

Kirchhoff Theory

- Relation to Huygens' principle
 - Wavefield extrapolation
 - Application: modelling of reflected wavefields
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- Reading:
 - › Shearer, 7.6

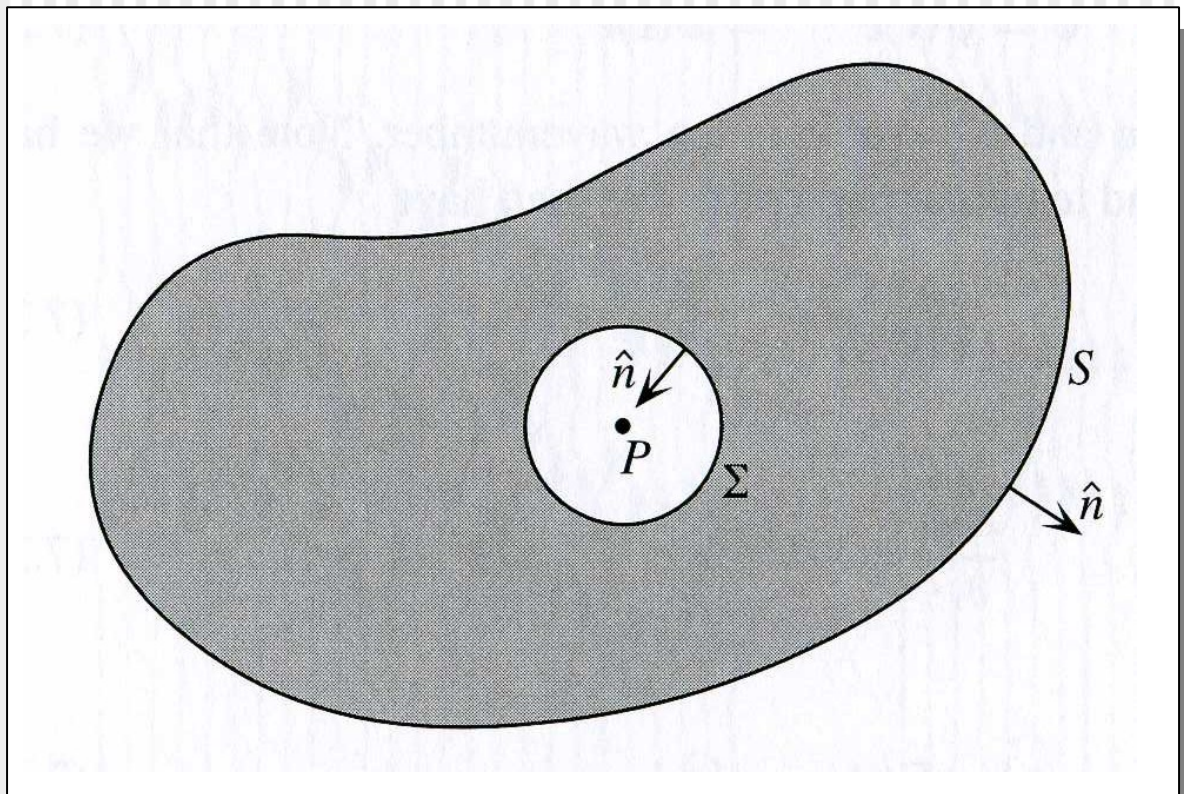
Kirchhoff theory

A rigorous form of Huygens' principle

- For a field satisfying the wave equation:

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

- We want to express the value of ϕ at point P through an integral over a surface (compare to Huygens' principle!):



Kirchhoff theory (cont.)

- Now consider a harmonic form of ϕ (that is, apply the Fourier transform to ϕ):

$$\phi(t, \vec{r}) = \phi(\vec{r}) e^{-i\omega t} = \phi(\vec{r}) e^{-i k c t}$$

- Here, $k = \omega/c$ is the wavenumber.
- The wave equation then becomes the time-independent *Helmholtz's equation*:

$$\nabla^2 \phi = -k^2 \phi$$

- Now recall the Green's theorem from field calculus:

$$\int_V (\phi_2 \nabla^2 \phi_1 - \phi_1 \nabla^2 \phi_2) dV = \int_{S+\Sigma} \left(\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n} \right) dS$$

- this holds for any two continuous functions with continuous derivatives

Kirchhoff theory (cont.)

- If both ϕ_1 and ϕ_2 satisfy the Helmholtz's equation, then the l.h.s. of the Green's equation vanishes, and therefore:

$$\int_{S+\Sigma} \left(\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n} \right) dS = 0$$

- Now consider: $\phi_1 = \phi(\mathbf{r})$ to be our field and:

$$\phi_2 = \frac{e^{ikr}}{r} \quad \leftarrow \text{Point-source field}$$

(this is a point-source field from a source located at \mathbf{P})

- Then:

$$\int_{\Sigma+S} \left[\frac{e^{ikr}}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left(\frac{e^{ikr}}{r} \right) \right] dS = 0$$

Kirchhoff theory (cont.)

- For radius of $\Sigma \rightarrow 0$ (show this!):

$$\int_{\Sigma} \left[\frac{e^{ikr}}{r} \frac{\partial \phi}{\partial r} - \phi \frac{\partial}{\partial r} \left(\frac{e^{ikr}}{r} \right) \right] dS = -4\pi \phi(\vec{P})$$

- and therefore $\phi(\mathbf{P})$ can be expressed as an integral of the wavefield and its gradient over the enclosing surface S :

$$\phi(\vec{P}) = \frac{1}{4\pi} \int_S \left[\frac{e^{ikr}}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left(\frac{e^{ikr}}{r} \right) \right] dS$$

- Returning to the time-dependent (still harmonic) function:

$$\phi(t, \vec{P}) = \frac{1}{4\pi} \int_S \left[\frac{e^{ik(r-ct)}}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left(\frac{e^{ik(r-ct)}}{r} \right) \right] dS$$

Note that the surface field is taken at *retarded times*:

$$t_{Ret} = t - r/c$$

Kirchhoff theory (cont.)

- Finally, after the Fourier transform in ω (i.e., time):

$$\phi(t, \vec{P}) = \frac{1}{4\pi} \int_S \left\{ \frac{1}{r} \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \left(\frac{1}{r} \right) [\phi] + \frac{1}{cr} \frac{\partial r}{\partial n} \left[\frac{\partial \phi}{\partial t} \right] \right\} dS$$

- here, the [...] quantities are taken at retarded times $t_{Ret} = t - r/c$

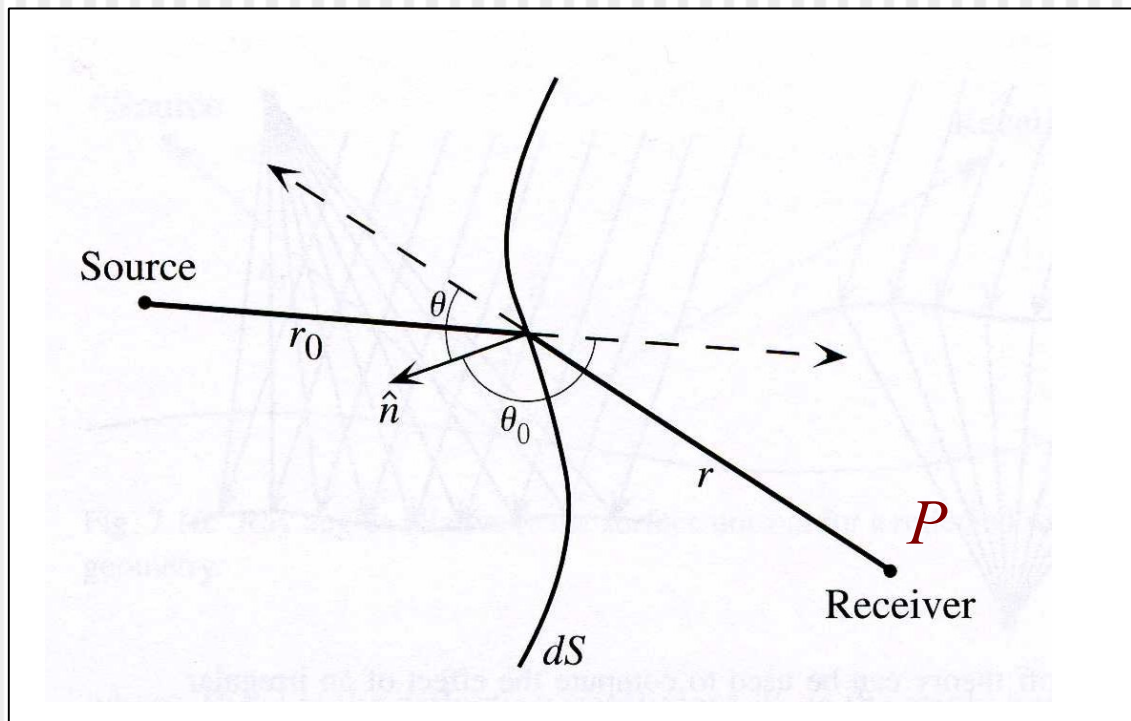
Kirchhoff theory (cont.)

- For a practical application, assume that the field at the surface comes from some distant source:

$$\phi = \frac{f\left(t - \frac{r_0}{c}\right)}{r_0}$$

- Then (see Shearer 7.6):

$$\phi(t, \vec{P}) = \frac{1}{4\pi c} \int_S \delta\left(t - \frac{r+r_0}{c}\right) \frac{1}{r r_0} (\cos\theta - \cos\theta_0) dS * f'(t)$$



Modelling reflected field using Kirchhoff integral

- High-frequency (far-field, $r \gg \lambda$) reflected field from an arbitrary irregular reflector:

$$\phi(t, \vec{P}) = \frac{1}{4\pi c} \int_S \delta\left(t - \frac{r+r_0}{c}\right) \frac{1}{rr_0} (\cos\theta + \cos\theta_0) dS * f'(t)$$

- ♦ this describes both reflections and diffractions, phase shifts, angle and frequency dependence
- Inversion of this formula is the basis for "Kirchhoff migration"

