Ray theory

- Ray-tracing
  - Travel times
  - Amplitudes
- WKBJ approximation
- Eikonal equation
- Practical travel-time modelling methods

- Reading:
  - Shearer, 4 and 6

Rays and wavefronts

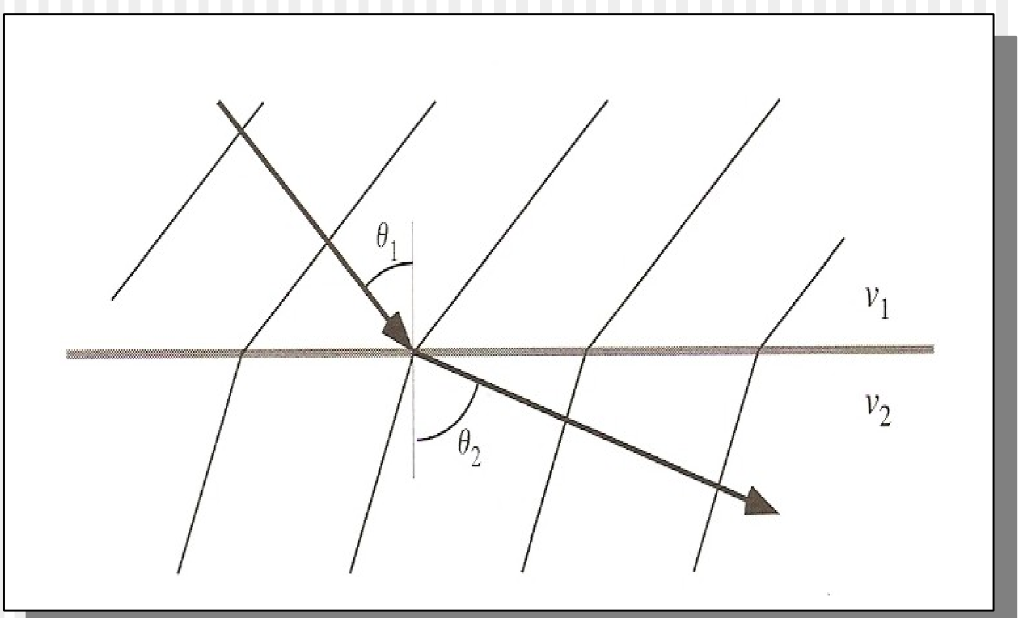
- Rays and wavefronts only represent attributes of "the travel time field"
- If $t(\mathbf{x})$ is the time at which certain wave reaches point $\mathbf{x}$, then:
  - Wavefronts are surfaces $t(\mathbf{x})$ = const
  - Rays are flow lines of the gradient of $t(\boldsymbol{x})$
- Ray theory corresponds to the high-frequency limit:
  - wavefronts and rays are smooth but bend sharply on discontinuities

Snell's law

- The travel time field and consequently wavefronts are continuous across a velocity contrast
- Propagation velocities (and slownesses) along the boundary are the same:

$$p = s_1 \sin\theta_1 = s_2 \sin\theta_2$$

Slowness = $1/V_1$



WKBJ approximation

- Originates from Liouville and Green (~1837)
- Named after Wentzel, Kramers, Brillouin, and Jeffreys (~1923-26)
- Gives approximate solutions of the differential equation with small parameter $\epsilon \ll 1$ in the leading derivative:

$$\epsilon \frac{d^n y}{d x^n} + a_{n-1} \frac{d^{n-1} y}{d x^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0$$

- The solution is sought in the form:

$$y(x) = \exp\left[\frac{1}{\delta} \sum_{k=0}^{\infty} \delta^k S_k(x)\right]$$

where $\delta \ll 1$ as well.

WKBJ approximation of wave equation

- Consider the wave equation for a harmonic wave (Helmholtz equation) in variable wave speed *c*(*x*):

$$\frac{1}{c(x)^2}\frac{\partial^2 u}{\partial^2 t}-\frac{\partial^2 u}{\partial^2 x}=\frac{-\omega^2}{c(x)^2}u-\frac{\partial^2 u}{\partial^2 x}=0$$

that is:

$$\left(\frac{c_0}{\omega}\right)^2\frac{\partial^2 u}{\partial^2 x}=-\left(\frac{c_0}{c}\right)^2 u$$

Denote this ϵ^2

and c_0 is some characteristic value of *c*(*x*)

- Let's look for a solution like this:

$$u(x)=\exp\left[\frac{1}{\delta}\sum_{k=0}^{\infty}\delta^k S_k(x)\right]$$

WKBJ approximation of wave equation

- The wave equation becomes:

$$\epsilon^2\left[\frac{1}{\delta^2}\left(\sum_{k=0}^{\infty}\delta^k S'_k(x)\right)^2+\frac{1}{\delta}\sum_{k=0}^{\infty}\delta^k S''_k(x)\right]=-\left(\frac{c_0}{c}\right)^2$$

- To the leading order, with $\delta\to 0$:

This is the "eikonal equation"

$$\frac{1}{\delta^2}\left(\frac{c_0}{\omega}\right)^2 S'^2_0=-\left(\frac{c_0}{c}\right)^2$$

- Thus, δ is proportional to $\in$, and we can take:

$$\delta=\frac{c_0}{\omega},\ S_0(x)=\pm i\int\frac{c_0\,dx'}{c(x')}+const$$

and the solution becomes:

$$u(x)\approx A\exp\left[\pm i\omega\int_{x_0}^{x}\frac{dx'}{c(x')}\right]$$

Ray travel time from x_0 to x

WKBJ approximation of wave equation (end)

- The above solution:

$$u(x) \approx A \exp\left[\pm i\omega \int_{x_0}^{x} \frac{dx'}{c(x')}\right]$$

  only gives the ray-theoretical phase of the wave
  - It is equivalent to the solution of the eikonal equation
- The amplitude can be estimated by the <u>second-order</u> WKBJ approximation
  - This is called the transport equation

Eikonal equation

- From German *Eikonal*, which comes from Greek $\varepsilon\iota\kappa\omega\nu$, image (that is, "icon")
- Provides the link between the wave and geometrical optics (and acoustics)
- If $t(\mathbf{x})$ is the time at which certain wave reaches point **x**, then in the geometrical (high-frequency) limit, it must satisfy:

$$|\vec{\nabla} t(\mathbf{x})| = \frac{1}{V(\mathbf{x})}$$

- This is the eikonal equation for seismic travel times
  - Broadly used in fast 2-D and 3-D wavefront- and ray-tracing algorithms

Travel-time modelling methods

- Ray tracing (shooting)
  - τ-p methods (in layered structures)
- Eikonal-equation based wavefront propagation
- Ray bending
- Shortest-time ray methods

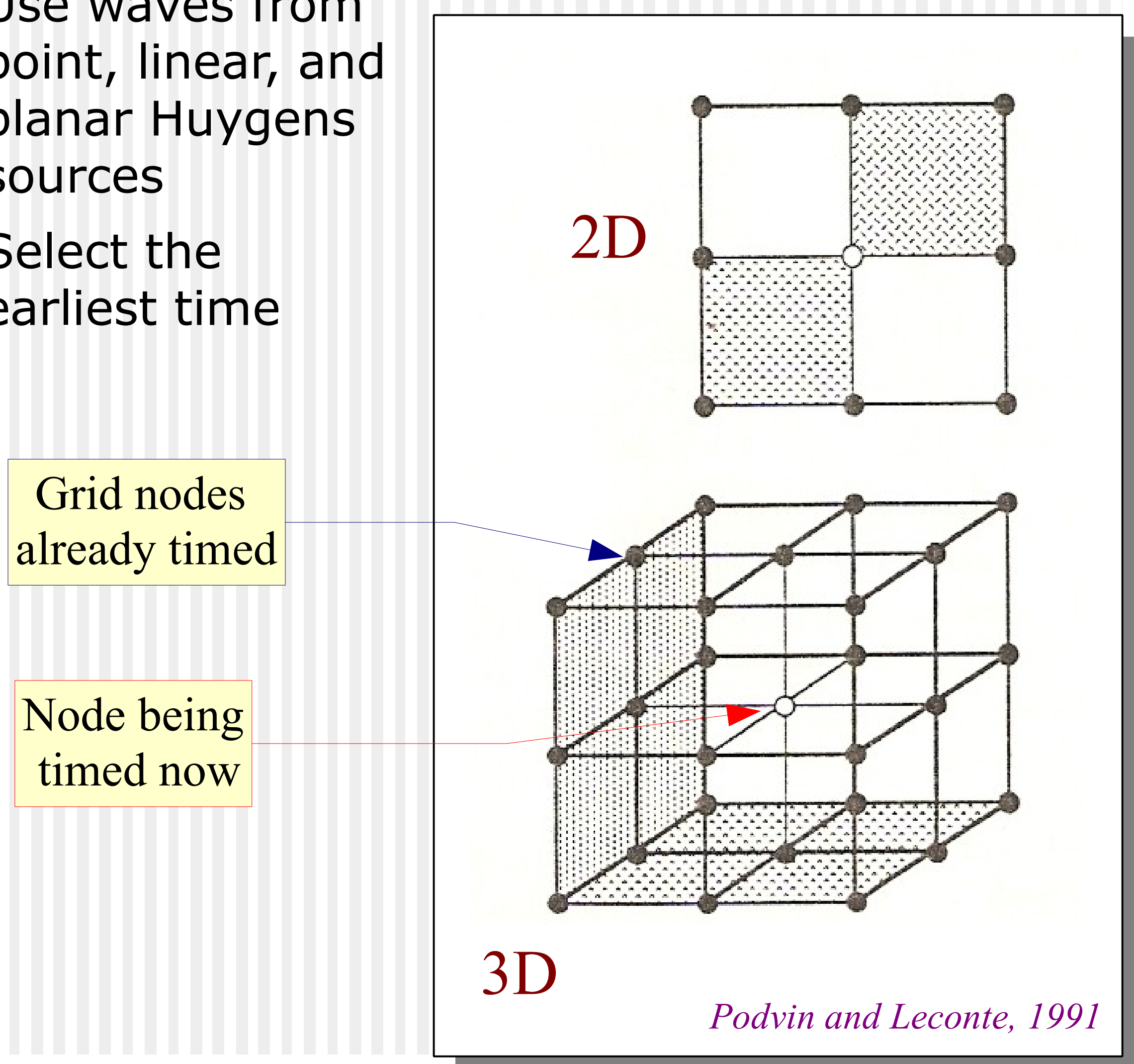
We will not discuss them here
See Shearer

Ray shooting
(A simple approach in 2D)

- Velocity model is split into triangular cells
- In each cell, the velocity has a constant gradient
- In a constant velocity gradient, the ray is always a circular arc (we will see this later)
- Starting from the source the ray is constructed by combining such arcs
- Accurate, but complex method
  - Computationally intensive when many rays are needed
  - May have problems in complex structures

Eikonal first-arrival time calculation

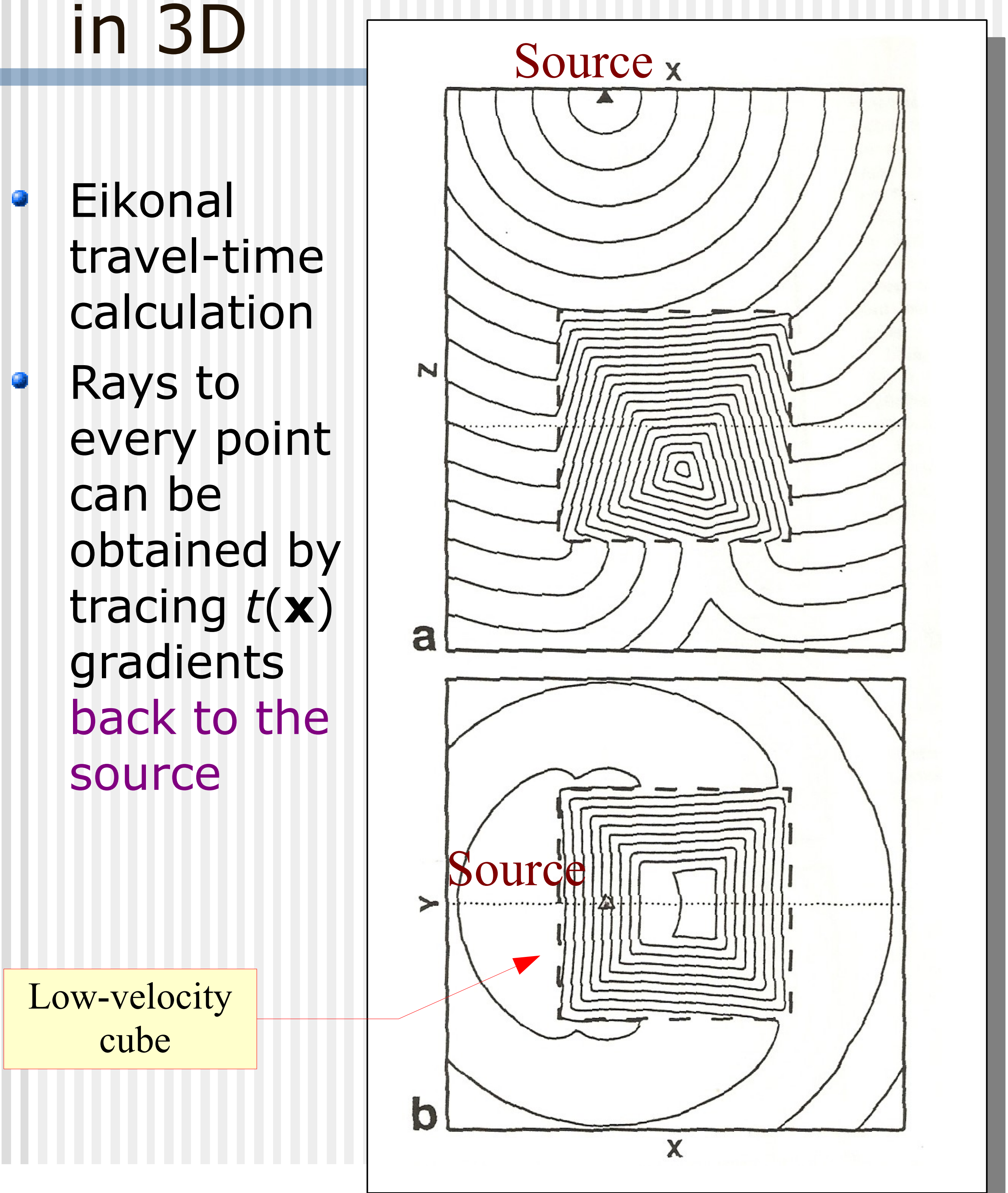
- Initialize the near-source times
- At each iteration try timing each node by using the adjacent nodes
  - Use waves from point, linear, and planar Huygens sources
  - Select the earliest time



Example

First-arrival travel times in 3D

- Eikonal travel-time calculation
- Rays to every point can be obtained by tracing *t*(**x**) gradients back to the source

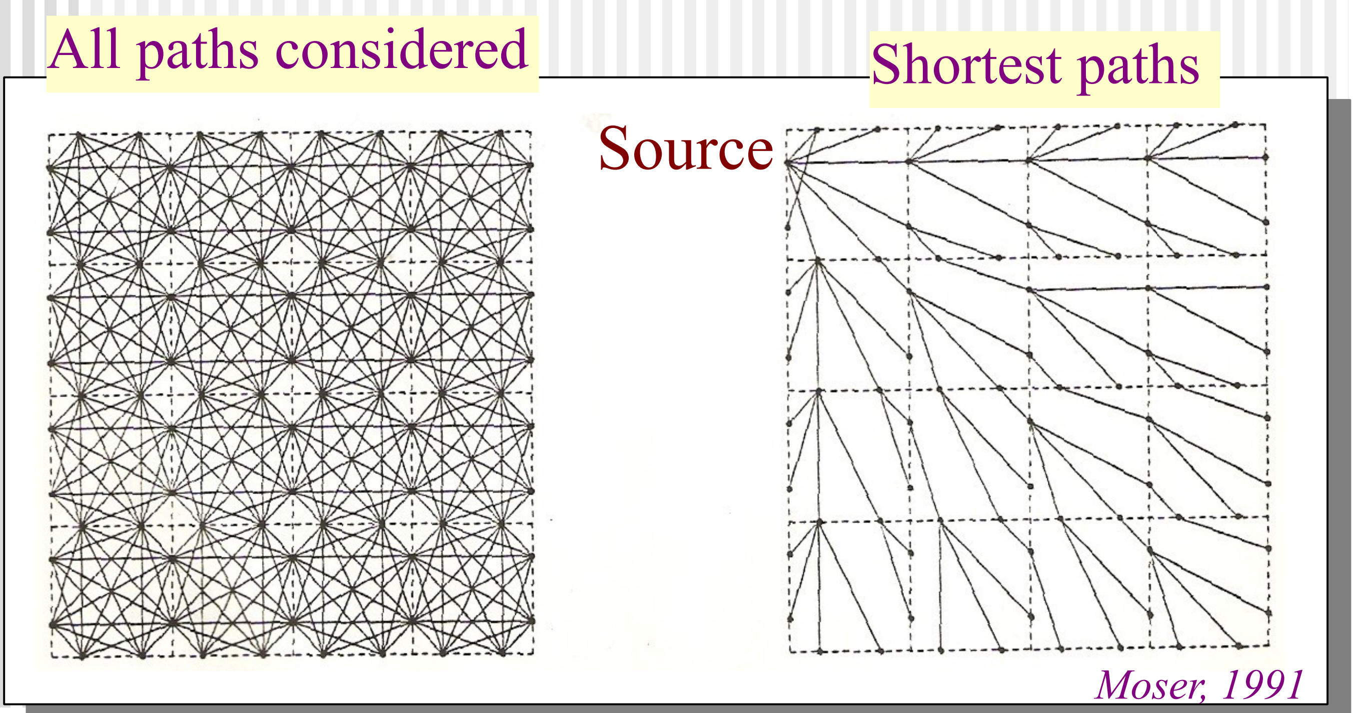


Ray bending

- Directly employing Fermat principle
- Connecting the source and receiver by a shortest-time ray
  - Accurate and stable
  - Works only for selected source-receiver pairs
  - Computationally intensive

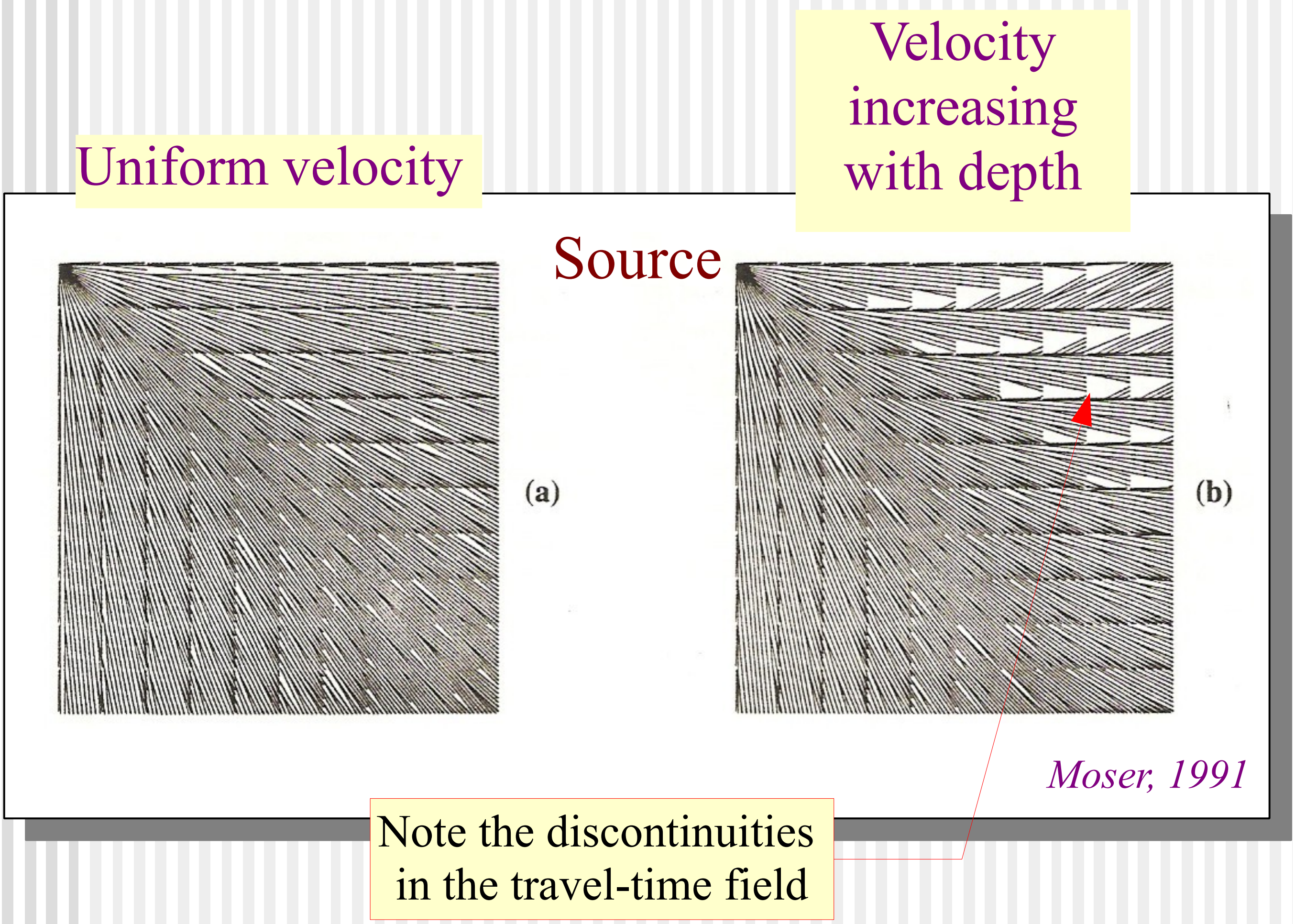
Shortest-path ray tracing

- A suitable grid of possible ray paths is created
  - Including reasonable dips and structures
- Starting from the source, shortest-time paths are identified
- Fast and stable method
  - Good for quick general assessment of time field
  - Can be followed by ray bending for accuracy



Moser, 1991

Example:
Shortest-time paths

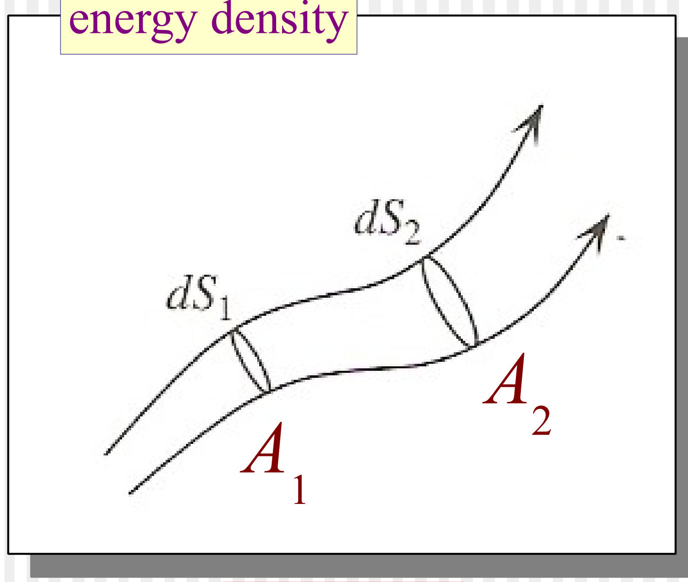


Ray-based amplitudes

- Amplitudes can be estimated from ray flux tubes
  - For example, the *Geometrical spreading* is often modelled in this way
- If energy flux remains constant:

$$E_{flux} = c\left(\frac{\rho}{2}A^2\omega^2\right)$$

Kinetic energy density



amplitude varies as:

$$\frac{A_2}{A_1} = \sqrt{\frac{dS_1}{dS_2}}\sqrt{\frac{\rho_1 c_1}{\rho_2 c_2}}$$

"Geometrical spreading"

Ratio of impedances