

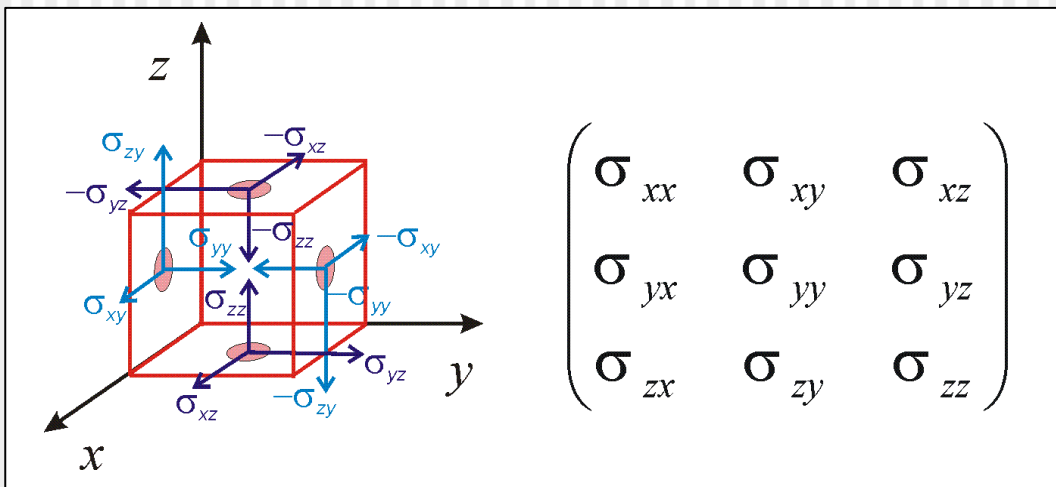
Elasticity and seismic waves

- Recap of theory
 - Equations of motion
 - Wave equations
 - P- and S-waves
 - Impedance
 - Wave potentials
 - Energy of a seismic wave
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- Reading:
 - › Telford *et al.*, Section 4.2
 - › Shearer, 3
 - › Sheriff and Geldart, Sections 2.1-4

Forces acting on a small cube

- Consider a small volume ($dx \times dy \times dz = dV$) within the elastic body.
- *Force* applied to the parallelepiped from the outside is:

$$F_i = -\partial_j \sigma_{ij} dV$$



Equations of Motion

(Motion of the elastic body with time)

- Uncompensated net force will result in *acceleration* (second Newton's law):

Newton's law: $\rho \delta V \frac{\partial^2 U_i}{\partial t^2} = F_i$

$$\rho \frac{\partial^2 U_i}{\partial t^2} = \left(\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z} \right)$$

$$\begin{aligned} \rho \frac{\partial^2 U_x}{\partial t^2} &= \frac{\partial}{\partial x} \left(\lambda' \Delta + 2\mu \frac{\partial U_x}{\partial x} \right) + \mu \frac{\partial}{\partial y} \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) + \mu \frac{\partial}{\partial z} \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right) \\ &= \lambda' \frac{\partial \Delta}{\partial x} + \mu \frac{\partial}{\partial x} \left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \right) + \mu \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right) \\ &= (\lambda' + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 U_x \end{aligned}$$

- Therefore, the *equations of motion* for the components of **U**:

$$\rho \frac{\partial^2 U_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x_i} + \mu \nabla^2 U_i$$

Wave potentials

Compressional and Shear waves

- These equations describe two types of waves.
- The general solution has the form (“**Lamé theorem**”):

$$\vec{U} = \vec{\nabla} \phi + \vec{\nabla} \times \vec{\psi}. \quad (\text{or } U_i = \partial_i \phi + \epsilon_{ijk} \partial_j \psi_k)$$

$$\vec{\nabla} \cdot \vec{\psi} = 0.$$

Because there are 4 components in ψ and ϕ only 3 in \mathbf{U} , we need to constrain ψ .

- Exercise: substitute the above into the equation of motion:

$$\rho \frac{\partial^2 U_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x_i} + \mu \nabla^2 U_i$$

and show:

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu) \nabla^2 \phi, \quad \leftarrow \text{P-wave (scalar) potential.}$$

$$\rho \frac{\partial^2 \psi_i}{\partial t^2} = \mu \nabla^2 \psi_i, \quad \leftarrow \text{S-wave (vector) potential.}$$

Wave velocities

Compressional and Shear waves

- These are wave equations; compare to the general form of equation describing wave processes:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] f(x, y, z, t) = 0$$

- *Compressional (P)* wave velocity:

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

- *Shear (S)* wave velocity:

♦ $V_S < V_P$

♦ for $\sigma=0.25$: $V_P/V_S = \sqrt{3}$

$$v_S = \sqrt{\frac{\mu}{\rho}}$$

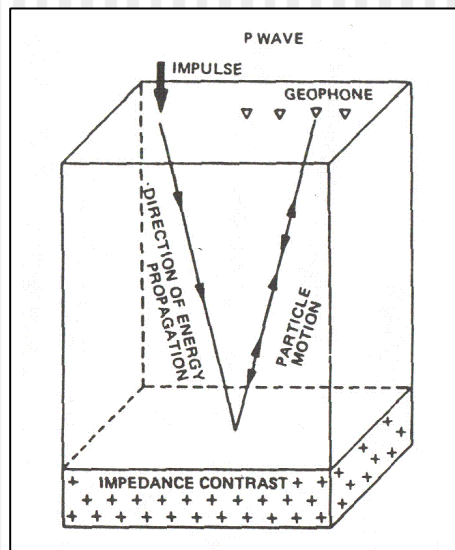
- Note that the V_P/V_S depends on the Poisson's ratio alone:

$$\frac{V_S}{V_P} = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1/2 - \sigma}{1 - \sigma}}$$

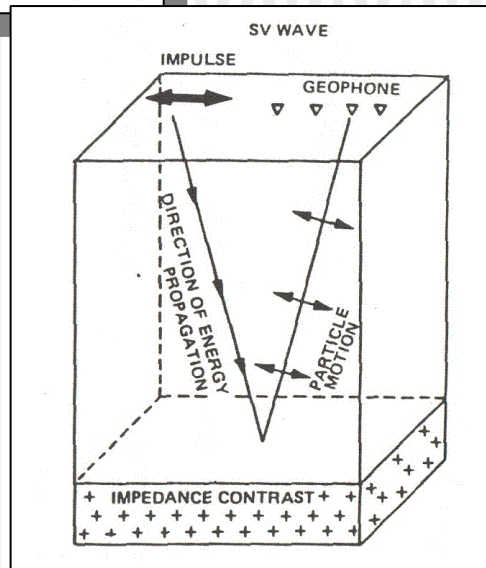
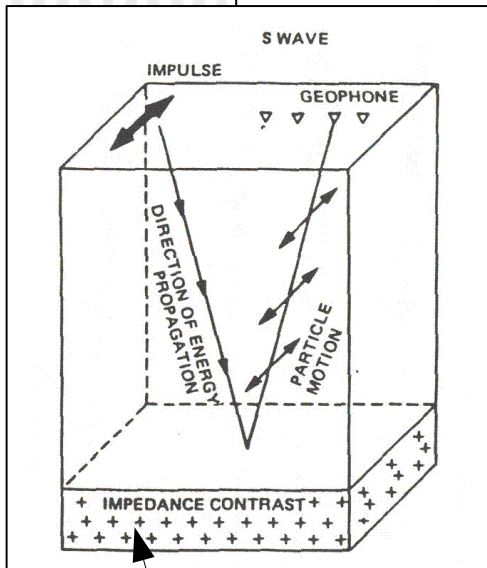
Wave Polarization

- Elastic solid supports **two types of body waves**:

P



S



Note that this is still an **ISOTROPIC** reflector. In general, reflection will intermix the S-wave polarization modes, and P-wave will convert into SV upon reflection.

Notes on the use of potentials

- Wave potentials are very useful for solving elastic wave problems
- Just take ϕ or ψ satisfying the wave equation, e.g.:

$$\phi(\vec{r}, t) = Ae^{i\omega(t - \frac{\vec{r}\vec{n}}{V_P})} \quad \text{(plane wave)}$$

...and use the equations for potentials to derive the displacements:

$$\vec{U} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{\psi}$$

...and stress from Hooke's law:

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\epsilon_{ij}$$

- *Displacement amplitude* = $\omega \times (\text{potential amplitude}) / V$
- *Power* = $\frac{1}{2}\rho[\omega^2 \times (\text{potential amplitude}) / V]^2$

.....

 This is *velocity* amplitude

Example:

Compressional (P) wave

- Scalar potential for *plane harmonic* wave:

$$\phi(\vec{r}, t) = A e^{i\omega(t - \frac{\vec{r}\vec{n}}{V_P})}.$$

- Displacement:

$$u_i(\vec{r}, t) = \partial_i \phi(\vec{r}, t) = \frac{-i\omega n_i}{V_P} A e^{i\omega(t - \frac{\vec{r}\vec{n}}{V_P})}.$$

note that the **displacement is always along \mathbf{n}** .

- Strain:

$$\varepsilon_{ij}(\vec{r}, t) = \partial_i u_j(\vec{r}, t) = \frac{\omega^2 n_i n_j}{V_P^2} A e^{i\omega(t - \frac{\vec{r}\vec{n}}{V_P})}.$$

- Dilatational strain:

$$\Delta = \varepsilon_{ii}(\vec{r}, t) = \frac{\omega^2}{V_P^2} A e^{i\omega(t - \frac{\vec{r}\vec{n}}{V_P})} = \frac{\omega^2}{V_P^2} \phi(\vec{r}, t).$$

- Stress:

$$\sigma_{ij}(\vec{r}, t) = \frac{\omega^2}{V_P^2} (\lambda \delta_{ij} + 2\mu n_i n_j) \phi(\vec{r}, t).$$

- **Question:** what wavefield would we have if used $\cos(\dots)$ or $\sin(\dots)$ function instead of complex $\exp(\dots)$ in the expression for potential above?

Impedance

- In general, *the Impedance, Z*, is a measure of the amount of resistance to particle motion.
- In elasticity, impedance is the *ratio of stress to particle velocity*.
 - ♦ Thus, for a given applied stress, particle velocity is inversely proportional to impedance.
 - ♦ For *P* wave, in the direction of its propagation:

$$Z(\vec{r}, t) = \frac{\sigma_{nn}(\vec{r}, t)}{\dot{u}_n(\vec{r}, t)} = \frac{\lambda + 2\mu}{V_P} = \rho V_P.$$

- impedance does not depend on frequency but *depends on the wave type and propagation direction*.

Elastic Energy Density

- Recall that for a deformed elastic medium, the *energy density* is:

$$E = \frac{1}{2} \sigma_{ij} \epsilon_{ij}$$

Elastic Energy Density *in a plane wave*

- For a plane wave:

$$u_i = u_i(t - \vec{p} \cdot \vec{x})$$

$$\varepsilon_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) = -\frac{1}{2} (\dot{u}_i p_j + \dot{u}_j p_i).$$

...and therefore:

$$\frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} [(\lambda + \mu) (\vec{p} \cdot \vec{u})^2 + \mu (\vec{u} \cdot \vec{u}) (\vec{p} \cdot \vec{p})]$$

- For *P*- and *S*-waves, this gives:

$$\frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} (\lambda + 2\mu) p^2 \vec{u}^2 = \frac{1}{2} \rho \dot{u}^2 \quad \text{P-wave}$$

$$\frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} (\mu) p^2 \vec{u}^2 = \frac{1}{2} \rho \dot{u}^2 \quad \text{S-wave}$$

- Thus, in a wave, *strain energy equals the kinetic energy* Energy is *NOT* conserved locally!
- *Energy propagates at the same speed* as the wave pulse