Time and Spatial Series and Transforms

- Z- and Fourier transforms
- Gibbs' phenomenon
- Transforms and linear algebra
- Wavelet transforms

- Reading:
  - Sheriff and Geldart, Chapter 15

Z-Transform

- Consider a discretized record of *N* readings: $U=\{u_0, u_1, u_2, \ldots, u_{N-1}\}$. How can we represent this series differently?
- The *Z* transform simply associates with this time series a *polynomial function*:

$$\{u_i\} \to U(z) = u_0 + u_1 z + u_2 z^2 + u_3 z^3 + \ldots$$

  - For example, a 3-sample record of {1,2,5} is represented by a quadratic polynomial:

    $1 + 2z + 5z^2$.

- In *Z*-domain, the all-important operation of *convolution* of time series becomes simple multiplication of their Z-transforms:

$$u_1(t) * u_2(t) \to U_1(z) U_2(z)$$

Fourier Transform

- To describe a polynomial function of order *N*-1, it is sufficient to specify its values at *N* points in the plane of complex variable "*z*"
- The *Discrete Fourier transform* is obtained by taking the *Z*-transform at *N* points uniformly distributed around a unit circle on the complex plane of *z*:

$$U\left(k\right)=\sum_{m=1}^{N-1} e^{i\frac{2\pi k}{N}m} u\left(t_m\right) \qquad k=0,1,2,...,N-1$$

- Each term (k>0) in the sum above is a *periodic function* (a combination of *sin* and *cos*), with a period of N/k sampling intervals:

$$e^{i\alpha}=\cos\left(\alpha\right)+i\sin\left(\alpha\right)$$

- Thus, the Fourier transform expresses the signal as a sum of its frequency components,
  - Fourier transform also has the property of the *Z*-transform regarding convolution

New!

Matrix form of Fourier Transform

- Note that the Fourier transform can be written as matrix multiplication:

$$\begin{pmatrix} U(\omega_1) \\ U(\omega_2) \\ U(\omega_3) \\ \dots \end{pmatrix} = \mathbf{F} \begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \\ \dots \end{pmatrix}$$

$$\mathbf{F} = \begin{bmatrix} e^{i\omega_1 t_1} & e^{i\omega_1 t_2} & e^{i\omega_1 t_3} & \dots \\ e^{i\omega_2 t_1} & e^{i\omega_2 t_2} & e^{i\omega_2 t_3} & \dots \\ e^{i\omega_3 t_1} & e^{i\omega_3 t_2} & e^{i\omega_3 t_3} & \dots \\ \dots & \dots & \dots & \ddots \end{bmatrix}$$

- Inverse:

$$\mathbf{F}^{-1} = \frac{\bar{\mathbf{F}}^T}{N} = \frac{1}{N} \begin{bmatrix} e^{-i\omega_1 t_1} & e^{-i\omega_2 t_1} & e^{-i\omega_3 t_1} & \dots \\ e^{-i\omega_1 t_2} & e^{-i\omega_2 t_2} & e^{-i\omega_3 t_2} & \dots \\ e^{-i\omega_1 t_3} & e^{-i\omega_2 t_3} & e^{-i\omega_3 t_3} & \dots \\ \dots & \dots & \dots & \ddots \end{bmatrix}$$

New!

Resolution of Fourier Transform

- Resolution matrix:

$$\mathbf{R}_F = \mathbf{F}^{-1}\mathbf{F}$$

- If all N frequencies are used to reproduce the Fourier-transformed signal, the recovery is accurate:

$$\mathbf{R}_F = \mathbf{I}$$

- If fewer than N frequencies are used for recovering the signal (Gibbs phenomenon), the resolution is incomplete:

$$\mathbf{R}_F \neq \mathbf{I}$$

New!

Integral Fourier Transform

- For continuous time and frequency (infinitesimal sampling interval and infinite recording time), Fourier transform reads:
  - Forward: $$U(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt u(t) e^{i\omega t}$$
  - Inverse: $$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega U(\omega) e^{-i\omega t}$$
- In practice, the bandwidth (and time) is always limited, and so the actual combination of the forward and inverse transforms is rather:

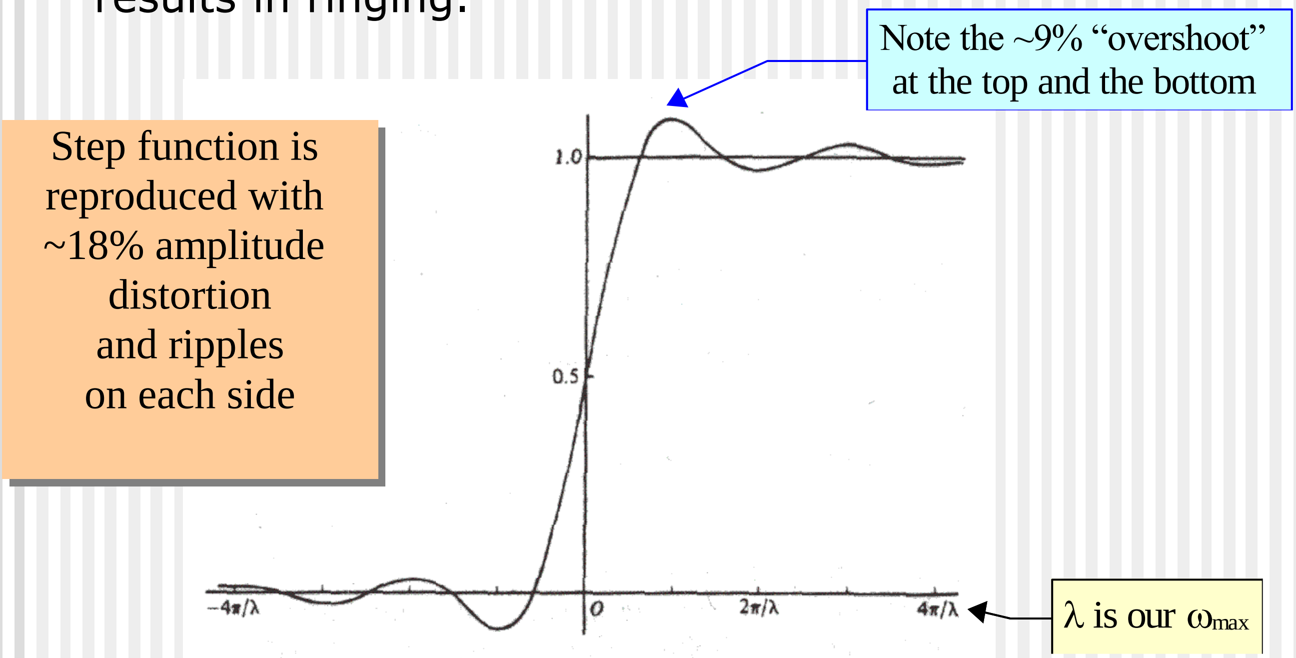
$$u_{\text{band-limited}}(t) = \frac{1}{2\pi} \int_{-\omega_{\max}}^{\omega_{\max}} d\omega \left[\int_{-\infty}^{\infty} d\tau u(\tau) e^{i\omega\tau} \right] e^{-i\omega t}$$

$$u_{\text{band-limited}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau u(\tau) \left[\int_{-\omega_{\max}}^{\omega_{\max}} d\omega e^{i\omega(t-\tau)} \right]$$

New!

Gibbs' phenomenon

- At a discontinuity, application of the Fourier forward and inverse transform (with a limited bandwidth), results in ringing.

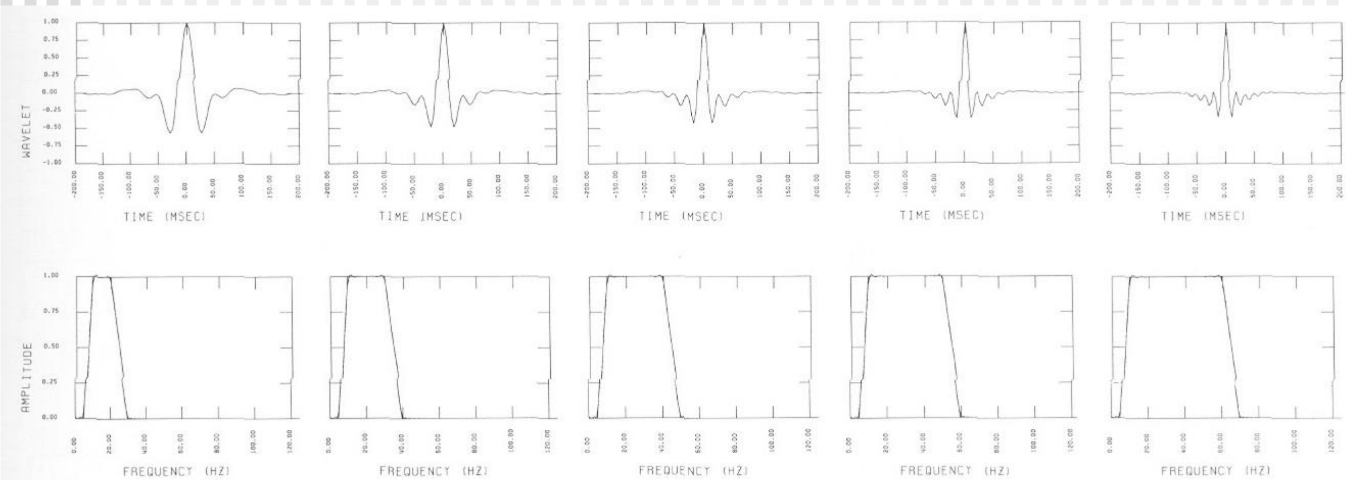


- This is important for constructing time and frequency windows
  - Boxcar windows create ringing at their edges.
  - "Hanning" (cosine) windows are often used to reduce ringing:

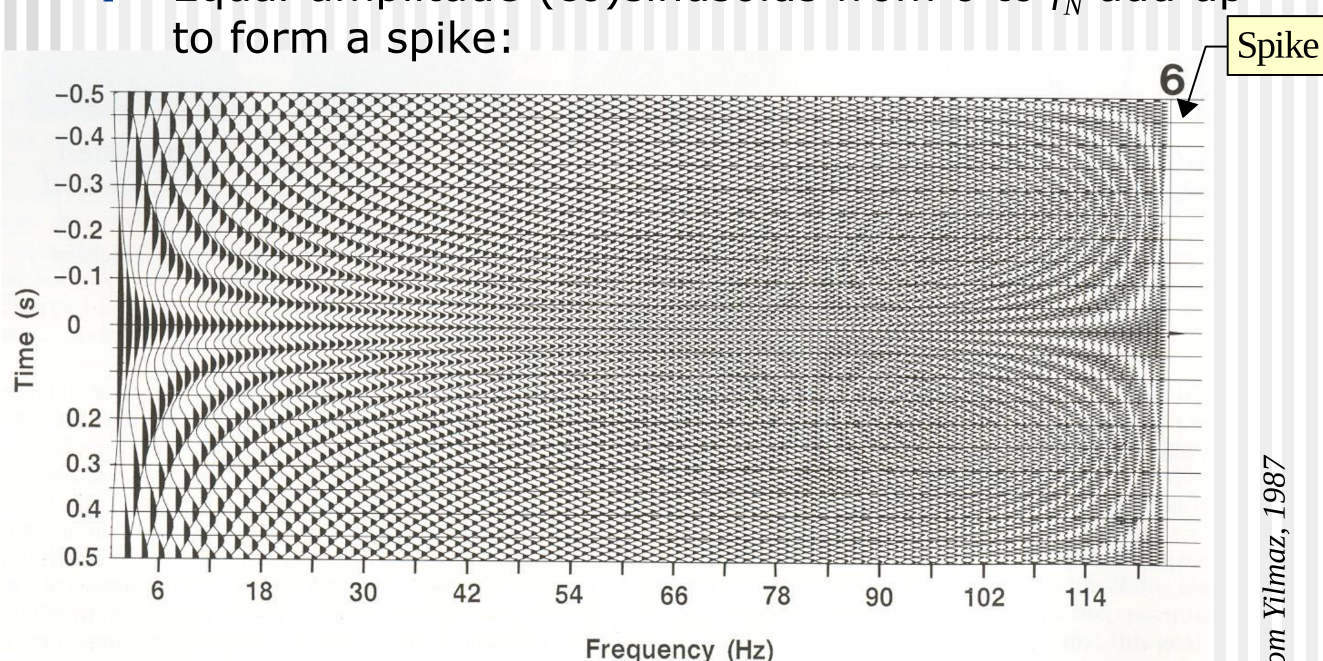
$$H_{\Delta t}(t) = \frac{1}{2}\left(1 - \cos\frac{\pi t}{\Delta t}\right)$$

Spectra of Pulses

- For a pulse of width *T* s, its spectrum is about 1/*T* Hz in width:

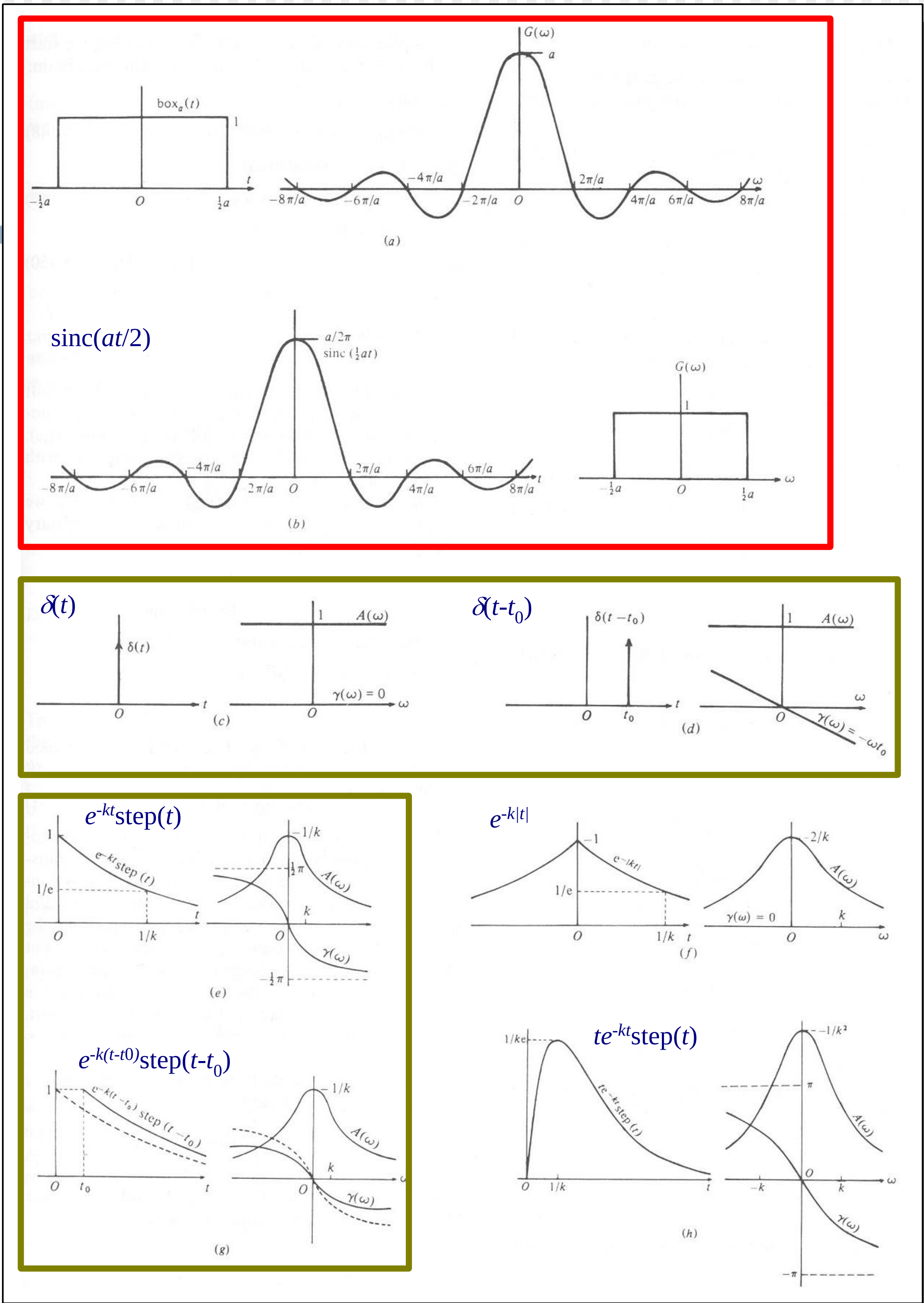


- Equal-amplitude (co)sinusoids from 0 to f_N add up to form a spike:



From Yilmaz, 1987

Sample Fourier Transforms



From Sheriff, Geldart, 1995

- Compare the transforms within the boxes...

New!

Wavelet transforms

- Like the inverse Fourier transform, *wavelet decomposition* represents the time-domain signal by a combination of *wavelets* of some desired shapes:

$$\begin{pmatrix} u(t_1) \\ u(t_2) \\ u(t_3) \\ \ldots \end{pmatrix} = \begin{bmatrix} f_1(t_1) & f_2(t_1) & f_3(t_1) & \ldots \\ f_1(t_2) & f_2(t_2) & f_3(t_2) & \ldots \\ f_1(t_3) & f_2(t_3) & f_3(t_3) & \ldots \\ \ldots & \ldots & \ldots & \ddots \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \ldots \end{pmatrix}$$

Wavelet shapes

Wavelet amplitudes

- Ideally, wavelets should form a *complete orthonormal basis*:

$$\sum_{k=0}^{N-1} f_i(t_k)\, f_j(t_k) = \delta_{ij}$$

exp(...) functions used in Fourier transforms satisfy this property

although this is not always necessary

- Usually, functions $f(t)$ represent time-scaled and shifted versions of some "wavelet" $W(t)$