Elasticity and seismic waves

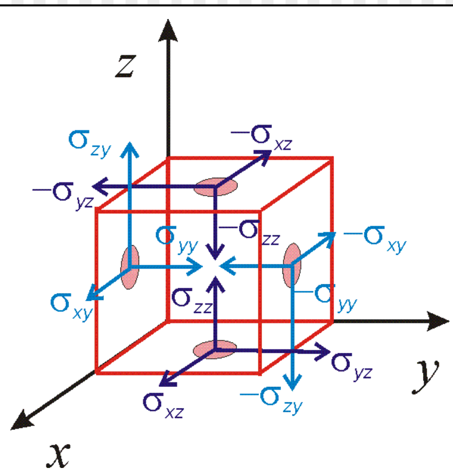
- Recap of theory
- Equations of motion
- Wave equations
- P- and S-waves
- Impedance
- Wave potentials
- Energy of a seismic wave

- Reading:
  - Telford *et a*l., Section 4.2
  - Shearer, 3
  - Sheriff and Geldart, Sections 2.1-4

Forces acting on a small cube

- Consider a small volume ($dx \times dy \times dz = dV$) within the elastic body.
- *Force* applied to the parallelepiped from the outside is:

$$F_i = -\partial_j \sigma_{ij} dV$$



$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Equations of Motion

(Motion of the elastic body with time)

- Uncompensated net force will result in *acceleration* (second Newton's law):

Newton's law: $$\rho\,\delta V \frac{\partial^2 U_i}{\partial t^2} = F_i$$

$$\rho \frac{\partial^2 U_i}{\partial t^2} = \left(\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z} \right)$$

$$\rho \frac{\partial^2 U_x}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda' \Delta + 2\mu \frac{\partial U_x}{\partial x} \right) + \mu \frac{\partial}{\partial y}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) + \mu \frac{\partial}{\partial z}\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right)$$

$$= \lambda' \frac{\partial \Delta}{\partial x} + \mu \frac{\partial}{\partial x}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \right) + \mu \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right)$$

$$= (\lambda' + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 U_x$$

- Therefore, the *equations of motion* for the components of **U**:

$$\rho \frac{\partial^2 U_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x_i} + \mu \nabla^2 U_i$$

Wave potentials

Compressional and Shear waves

- These equations describe two types of waves.
- The general solution has the form ("Lamé theorem"):

$$\vec{U} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{\psi}. \quad \left(\text{or} \quad U_i = \partial_i \phi + \epsilon_{ijk}\partial_j \psi_k \right)$$

$$\vec{\nabla}\cdot\vec{\psi} = 0.$$

Because there are 4 components in ψ and ϕ only 3 in **U**, we need to constrain ψ.

- Exercise: substitute the above into the equation of motion:

$$\rho \frac{\partial^2 U_i}{\partial t^2} = (\lambda+\mu)\frac{\partial \Delta}{\partial x_i} + \mu \nabla^2 U_i$$

and show:

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu)\nabla^2 \phi,$$

P-wave (scalar) potential.

$$\rho \frac{\partial^2 \psi_i}{\partial t^2} = \mu \nabla^2 \psi_i,$$

S-wave (vector) potential.

Wave velocities

Compressional and Shear waves

- These are wave equations; compare to the general form of equation describing wave processes:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]f(x,y,z,t)=0$$

- *Compressional* (*P*) wave velocity:

$$v_P=\sqrt{\frac{\lambda+2\mu}{\rho}}=\sqrt{\frac{K+\frac{4}{3}\mu}{\rho}}$$

- *Shear* (*S*) wave velocity:
  - $V_S < V_P$,
  - for $\sigma=0.25$: $V_P/V_P=\sqrt{3}$

$$v_S=\sqrt{\frac{\mu}{\rho}}$$

- Note that the V_P/V_S depends on the Poisson's ratio alone:

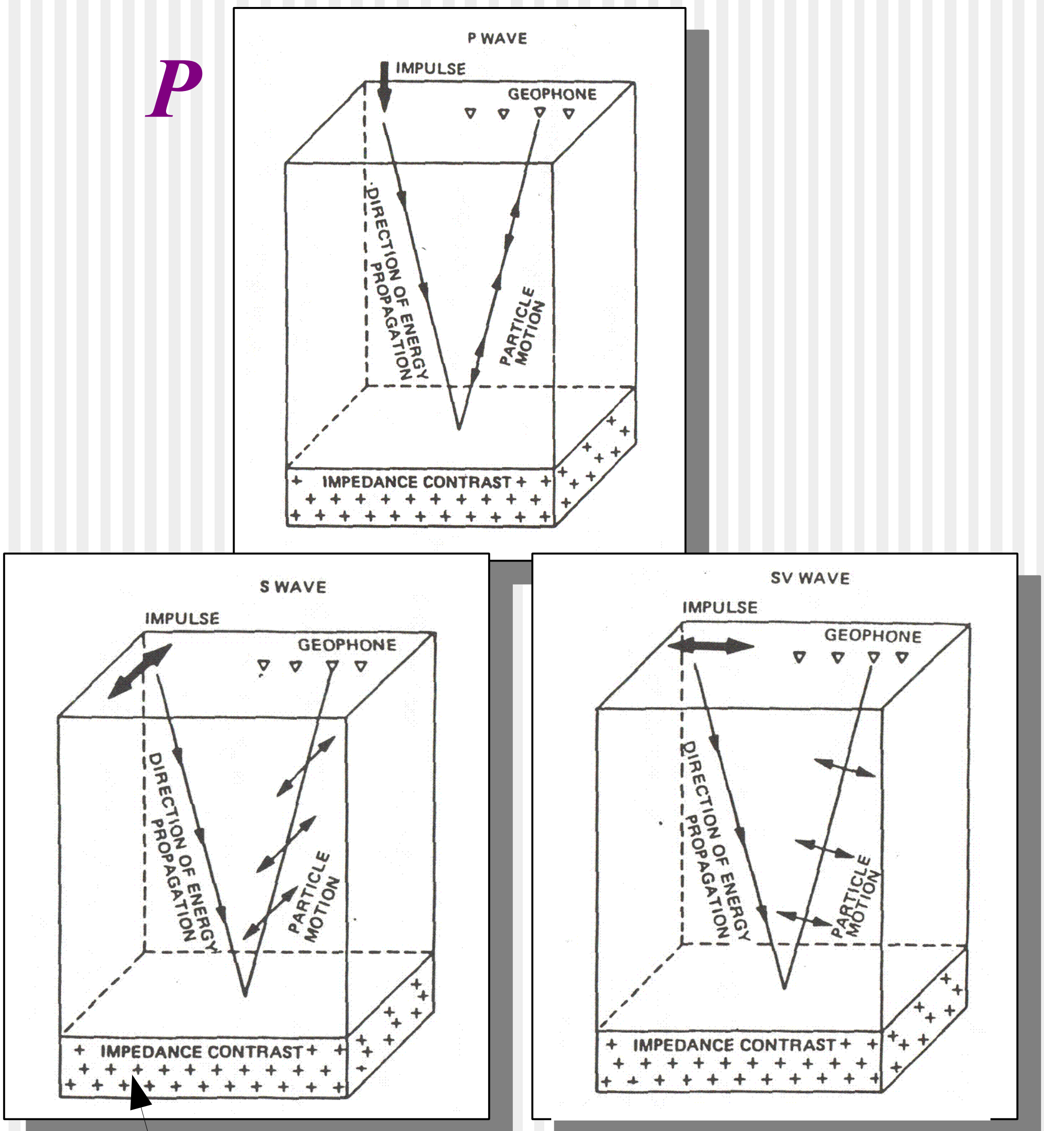
$$\frac{V_S}{V_P}=\sqrt{\frac{\mu}{\lambda+2\mu}}=\sqrt{\frac{1/2-\sigma}{1-\sigma}}.$$

Wave Polarization

- Elastic solid supports two types of *body waves*:

P

S



Note that this is still an **ISOTROPIC** reflector.
In general, reflection will intermix
the S-wave polarization modes,
and P-wave will convert into SV upon reflection.

Notes on the use of potentials

- Wave potentials are very useful for solving elastic wave problems
- Just take ϕ or ψ satisfying the wave equation, e.g.:

$$\phi(\vec{r},t) = Ae^{i\omega\left(t - \frac{\vec{r}\vec{n}}{V_P}\right)}$$ (plane wave)

...and use the equations for potentials to derive the displacements:

$$\vec{U} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{\psi}$$

...and stress from Hooke's law:

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\varepsilon_{ij}$$

- *Displacement amplitude =*

  ω×(potential amplitude)/V

Example:
Compressional (*P*) wave

- Scalar potential for *plane harmonic* wave:

$$\phi(\vec{r},t) = Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}.$$

- Displacement:

$$u_i(\vec{r},t) = \partial_i\phi(\vec{r},t) = \frac{-i\omega n_i}{V_P}Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}.$$

note that the displacement is always along **n**.

- Strain:

$$\varepsilon_{ij}(\vec{r},t) = \partial_i u_j(\vec{r},t) = \frac{-\omega^2 n_i n_j}{V_P^2}Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)}.$$

- Dilatational strain:

$$\Delta = \varepsilon_{ii}(\vec{r},t) = \frac{-\omega^2}{V_P^2}Ae^{i\omega\left(t-\frac{\vec{r}\vec{n}}{V_P}\right)} = \frac{-\omega^2}{V_P^2}\phi(\vec{r},t).$$

- Stress:

$$\sigma_{ij}(\vec{r},t) = \frac{-\omega^2}{V_P^2}\left(\lambda\delta_{ij} + 2\mu n_i n_j\right)\phi(\vec{r},t).$$

- Question: what wavefield would we have if used cos(...) or sin(...) function instead of complex exp(...) in the expression for potential above?

Impedance

- In general, *the Impedance, Z,* is a measure of the amount of resistance to particle motion.
- In elasticity, impedance is the *ratio of stress to particle velocity*.
  - Thus, for a given applied stress, particle velocity is inversely proportional to impedance.
  - For *P* wave, in the direction of its propagation:

$$Z(\vec{r},t)=\frac{\sigma_{nn}(\vec{r},t)}{\dot{u}_n(\vec{r},t)}=\frac{\lambda+2\mu}{V_P}=\rho V_P.$$

    - impedance does not depend on frequency but *depends on the wave type and propagation direction.*

Elastic Energy Density

- Recall that for a deformed elastic medium, the *energy density* is:

$$E = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$$

Elastic Energy Density
in a plane wave

- For a plane wave:

$$u_i = u_i(t - \vec{p}\cdot\vec{x})$$

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) = -\frac{1}{2}(\dot{u}_i p_j + \dot{u}_j p_i).$$

...and therefore:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left[(\lambda+\mu)(\vec{p}\cdot\vec{u})^2 + \mu(\vec{u}\cdot\vec{u})(\vec{p}\cdot\vec{p})\right]$$

- For *P*- and *S*-waves, this gives:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}(\lambda+2\mu)p^2\vec{u}^2 = \frac{1}{2}\rho\dot{\vec{u}}^2 \qquad P\text{-wave}$$

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}(\mu)p^2\vec{u}^2 = \frac{1}{2}\rho\dot{\vec{u}}^2 \qquad S\text{-wave}$$

- Thus, in a wave, *strain energy equals the kinetic energy*

Energy is *NOT* conserved locally!

- *Energy travels at the same speed* as the wave pulse