

Ray theory

- Ray-tracing
 - ◆ Travel times
 - ◆ Amplitudes
- WKB approximation
- Eikonal equation
- Practical travel-time modelling methods

- Reading:
 - Shearer, 4 and 6

Rays and wavefronts

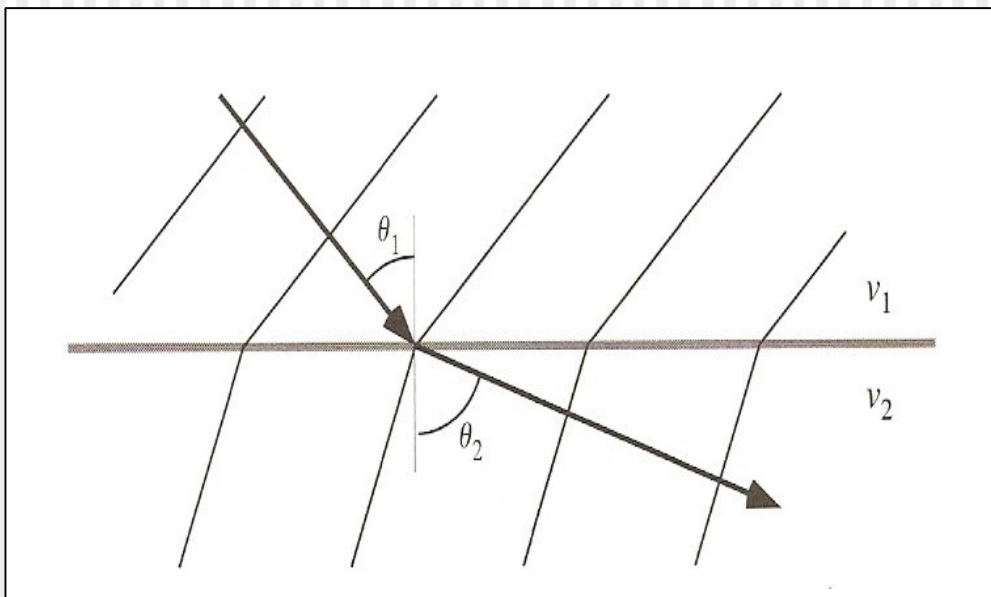
- Rays and wavefronts only represent attributes of “the travel time field”
- If $t(\mathbf{x})$ is the time at which certain wave reaches point \mathbf{x} , then:
 - ♦ Wavefronts are surfaces $t(\mathbf{x}) = \text{const}$
 - ♦ Rays are flow lines of the gradient of $t(\mathbf{x})$
- Ray theory corresponds to the high-frequency limit:
 - ♦ wavefronts and rays are smooth but bend sharply on discontinuities

Snell's law

- The travel time field and consequently **wavefronts** are continuous across a velocity contrast
- Propagation velocities (and **slownesses**) along the boundary are the same:

$$p = s_1 \sin \theta_1 = s_2 \sin \theta_2$$

Slowness = $1/V_1$



WKBJ approximation

- Originates from Liouville and Green (~1837)
- Named after Wentzel, Kramers, Brillouin, and Jeffreys (~1923-26)
- Gives approximate solutions of the differential equation with small parameter $\epsilon \ll 1$ in the leading derivative:

$$\epsilon \frac{d^n y}{d x^n} + a_{n-1} \frac{d^{n-1} y}{d x^{n-1}} + \dots + a_1 \frac{dy}{d x} + a_0 y = 0$$

- The solution is sought in the form:

$$y(x) = \exp \left[\frac{1}{\delta} \sum_{k=0}^{\infty} \delta^k S_k(x) \right]$$


where $\delta \ll 1$ as well.

WKBJ approximation of wave equation

- Consider the wave equation for a harmonic wave (**Helmholtz equation**) in variable wave speed $c(x)$:

$$\frac{1}{c(x)^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{-\omega^2}{c(x)^2} u - \frac{\partial^2 u}{\partial x^2} = 0$$

that is:

Denote this ϵ^2 

$$\left(\frac{c_0}{\omega}\right)^2 \frac{\partial^2 u}{\partial x^2} = -\left(\frac{c_0}{c}\right)^2 u$$

and c_0 is some characteristic value of $c(x)$

- Let's look for a solution like this:

$$u(x) = \exp \left[\frac{1}{\delta} \sum_{k=0}^{\infty} \delta^k S_k(x) \right]$$

WKBJ approximation of wave equation

- The wave equation becomes:

$$\epsilon^2 \left[\frac{1}{\delta^2} \left(\sum_{k=0}^{\infty} \delta^k S'_k(x) \right)^2 + \frac{1}{\delta} \sum_{k=0}^{\infty} \delta^k S''_k(x) \right] = - \left(\frac{c_0}{c} \right)^2$$

- To the leading order, with $\delta \rightarrow 0$:

This is the “eikonal equation”

$$\frac{1}{\delta^2} \left(\frac{c_0}{\omega} \right)^2 S'_0{}^2 = - \left(\frac{c_0}{c} \right)^2$$

- Thus, δ is proportional to ϵ , and we can take:

$$\delta = \frac{c_0}{\omega}, \quad S_0(x) = \pm i \int \frac{c_0 dx'}{c(x')} + const$$

and the solution becomes:

$$u(x) \approx A \exp \left[\pm i \omega \int_{x_0}^x \frac{dx'}{c(x')} \right]$$

Ray travel time from x_0 to x

WKBJ approximation of wave equation (end)

- The above solution:

$$u(x) \approx A \exp \left[\pm i \omega \int_{x_0}^x \frac{dx'}{c(x')} \right]$$

only gives the ray-theoretical **phase** of the wave

- It is equivalent to the solution of the **eikonal equation**
- The amplitude can be estimated by the second-order WKBJ approximation
 - This is called the **transport equation**

Eikonal equation

- From German *Eikonal*, which comes from Greek *εικων*, image (that is, “icon”)
- Provides the link between the **wave** and **geometrical** optics (and acoustics)
- If $t(\mathbf{x})$ is the time at which certain wave reaches point \mathbf{x} , then in the geometrical (high-frequency) limit, it must satisfy:

$$|\vec{\nabla} t(\mathbf{x})| = \frac{1}{V(\mathbf{x})}$$

- This is the **eikonal equation** for seismic travel times
 - ◆ Broadly used in fast 2-D and 3-D wavefront- and ray-tracing algorithms

Travel-time modelling methods

- Ray tracing (shooting)
 - τ -p methods (in layered structures)
- Eikonal-equation based wavefront propagation
- Ray bending
- Shortest-time ray methods

We will not discuss them here
See Shearer

Ray shooting

(A simple approach in 2D)

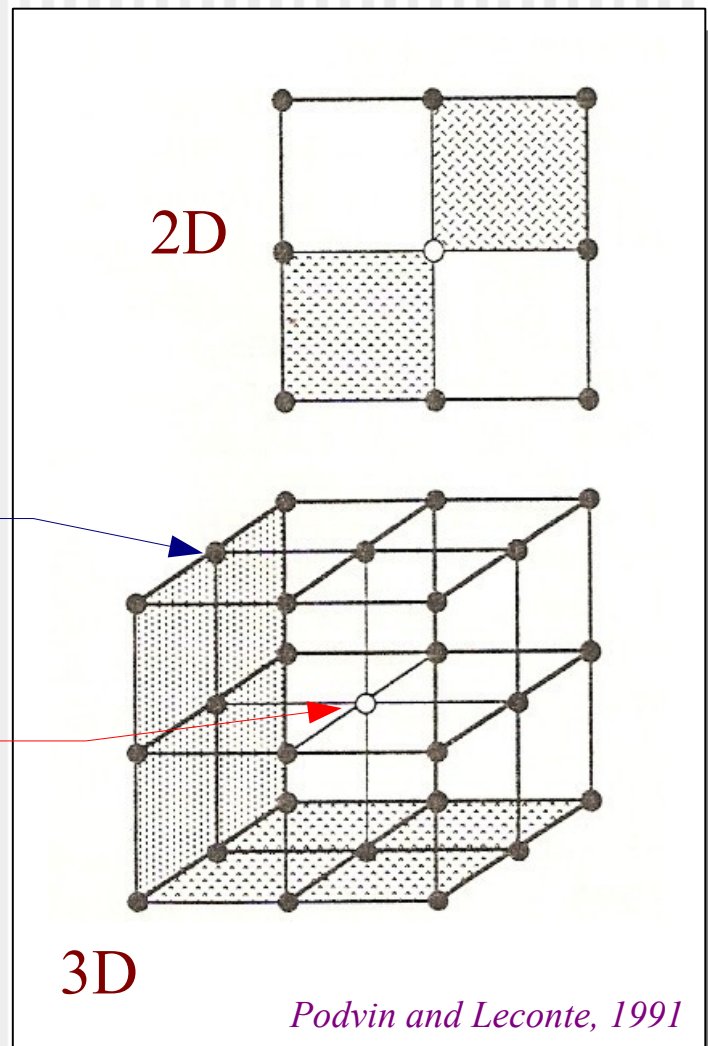
- Velocity model is split into triangular cells
- In each cell, the velocity has a constant gradient
- In a **constant velocity gradient**, the ray is always a **circular arc** (we will see this later)
- Starting from the source the ray is constructed by combining such arcs
- Accurate, but complex method
 - Computationally intensive when many rays are needed
 - May have problems in complex structures

Eikonal first-arrival time calculation

- Initialize the near-source times
- At each iteration try timing each node by using the adjacent nodes
 - ◆ Use waves from point, linear, and planar Huygens sources
 - ◆ Select the earliest time

Grid nodes
already timed

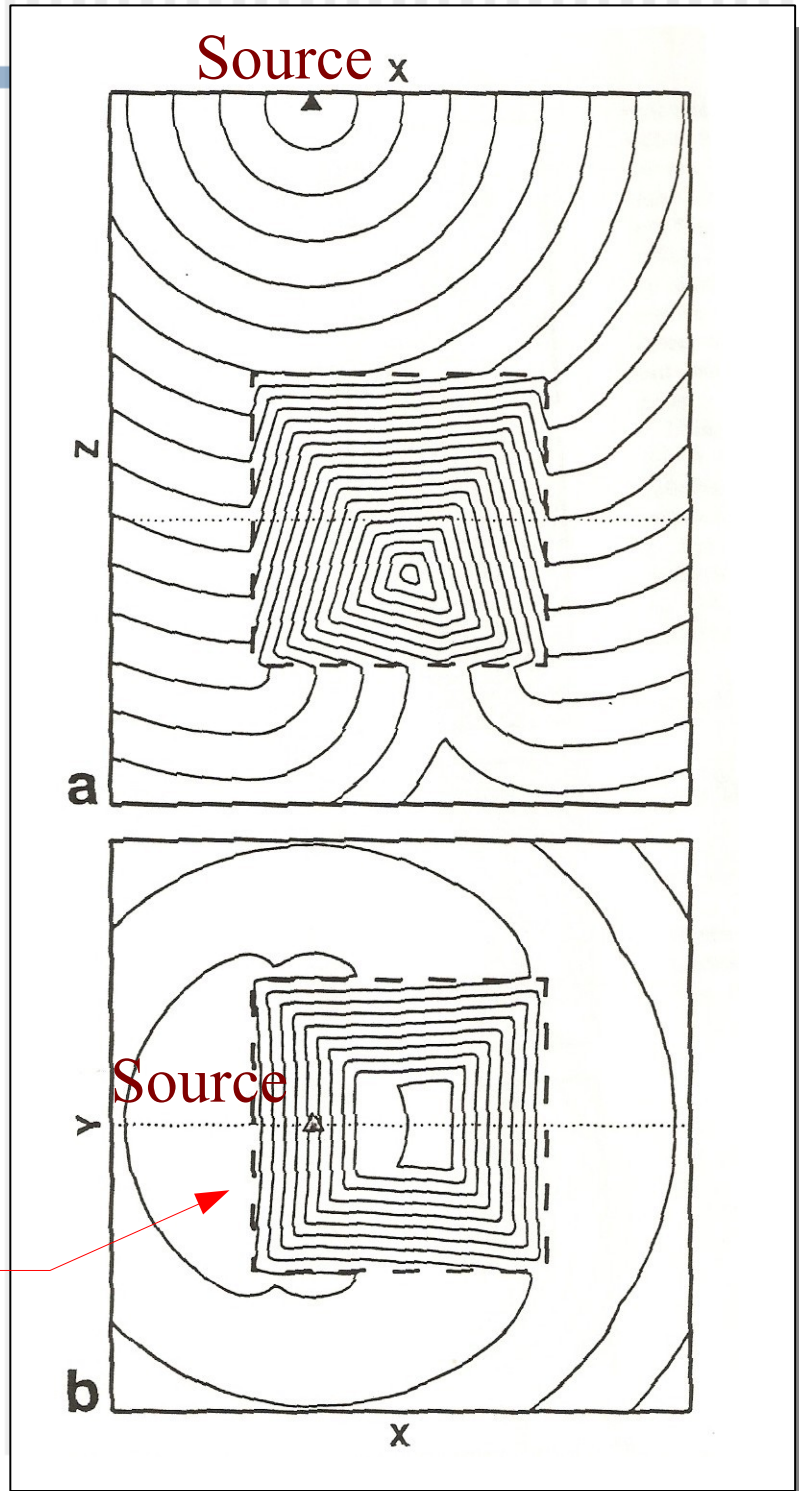
Node being
timed now



Example

First-arrival travel times in 3D

- Eikonal travel-time calculation
- Rays to every point can be obtained by tracing $t(\mathbf{x})$ gradients back to the source



Low-velocity
cube

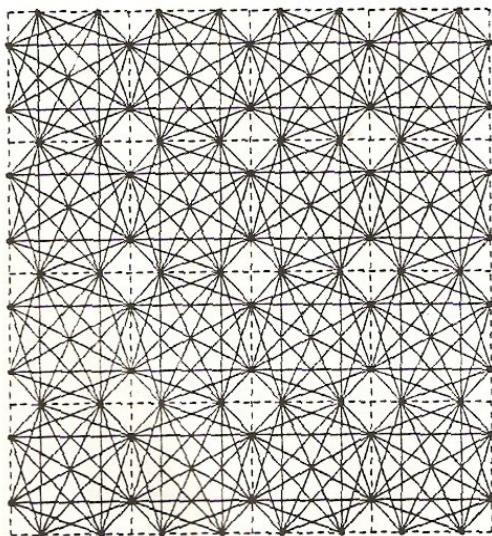
Ray bending

- Directly employing Fermat principle
- Connecting the source and receiver by a shortest-time ray
 - ◆ Accurate and stable
 - ◆ Works only for selected source-receiver pairs
 - ◆ Computationally intensive

Shortest-path ray tracing

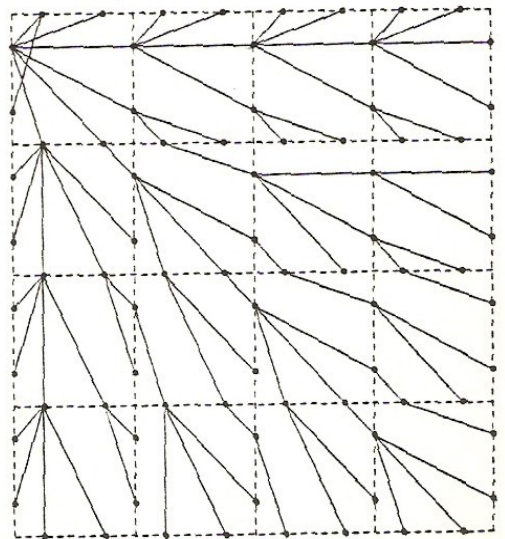
- A suitable grid of possible ray paths is created
 - ◆ Including reasonable dips and structures
- Starting from the source, shortest-time paths are identified
- Fast and stable method
 - ◆ Good for quick general assessment of time field
 - ◆ Can be followed by ray bending for accuracy

All paths considered



Shortest paths

Source

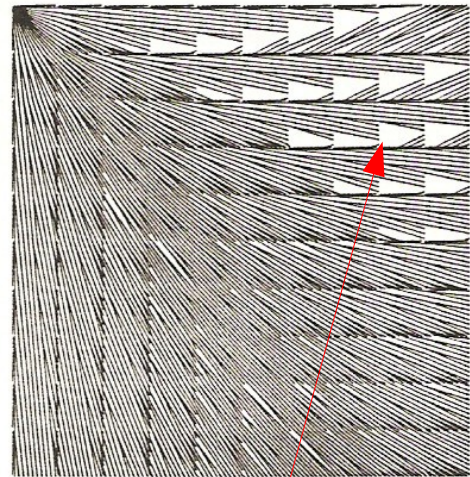
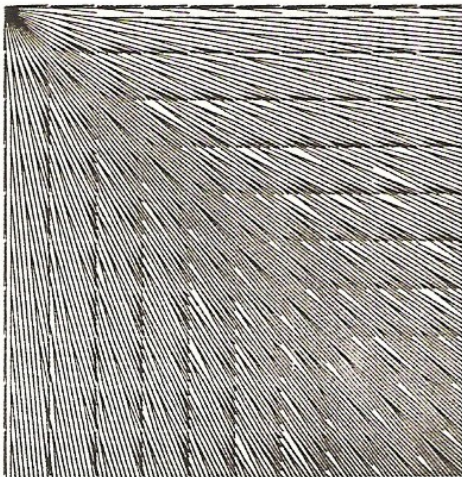


Example: Shortest-time paths

Uniform velocity

Velocity increasing with depth

Source



Moser, 1991

Note the discontinuities in the travel-time field

Ray-based amplitudes

- Amplitudes can be estimated from ray flux tubes
 - ◆ For example, the *Geometrical spreading* is often modelled in this way
- If energy flux remains constant:

$$E_{flux} = c \left(\frac{\rho}{2} A^2 \omega^2 \right)$$

Kinetic energy density

amplitude varies as:

$$\frac{A_2}{A_1} = \sqrt{\frac{dS_1}{dS_2}} \sqrt{\frac{\rho_1 c_1}{\rho_2 c_2}}$$

“Geometrical spreading”

Ratio of impedances

