}$$

- δ_{ij} is the "Kronecker symbol" (unit tensor) equal 1 for $i = j$ and 0 otherwise;
 - λ and μ are called the *Lamé constants*.
- Question: what are the units for λ and μ ?

Hooke's law (anisotropic)

- For an *anisotropic* medium, there are 36 coefficients of proportionality between six independent σ_{ij} and six ε_{ij} :

$$\sigma_{ij} = \Lambda_{ij,kl} \varepsilon_{kl}$$



This is the
instantaneous-response
case only

Elastic moduli

- Although λ and μ provide a natural mathematical parametrization for $\sigma(\varepsilon)$, they are typically intermixed in experiment environments
 - ♦ Their combinations, called “*elastic moduli*” are typically directly measured.
 - ♦ For example, *P*-wave speed is sensitive to $M = \lambda + 2\mu$, which is called the “*P-wave modulus*”
- Two important pairs of elastic moduli are:
 - ♦ Young's and Poisson's
 - ♦ Bulk and shear

Young's and Poisson's moduli

- Depending on boundary conditions (i.e., experimental setup), different combinations of λ' and μ may be convenient. These combinations are called *elastic constants*, or **moduli**:

- ◆ **Young's modulus** and **Poisson's ratio**:

- Consider a cylindrical sample uniformly squeezed along axis X:

$$\sigma_{xx} = \lambda' \Delta + 2\mu \varepsilon_{xx},$$

$$\sigma_{yy} = \lambda' \Delta + 2\mu \varepsilon_{yy} = 0,$$

$$\sigma_{zz} = \lambda' \Delta + 2\mu \varepsilon_{zz} = 0 \Rightarrow \varepsilon_{yy} = \varepsilon_{zz} = \frac{-\lambda' \Delta}{2\mu}.$$

Young's modulus:

Poisson's ratio:

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

$$\nu = \frac{\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda}{2(\lambda + \mu)}$$

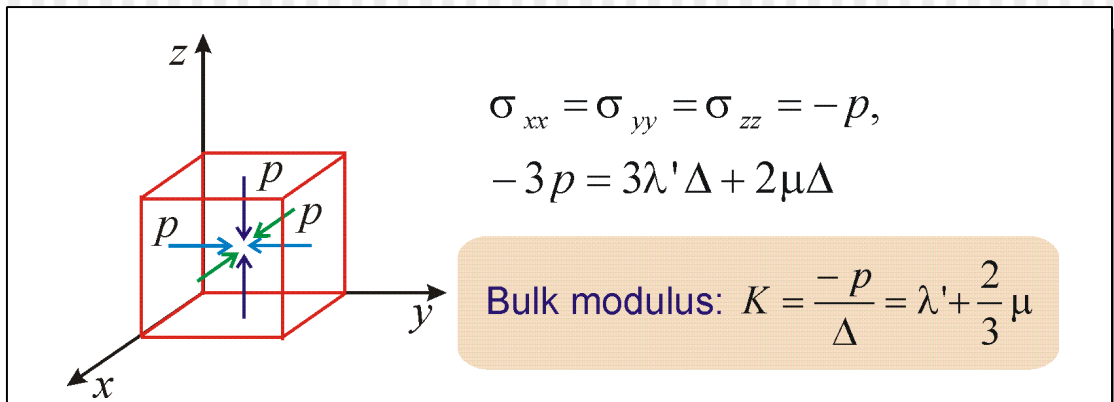
- Note: The Poisson's ratio is more often denoted σ
- It measures the ratio of λ and μ :

$$\frac{\mu}{\lambda} = \frac{1}{2\sigma} - 1$$

Bulk and Shear Moduli

◆ Bulk modulus, K

- Consider a cube subjected to hydrostatic pressure



- ◆ The Lamé constant μ complements K in describing the shear rigidity of the medium. Thus, μ is also called the '*rigidity modulus*'
- ◆ For rocks:
 - Generally, $10 \text{ Gpa} < \mu < K < E < 200 \text{ Gpa}$
 - $0 < \nu < \frac{1}{2}$ always; for rocks, $0.05 < \nu < 0.45$, for most "hard" rocks, ν is near 0.25.
- ◆ For fluids, $\nu = \frac{1}{2}$ and $\mu = 0$ (no shear resistance)

Empirical relations:

$$K, \lambda, \mu(\rho, V_p)$$

- From expressions for wave velocities, we can estimate the elastic moduli:

$$\mu = \rho V_S^2$$

$$\lambda = \rho (V_P^2 - 2 V_S^2)$$

$$K = \lambda + \frac{2}{3} \mu = \rho \left(V_P^2 - \frac{4}{3} V_S^2 \right)$$

Empirical relations: *density*(V_p)

(See the plot is in the following slide)

- Nafe-Drake curve for a wide variety of sedimentary and crystalline rocks (Ludwig, 1970):

$$\rho [g/cm^3] = 1.6612 V_p - 0.4721 V_p^2 + 0.0671 V_p^3 - 0.0043 V_p^4 + 0.000106 V_p^5$$

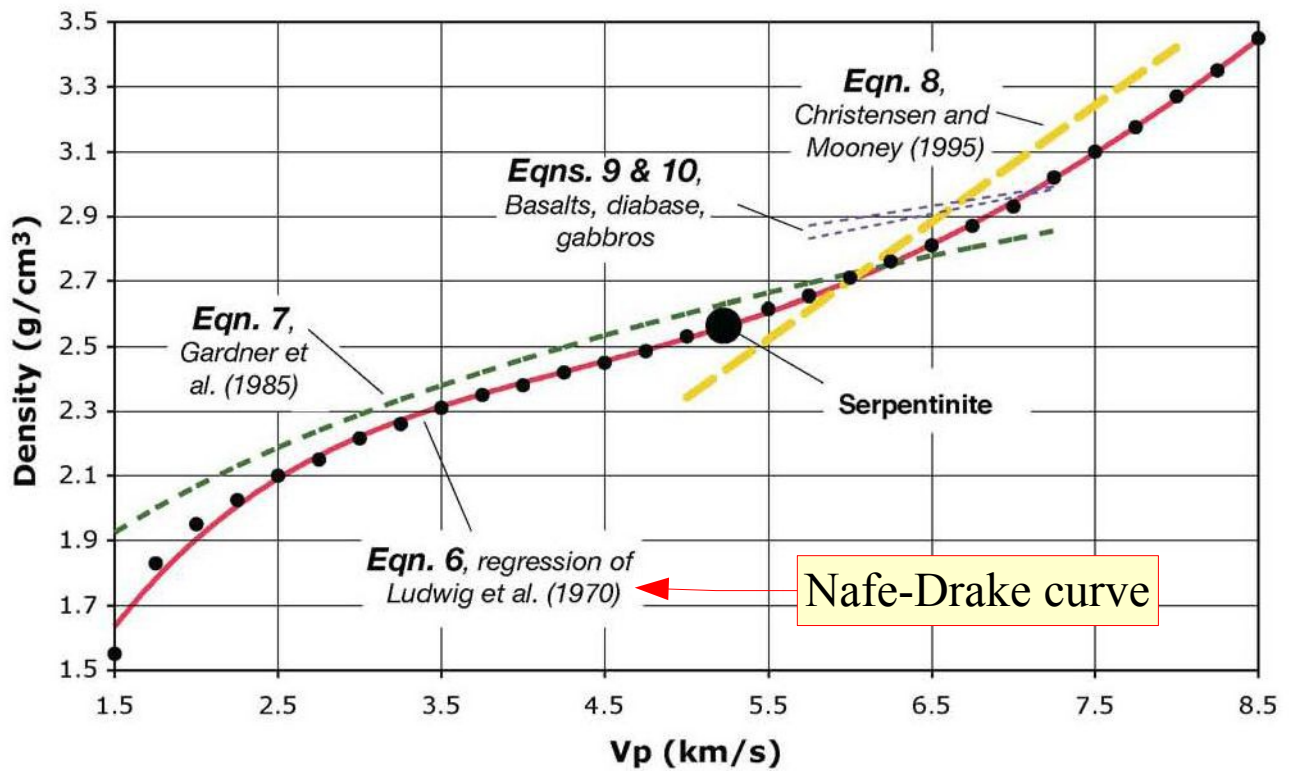
- Gardner's rule for sedimentary rocks and $1.5 < V_p < 6.1$ km/s (Gardner et al., 1984):

$$\rho [g/cm^3] = 1.74 V_p^{0.25}$$

- For crystalline rocks $5.5 < V_p < 7.5$ km/s (Christensen and Mooney., 1995):

$$\rho [g/cm^3] = 0.541 + 0.3601 V_p$$

Empirical relations: *density*(V_p)



From T. Brocher, USGS Open File Report 05-1317, 2005

Empirical relations:

$$V_S(V_P)$$

(See the plot is in the following slide)

- “Mudline” for clay-rich sedimentary rocks (Castagna et al., 1985)

$$V_S [km/s] = (V_P - 1.36) / 1.16$$

- Extension for higher velocity crustal rocks ($1.5 < V_P < 8$ km/s; California, Brocher, 2005):

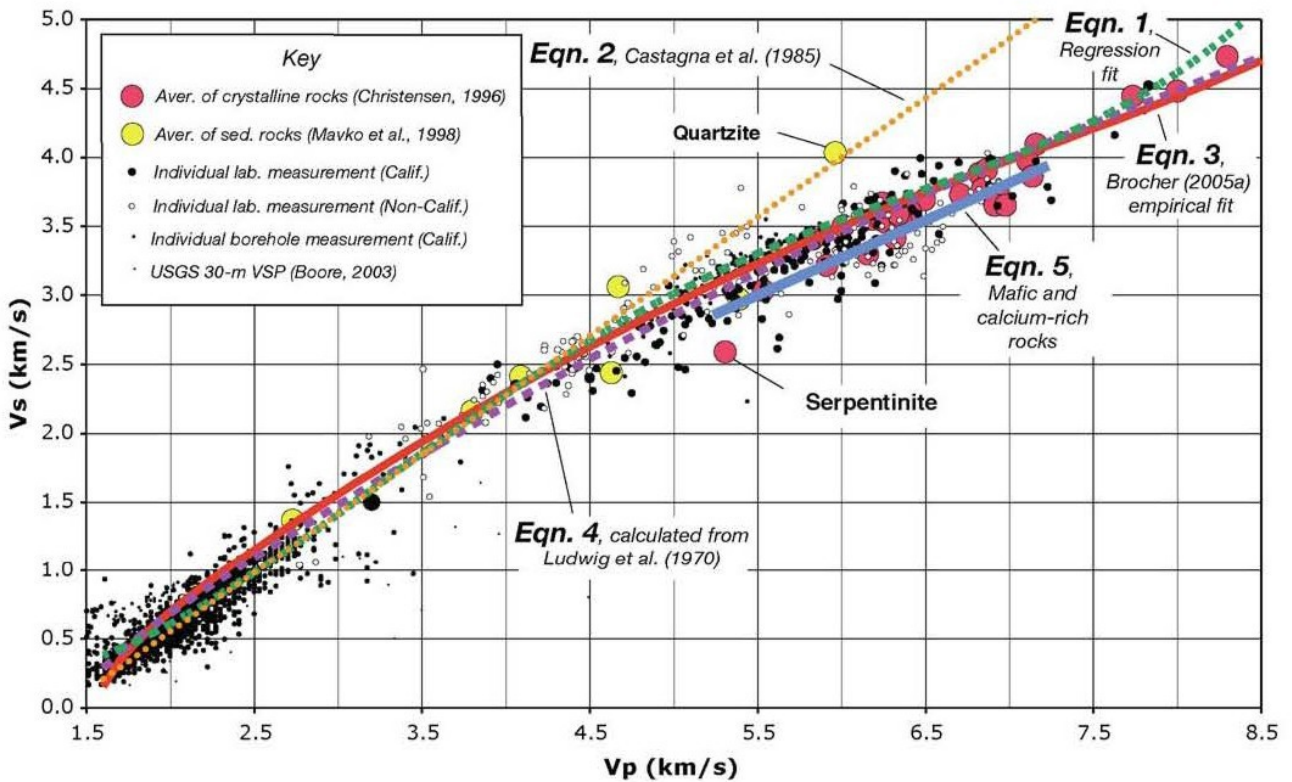
$$V_S [km/s] = 0.7858 - 1.2344 V_P + 0.7949 V_P^2 - 0.1238 V_P^3 + 0.0064 V_P^4$$

- “Mafic line” for calcium-rich rocks (dolomites), mafic rocks, and gabbros (Brocher, 2005):

$$V_S [km/s] = 2.88 + 0.52(V_P - 5.25)$$

Empirical relations:

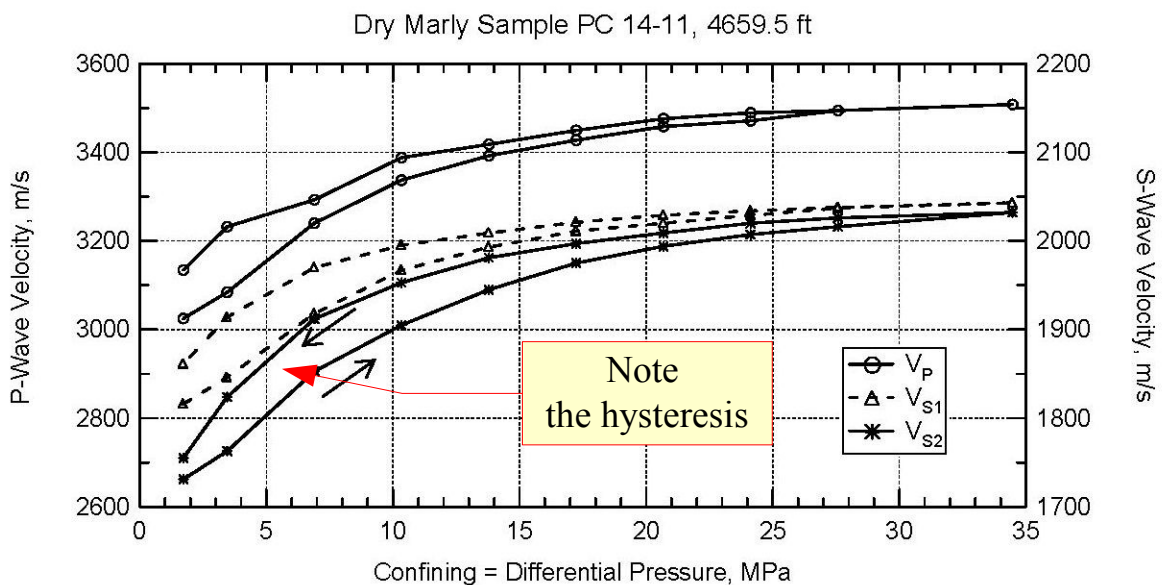
$$V_S(V_P)$$



From T. Brocher, USGS Open File Report 05-1317, 2005

Effect of pressure

- Differential pressure is the difference of confining and pore pressures (related to pressure within rock matrix)
- Differential pressure closes cracks and generally increases the moduli, especially K :



Dry carbonate sample from Weyburn reservoir

Effects of porosity

- Pores in rock
 - ♦ Reduce average density, ρ
 - ♦ Reduce the total elastic energy stored, and thus reduce the K and μ
- Fractures have similar effects, but these effects are always anisotropic

Effects of pore fluids

- Fluid in rock-matrix pores increases the bulk modulus
 - ◆ Gassmann's model (next page)
 - ◆ It has no effect on shear modulus

- Relative to rock-matrix ρ , density effectively decreases:

$$\rho \rightarrow \rho + \rho_f(1 - a)$$

where ρ_f is fluid density, and $a > 1$ is the "tortuosity" of the pores

- In consequence, V_p and V_s increase, and the Poisson's ratio and V_p/V_s decrease

Gassmann's equation

("fluid substitution")

- Relates the elastic moduli of fluid-saturated rock (K_s, μ_s) to those of dry porous rock (K_d, μ_d):

$$K_s = K_d + \frac{K_0(1 - K_d/K_0)^2}{1 - K_d/K_0 - \phi(1 - K_0/K_f)}$$

$$\mu_s = \mu_d$$

where K_f is fluid bulk modulus, and K_0 is the bulk modulus of the matrix

♦ Note: $K_f < K_d \leq K_s \leq K_0$

- Assumptions:
 - ♦ Isotropic, homogeneous, elastic, monomineralic medium;
 - ♦ Pore space is well-connected and in pressure equilibrium;
 - ♦ Closed system with no fluid movement across boundaries;
 - ♦ No chemical reactions.

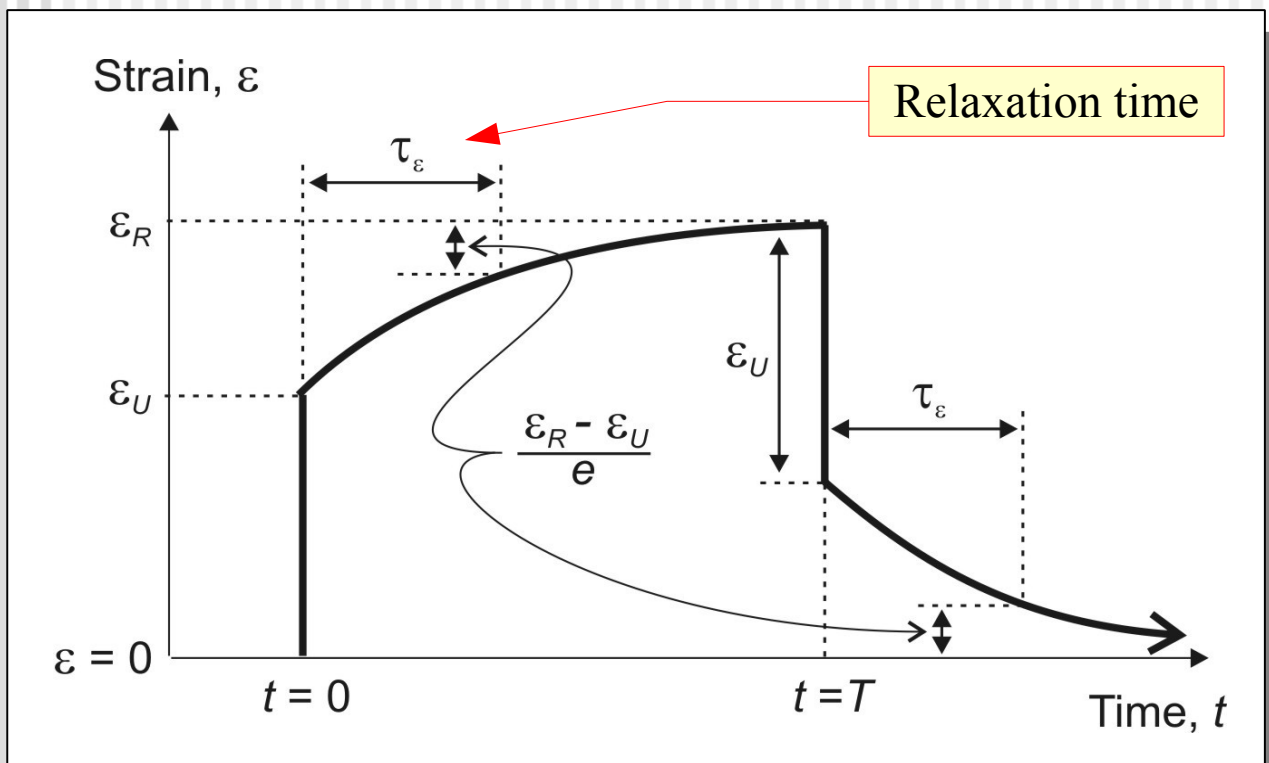
Creep

- When step-function stress $\sigma(t) = \sigma_0 \theta(t)$ is applied to a solid, it exhibits *creep*:

$$\varepsilon(t) = \frac{\sigma_0}{M_U} [1 + \phi(t)]$$

“Creep function”

“Unrelaxed modulus”



Viscoelasticity

- It is thought that creep-like processes also explain:
 - ◆ Attenuation of seismic waves (0.002 – 100 Hz)
 - ◆ Attenuation of Earth's free oscillations (periods ~1 hour)
 - ◆ Chandler wobble (period ~433 days)
- General *viscoelastic* model: stress depends on the *history of strain rate*

$$\sigma(t) = \int_{-\infty}^t M(t-\tau) \dot{\epsilon}(\tau) d\tau$$

Viscoelastic modulus

- Constitutive equation for the "*standard linear solid*" (Zener, 1949):

$$\sigma + \tau_{\sigma} \dot{\sigma} = M (\epsilon + \tau_{\epsilon} \dot{\epsilon})$$

Elastic Energy Density

- Mechanical work is required to deform an elastic body; as a result, elastic energy is accumulated in the strain/stress field
- When released, this energy gives rise to earthquakes and seismic waves
- For a loaded spring (1-D elastic body), $E = \frac{1}{2}kx^2 = \frac{1}{2}Fx$
- Similarly, for a deformed elastic medium, the *elastic energy density* is:

$$E_{elastic} = \frac{1}{2} \sigma_{ij} \epsilon_{ij}$$

Energy Flux in a Wave

- Later, we will see that in a wave, the kinetic energy density equals the elastic energy:

$$\frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \rho \dot{u}^2$$

- and so the total energy density:

$$E = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} + \frac{1}{2} \rho \dot{u}^2 = \rho \dot{u}^2$$

- The energy propagates with wave speed V , and so the **average energy flux** equals:

$$J = VE = \rho V \langle \dot{u}^2 \rangle = \frac{1}{2} Z A_v^2$$

where $Z = \rho V$ is the impedance, and A_v is the particle-velocity amplitude

Lagrangian mechanics

- Instead of equations of motion, modern (*i.e.*, 18th century!) “analytical mechanics” is described in terms of **energy functions** of **generalized coordinates** x and **velocities** \dot{x} :

- ◆ Kinetic: (for example) $E_k = \frac{1}{2} m \dot{x}^2$

- ◆ Potential: $E_p = \frac{1}{2} k x^2$

- These are combined in the **Lagrangian function**:

$$L(x, \dot{x}) = E_k - E_p$$

- Equations of motion become:

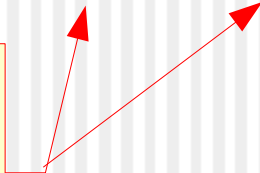
$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{dL}{dx} = 0$$

Lagrangian mechanics of elastic medium

- Lagrangian of isotropic elastic field:

$$L(\mathbf{u}, \dot{\mathbf{u}}) = \int dV \left[\frac{1}{2} \rho \dot{u}_i \dot{u}_i - \left(\frac{1}{2} \lambda \varepsilon_{ii}^2 + \mu \varepsilon_{ij} \varepsilon_{ij} \right) \right]$$

These are the only two second-order combinations of ε that are **scalar** and **invariant with respect to rotations**



- This shows the true meanings of *Lamé parameters*
 - ♦ They correspond to the contributions of two different types of deformation (compression and shear) to the potential energy

Lagrangian mechanics

- **Exercise:** use the Hooke's law to show that

$$E_{elastic} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

is indeed equivalent to:

$$E_{elastic} = \frac{1}{2} \lambda \varepsilon_{ii}^2 + \mu \varepsilon_{ij} \varepsilon_{ij}$$