Tomography and Location

- Forward and Inverse travel-time problems
- Seismic tomography
- Least Squares inverse
- Generalised Linear Inverse
- Iterative inverse
  - Back-projection method
- Resolution
- Statistical testing of results
- Location of seismic sources
- Data norms

- Reading:

Shearer, 5.6-5.7

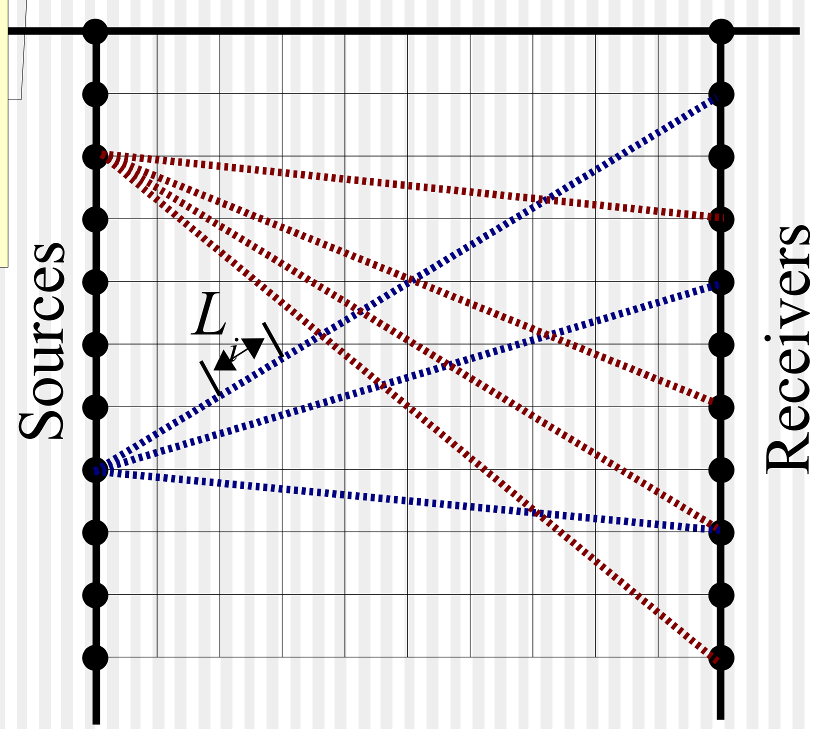
Seismic (velocity) tomography

- Tomography
  - The name derived from the Greek for “section drawing” - the idea is that the section appears *almost* automatically...
  - Using multitude of source-receive pairs with rays crossing the area of interest.
  - Looking for an unknown velocity structure.
  - Depending on the type of recording used, it could be:
    - *Transmission tomography* (nearly straight rays between boreholes);
    - *Reflection tomography* (reflected rays; in this case, positions of the reflectors could be also found);
    - *Diffraction tomography* (using least-time travel paths according to Fermat rather than Snell's law; this is actually more a waveform inversion technique).

Cross-well tomography

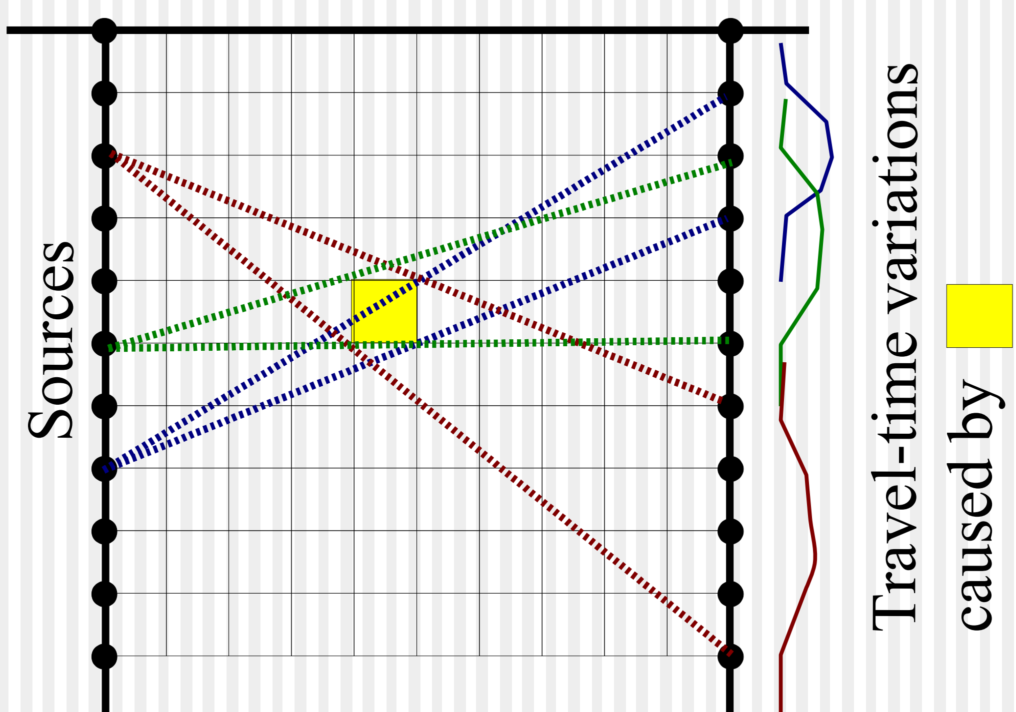
- Consider the case of transmission "cross-well" tomography
  - This is the simplest case – rays may be considered nearly straight, the data are abundant, and the coverage is *relatively uniform*.

These are the three principal concerns in tomography:
1) linearity of the problem;
2) density of data coverage;
3) good azimuthal coverage.



Principle

- Velocity perturbations are considered as small
  - Therefore, rays are approximated as straight
- Each velocity cell leads to characteristic travel-time variations at the receivers ("impulse response")
  - These are inverted for velocity value at



Travel-time inversion as a *linear inverse problem*

- First, we parameterize the velocity model
  - Typically, the parameterization is a grid of constant-velocity blocks (sometimes splines are used instead of the blocks).
  - This parameterization gives us a *model vector,* **m***.*

$$\boldsymbol{m} = \begin{pmatrix} s_1 = 1/V_1 \\ s_2 = 1/V_2 \\ \dots \\ s_N = 1/V_N \end{pmatrix}.$$

- Second, we measure all travel times and arrange them into a data vector:

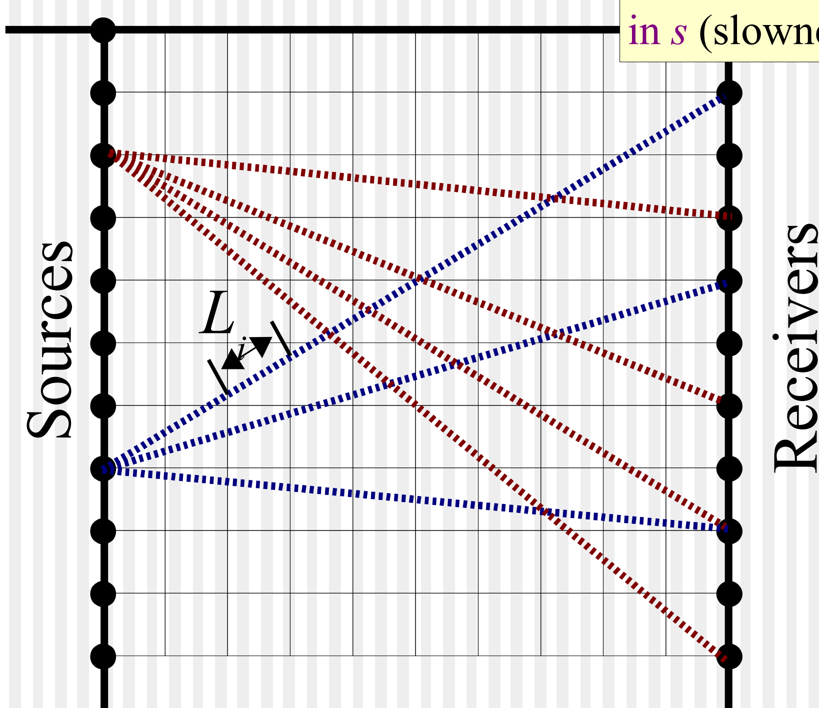
$$\boldsymbol{d}^{observed} = \begin{pmatrix} t_1 \\ t_2 \\ \dots \\ t_M \end{pmatrix}.$$

Forward model

- Third, we formulate the *forward model* to predict **d** from **m**. To achieve this, we need to *trace rays* through the model and measure the length of every ray's segment in each model block, L_{ij}.
  - The travel time for *i*-th ray is then:

$$t_i = \sum_j L_{ij} \frac{1}{V_j} = \sum_j L_{ij} s_j .$$

Note that the expression is non-linear in V but linear in s (slowness).



Generalized Linear Inverse

- The model for travel times: $t_i = \sum_j L_{ij} s_j$ can be written in matrix form:

$$\boldsymbol{d} = \boldsymbol{L}\, m$$

- Now, we want to substitute **d** = $\mathbf{d}^{\text{observed}}$ and solve for unknown **m**. This is called the *inverse problem*.
- Typically, matrix *L* is not invertible (it is not square), and so it is inverted in some *generalized* (averaged) sense.
- Any solution in the linear form

$$\boldsymbol{m} = \boldsymbol{L}_g^{-1}\, \boldsymbol{d}^{observed}$$

  is called the generalized linear inverse.
- The problem is thus in finding a suitable form for $\mathbf{L}_g^{-1}$.

Projection into model space

- Often, tomography problems are typically <u>overdetermined</u> (contain many more ray paths than grid model blocks).
- In such cases, the following approach to constructing $\mathbf{L}_g^{-1}$ works well:
  - multiply by transposed $\mathbf{L}^T$:

$$\boldsymbol{L}^T d^{observed} = \boldsymbol{L}^T \boldsymbol{L} \boldsymbol{m} ,$$

This operation "back-projects" the redundant data into "model space"

  - hence:

$$\boldsymbol{m} = (\boldsymbol{L}^T \boldsymbol{L})^{-1} \boldsymbol{L}^T \boldsymbol{d}^{observed} .$$

This is the "least-squares" solution
It is used
in the
well-known GLI3D
program
for refraction
statics

Least Squares Inverse

- Note that the solution is a linear combination of data values:

$$\mathbf{m} = (\mathbf{L}^T \mathbf{L})^{-1} \mathbf{L}^T \mathbf{d}^{observed} = \mathbf{L}_g^{-1} \mathbf{d}^{observed}.$$

$$\mathbf{L}_g^{-1} = (\mathbf{L}^T \mathbf{L})^{-1} \mathbf{L}^T.$$

The "generalized inverse" matrix

- The reason for its name of "Least Squares" is in minimizing the mean square of data misfits:

$$Misfit(m) = (\mathbf{d}^{observed} - \mathbf{L}\mathbf{m})^T (\mathbf{d}^{observed} - \mathbf{L}\mathbf{m}).$$

  - <u>Exercise</u>: show this!

Damped Least Squares

- Sometimes the matrix $\mathbf{L}^T\mathbf{L}$ is singular and its inverse is unstable.
  - This happens, *e.g.*, when some cells are not crossed by any rays, or there are groups of cells traversed by the same rays only.
- In such cases, the inversion can be *regularized* by adding a small positive diagonal term to $\mathbf{L}^T\mathbf{L}$:

$$\boldsymbol{m} = \left(\boldsymbol{L}^T\boldsymbol{L} + \varepsilon\boldsymbol{I}\right)^{-1}\boldsymbol{L}^T\boldsymbol{d}^{observed}.$$

  - This is called the *Damped Least Squares* solution.
  - ε is chosen such that stability is achieved and the non-zero contributions in $\mathbf{L}^T\mathbf{L}$ are affected only slightly.

Weighted Least Squares

- Often, different types of data are included in **d**
  - For example, different travel times, t_i, may be measured with different uncertainties δt_i
- In such cases, it is useful to apply weights to the equations:

$$\mathbf{W}\,\mathbf{d} = \mathbf{W}\,\mathbf{L}\,\mathbf{m}$$

where **W** is a diagonal weight matrix:

$$\mathbf{W} = diag\left(\frac{1}{\delta t_1}, \frac{1}{\delta t_2}, \frac{1}{\delta t_3}, \ldots\right)$$

Weighted Least Squares (cont.)

- This corresponds to a modified least-squares misfit function:

$$Misfit(m) = (\mathbf{d}^{\mathbf{observed}} - \mathbf{L}\,\mathbf{m})^T\, \mathbf{W}^T\, \mathbf{W}\, (\mathbf{d}^{\mathbf{observed}} - \mathbf{L}\,\mathbf{m})$$

and solution:

$$\mathbf{m} = \mathbf{L}_{\mathbf{g}}^{-1}\, d^{observed}$$

$$\mathbf{L}_{\mathbf{g}}^{-1} = (\mathbf{L}^T\, \mathbf{W}^T\, \mathbf{W}\, \mathbf{L} + \varepsilon\, \mathbf{I})^{-1}\, \mathbf{L}^T\, \mathbf{W}$$

Smoothness constraints

- When using finely-sampled models...
  - some cells may be poorly constrained;
  - solutions can become 'rough' (highly variable, noisy – see below)
- To remove roughness, additional 'smoothness constraint' equations can be added
  - These equations will be additional rows in **L**, for example:
    - $w\, m_i = w\, Average\left(Adjacent\; m_j\right)$
    - Zero Laplacian: $w \nabla^2 m = 0$
- These equations require small *weights* *w*

Simple Iterative Inverse

- Sometimes matrix $\mathbf{L}^T\mathbf{L}$ is also too large to invert, or even to store
- It can the be approximated by its diagonal:

$$\boldsymbol{m} = \left[diag\left(\boldsymbol{L}^T \boldsymbol{L}\right) + \varepsilon \boldsymbol{I} \right]^{-1} \boldsymbol{L}^T \boldsymbol{d}^{observed}.$$

  - The diagonal only contains one value per model cell (sum of squared *L*'s for all rays crossing it)
  - Contributions to **m** can be evaluated during a pass through all data and without storing matrices **L** or $\mathbf{L}^T\mathbf{L}$.
- Variants of this method are known as:
  - Back-projection method;
  - SIRT (Simultaneous Iterative Reconstruction technique)
  - ART (Algebraic Reconstruction Technique)

Simple Iterative Inverse (*how it works*)

- Iteration:

Travel times in "background model"

$$\delta_1 \boldsymbol{d} = \boldsymbol{d}^{observed} - d_{0,}$$

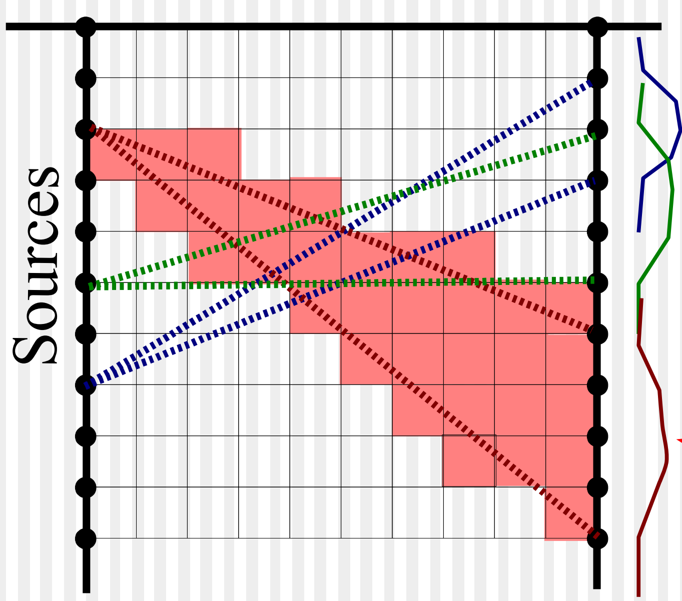
$$\delta_1 \boldsymbol{m} = \boldsymbol{L}_g^{-1} \delta_1 \boldsymbol{d}.$$

Approximate inverse of any kind

$$\delta_2 \boldsymbol{d} = \delta_1 \boldsymbol{d} - \boldsymbol{L} \delta_1 \boldsymbol{m},$$

$$\delta_2 \boldsymbol{m} = \boldsymbol{L}_g^{-1} \delta_2 \boldsymbol{d},$$

...



For each ray,
the observed travel-time perturbation
is thus "back-projected" into the slowness model

Resolution matrix

- Assessment of the *quality of inversion method* is often done by using the *Resolution Matrix*
  - Regardless of the selected form of the inverse, we can:
    1) Perturb 1 parameter (grid node) of the model;
    2) Perform forward modeling (generate synthetic data);
    3) Perform the inverse.
  - When repeated for each parameter, this process results in a resolution matrix:

$$\boldsymbol{R} = \boldsymbol{L}_g^{-1}\boldsymbol{L}$$

- Note that **R** *does not* depend on the data values but depends on sampling
  - Crossing rays are VERY important in tomography.

Checkerboard resolution test

- Test of the resolution in the model when computation of the *Resolution Matrix* is impossible or impractical
- Method:
  - Create an artificial model perturbation in the form of alternating positive and negative anomalies ("checkerboard")
  - Predict the data in this model:

$$\mathbf{d}' = \mathbf{L}\mathbf{m}_{checker}$$

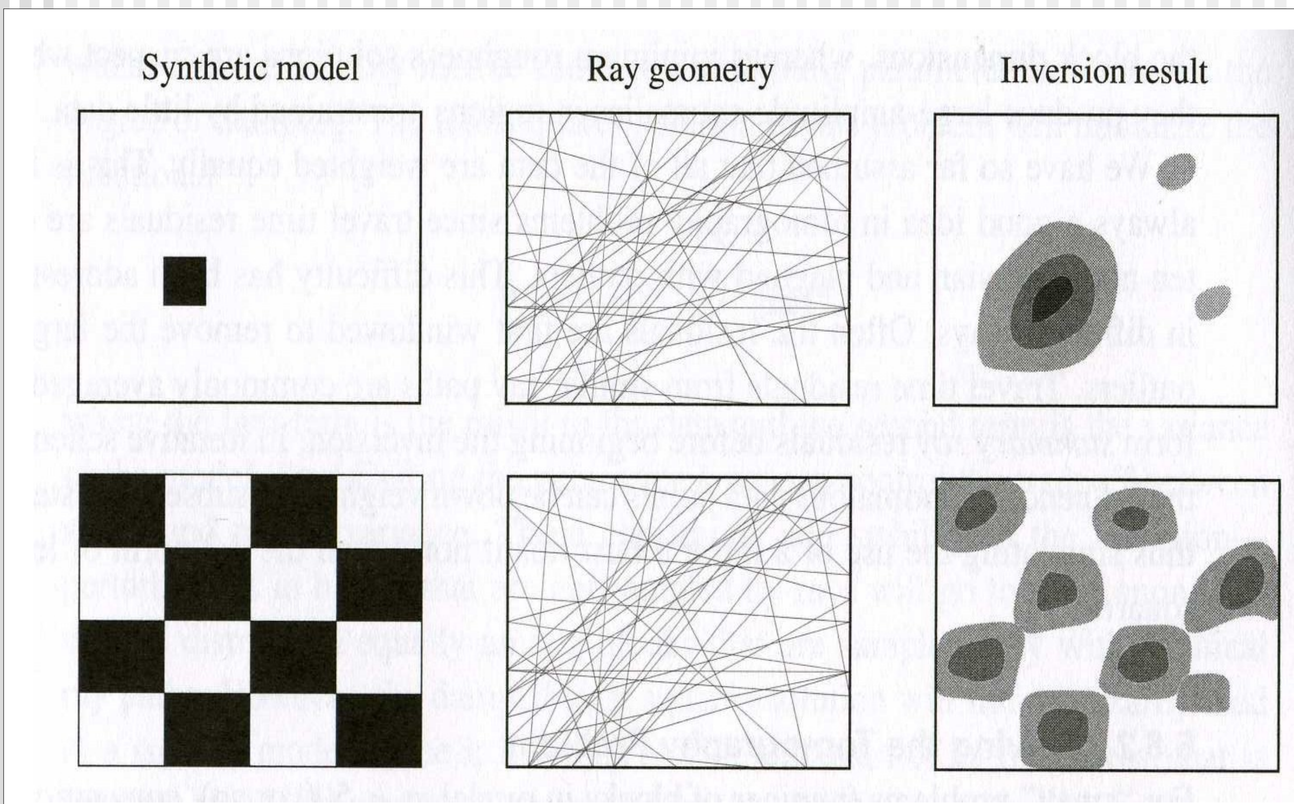
  - Invert the resulting synthetic data:

$$\mathbf{m}' = \mathbf{L}_g^{-1}\mathbf{d}' = \mathbf{L}_g^{-1}\mathbf{L}\mathbf{m}_{checker}$$

  - Compare the result to the input model
    - The degree of reproduction of the anomalies indicate the quality of inversion

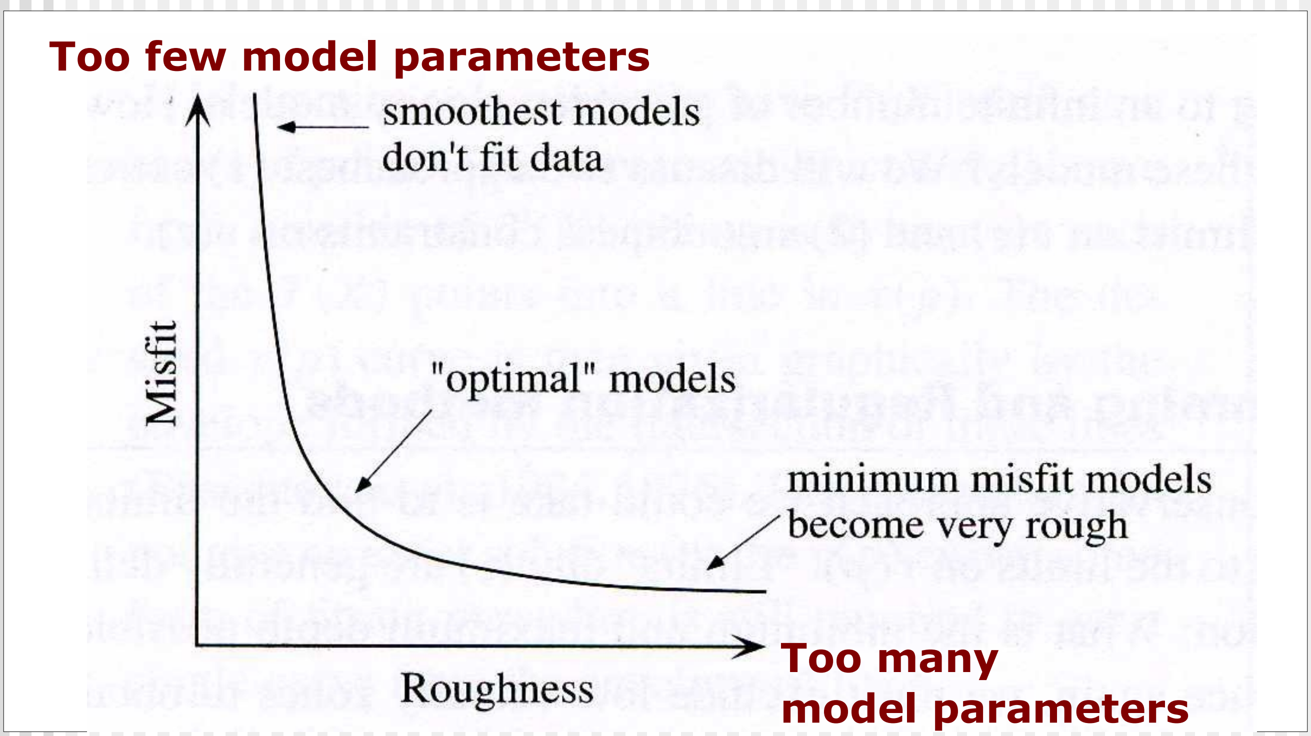
Checkerboard resolution test (cont.)

- Schematic example from travel-time tomography:



Trade-off between data fit and model simplicity

- Too simple models often cannot explain the data
- However, excessively detailed models can "over-fit" the data and result in overly complex model
  - This complexity may be spurious and caused by the noise
- We need to look for "optimally" complex models



Test for statistical significance

- How can we verify that the model fits the data within reasonable error?
  - Complex models (with large numbers of unknowns) would often fit the data well;
  - Because the data contains *noise*, we should not over-fit the data!
- The χ^2 test is commonly used to determine whether the remaining data misfit is likely to be random:

$$\chi^2 = \frac{\sum_{i=1}^{N} \left(t_i - t_i^{observed}\right)^2}{\sigma^2}$$

  - Here, σ is the estimated data uncertainty
  - It needs to be somehow measured from the data (see eq. 5.31 in Shearer)

χ^2 test (cont.)

- The p.d.f of χ^2 is controlled by $N_{df} = N_{data} - N_{model}$ ("number of data degrees of freedom").
- For a given N_{df}, tabulated percentage points of p.d.f.(χ^2) can be used to determine whether the residual data misfit is likely to be random:

N_{df}	At 95%	At 50%	At 5%
5	1.15	4.35	11.07
10	3.94	9.34	18.31
20	10.85	19.34	31.41
50	34.76	49.33	67.5
100	77.03	99.33	124.34

  - The 95-% level is commonly used.

Source Location Problem

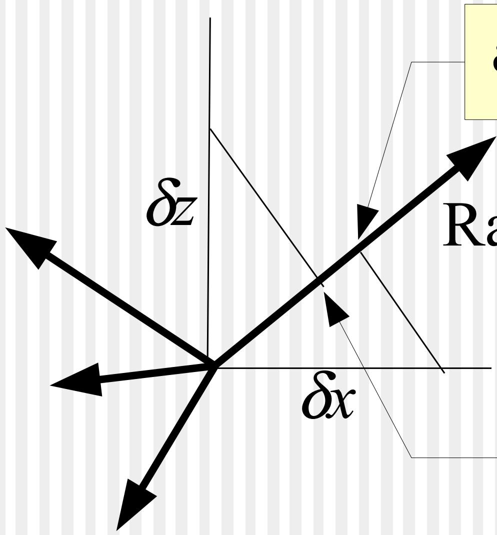
- When using a natural (impulsive) source, its location can also be determined by a similar approach.
  - This method is used for locating earthquakes worldwide
  - For monitoring creep of mine walls (potash exploration)
  - Monitoring reservoirs during injection (Weyburn)
- To solve this problem, we:
  - Start from some reasonable approximation for source coordinates and solve the velocity tomography problem.
  - Add the coordinates and time of the source to model vector:

$$\boldsymbol{m} = \begin{pmatrix} s_1 = 1/V_1 \\ s_2 = 1/V_2 \\ \dots \\ s_N = 1/V_N \\ x_{source} \\ z_{source} \\ t_{source} \end{pmatrix}.$$

(In two dimensions)

Source Location (cont.)

- Include into the matrix **L** time delays associated with shifting the source by δx or δz:



$$\delta t = \frac{-\cos\alpha}{V}\delta x \equiv L_{i,N+1}\delta x$$

$$\delta t = \frac{-\sin\alpha}{V}\delta z \equiv L_{i,N+2}\delta z$$

  - Now, when solved, the Generalized Inverse will yield the corrections to the location (δx, δz).

- This process is often <u>iterated</u>: with the new source location, velocities are recomputed, and sources relocated again, etc.

Measures of data misfit ("data norms")

- The Least-Squares ("L2") norm can be highly sensitive to data outliers:

$$\varepsilon_{L2} = \sum_{i=1}^{N} \left(t_i - t_i^{observed}\right)^2$$

  - However, it is the easiest to use (only for this norm, L^{-1}_g exists).

- Other useful norms:
  - L_n norms: $\varepsilon_{L_n} = \sum_{i=1}^{N} \left|t_i - t_i^{observed}\right|^n$
  - L_∞ norm: $\varepsilon_{L_\infty} = max_i \left|t_i - t_i^{observed}\right|$

- The "L_1" norm is less sensitive to outliers (*i.e.*, anomalous errors), and therefore also often used:

$$\varepsilon_{L_1} = \sum_{i=1}^{N} \left|t_i - t_i^{observed}\right|$$

L_1-norm inversion

- Solutions minimizing L_1 and similar norms are derived from L_2 by *iterative reweighting*:

  1) Use the least-squares inverse to minimize

  $$\varepsilon_{L2} = \sum_{i=1}^{N} \left(t_i - t_i^{observed}\right)^2$$

  2) Apply weights based on current data errors:

  $$W_i = \frac{1}{\sqrt{\left|t_i - t_i^{observed}\right|}}$$

  - The misfit then approximates ε_{L1}:

  $$\varepsilon_{L2} = \sum_{i=1}^{N} W_i^2 \left(t_i - t_i^{observed}\right)^2 \approx \sum_{i=1}^{N} \left|t_i - t_i^{observed}\right|$$

  3) <u>Iterate</u>