Kirchhoff Theory

- Relation to Huygens' principle
- Wavefield extrapolation
- Application: modelling of reflected wavefields

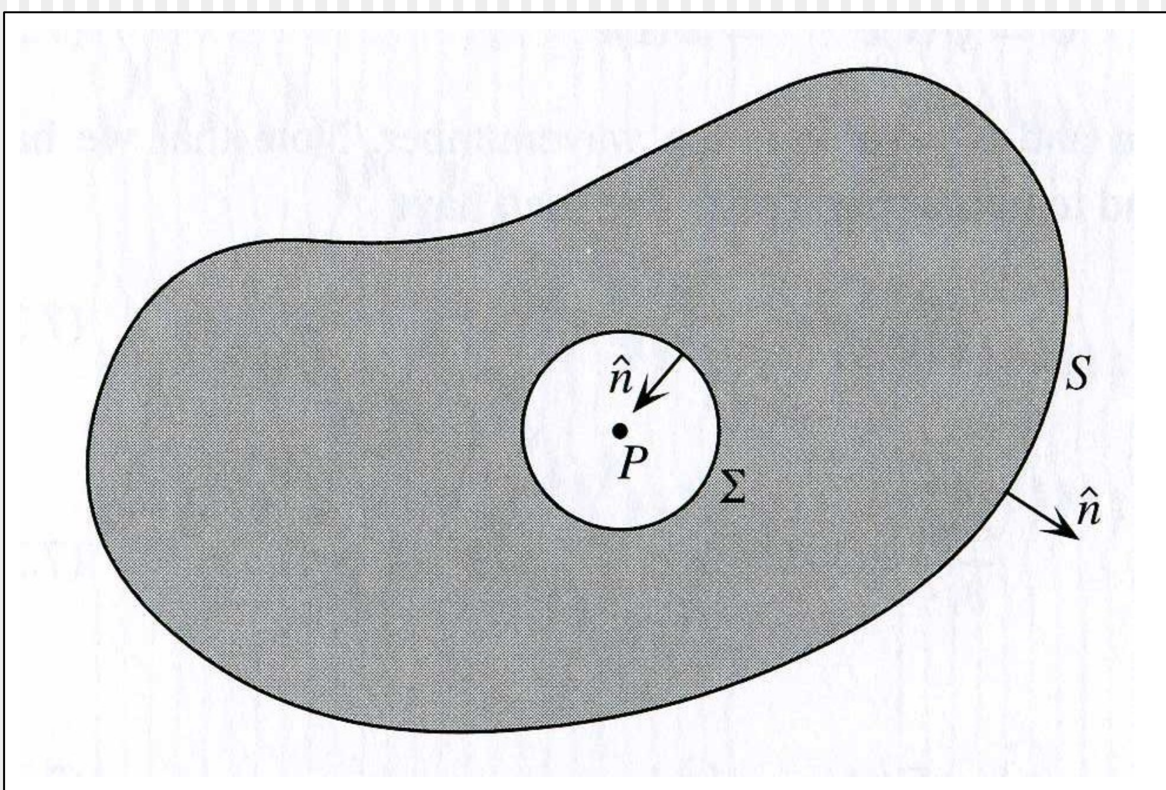
- Reading:
  - Shearer, 7.6

Kirchhoff theory
A rigorous form of Huygens' principle

- For a field satisfying the wave equation:

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

- We want to express the value of ϕ at point *P* through an integral over a surface (compare to Huygens' principle!):



Kirchhoff theory (cont.)

- Now consider a harmonic form of ϕ (that is, apply the Fourier transform to ϕ):

$$\phi(t, \vec{r}) = \phi(\vec{r}) e^{-i\omega t} = \phi(\vec{r}) e^{-ikct}$$

  - Here, $k=\omega/c$ is the wavenumber.
- The wave equation then becomes the time-independent *Helmholtz's equation*:

$$\nabla^2 \phi = -k^2 \phi$$

- Now recall the Green's theorem from field calculus:

$$\int_V (\phi_2 \nabla^2 \phi_1 - \phi_1 \nabla^2 \phi_2) dV = \int_{S+\Sigma} (\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n}) dS$$

  - this holds for any two continuous functions with continuous derivatives

Kirchhoff theory (cont.)

- If both ϕ_1 and ϕ_2 satisfy the Helmholtz's equation, then the l.h.s. of the Green's equation vanishes, and therefore:

$$\int_{S+\Sigma} \left(\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n}\right) dS = 0$$

- Now consider: $\phi_1 = \phi(\mathbf{r})$ to be our field and:

$$\phi_2 = \frac{e^{ikr}}{r}$$

Point-source field

(this is a point-source field from a source located at **P**)

- Then:

$$\int_{\Sigma+S} \left[\frac{e^{ikr}}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right)\right] dS = 0$$

Kirchhoff theory (cont.)

- For radius of $\Sigma \to 0$ (show this!):

$$\int_{\Sigma}\left[\frac{e^{ikr}}{r}\frac{\partial\phi}{\partial r}-\phi\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)\right]dS=-4\pi\phi(\vec{P})$$

- and therefore $\phi(\mathbf{P})$ can be expressed as an integral of the wavefield and its gradient over the enclosing surface *S*:

$$\phi(\vec{P})=\frac{1}{4\pi}\int_{S}\left[\frac{e^{ikr}}{r}\frac{\partial\phi}{\partial n}-\phi\frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right)\right]dS$$

- Returning to the time-dependent (still harmonic) function:

$$\phi(t,\vec{P})=\frac{1}{4\pi}\int_{S}\left[\frac{e^{ik(r-ct)}}{r}\frac{\partial\phi}{\partial n}-\phi\frac{\partial}{\partial n}\left(\frac{e^{ik(r-ct)}}{r}\right)\right]dS$$

Note that the surface field is taken at *retarded times*:

$$t_{Ret} = t - r/c$$

Kirchhoff theory (cont.)

- Finally, after the Fourier transform in ω (i.e., time):

$$\phi(t,\vec{P})=\frac{1}{4\pi}\int_S \left\{\frac{1}{r}\frac{\partial\phi}{\partial n}-\frac{\partial}{\partial n}\left(\frac{1}{r}\right)[\phi]+\frac{1}{cr}\frac{\partial r}{\partial n}\left[\frac{\partial\phi}{\partial t}\right]\right\} dS$$

  - here, the [...] quantities are taken at retarded times $t_{Ret} = t - r/c$

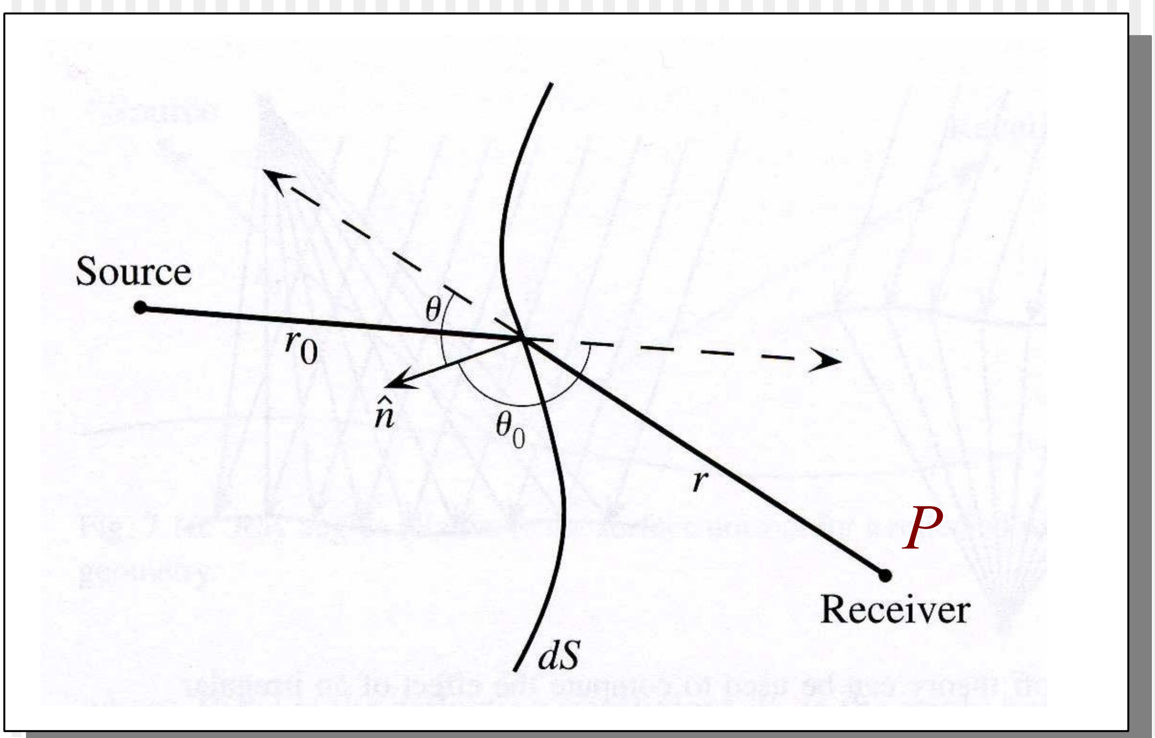
Kirchhoff theory (cont.)

- For a practical application, assume that the field at the surface comes from some distant source:

$$\phi = \frac{f\left(t - \frac{r_0}{c}\right)}{r_0}$$

- Then (see Shearer 7.6):

$$\phi(t, \vec{P}) = \frac{1}{4\pi c}\int_S \delta\left(t - \frac{r + r_0}{c}\right)\frac{1}{r r_0}(\cos\theta - \cos\theta_0)\, dS * f'(t)$$



Modelling reflected field using Kirchhoff integral

- High-frequency (far-field, r >> λ) reflected field from an arbitrary irregular reflector:

$$\phi(t,\vec{P})=\frac{1}{4\pi c}\int_S \delta\left(t-\frac{r+r_0}{c}\right)\frac{1}{r r_0}(\cos\theta+\cos\theta_0)\,dS * f'(t)$$

  - this describes both reflections and diffractions, phase shifts, angle and frequency dependence
- Inversion of this formula is the basis for "Kirchhoff migration"