Mathematical principles

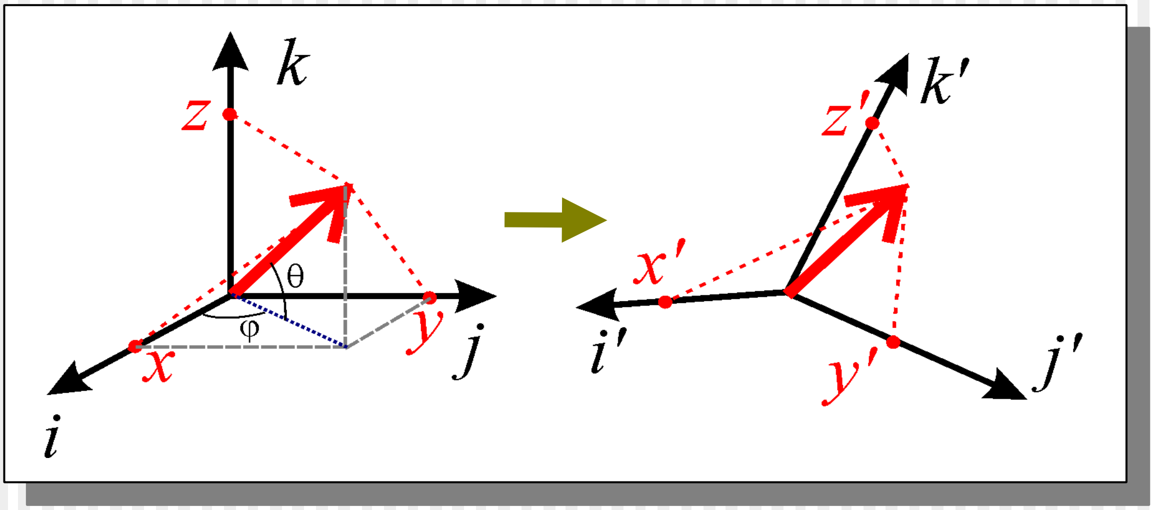
- Rotations
- Tensors, eigenvectors
- Wave equation
- Principle of superposition
- Boundary conditions

- Reading:
  - Telford *et al*., Sections A.2-3, A.5, A.7
  - Shearer, 2.1-2.2, 11.2, Appendix 2

Rotation (vector)

- When axes are rotated, the projections are transformed via an *axes rotation* matrix **R**:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

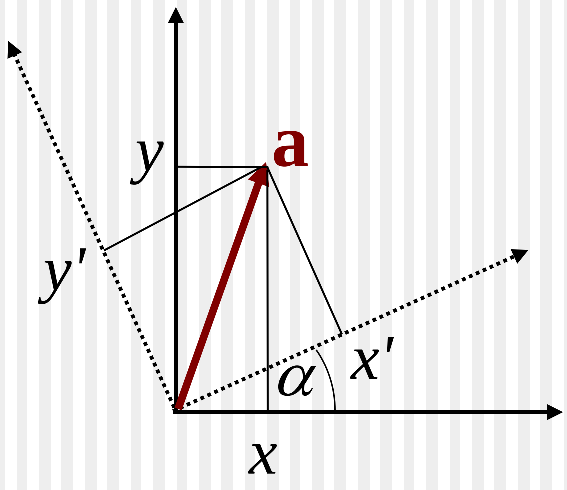


Two dimensional (2D) rotation

- Exercise: Derive the transformation for a counter-clockwise axes rotation by angle α:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{R} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Note that the matrix is anti-symmetric
- What is the matrix $\mathbf{R}^{-1}$ of the inverse transformation?



Rotation (tensor)

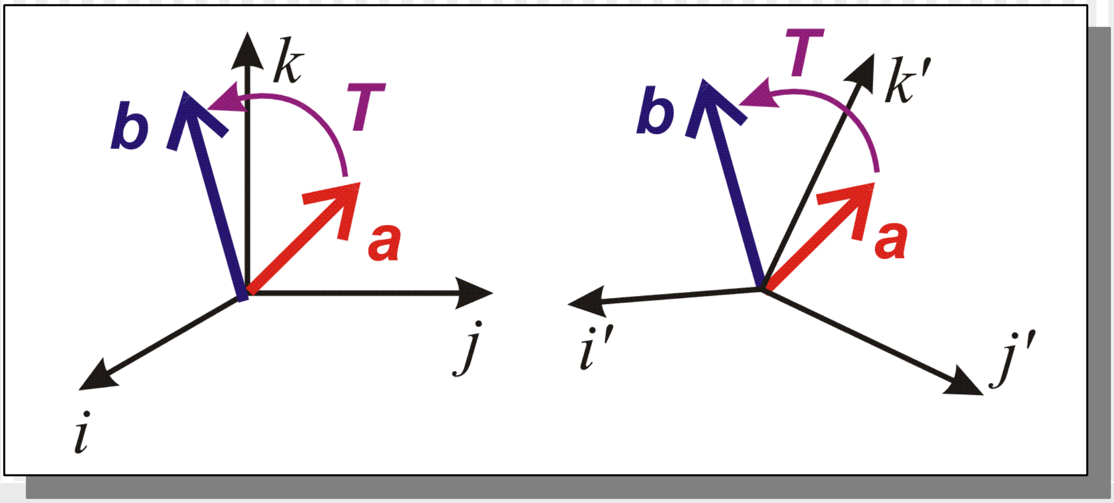
- Tensor is a bi-directional quantity:
  - Examples: Stress and strain in an elastic body; any operator transforming one vector (say, $\mathbf{a}$;) into another ($\mathbf{b}$);
  - Represented by a matrix:

$$b_i = \sum_{j=1}^{3} T_{ij} a_j \equiv T_{ij} a_j$$

Summation is assumed for repeated index (j) (Einstein's notation)

    - 3×3 in three-dimensional space, 2×2 in two dimensions, etc.
    - Transformed whenever the frame of reference is rotated:

$$T'_{ij} = \sum_{k,m} R_{ik} R^{-1}_{jm} T_{km}$$



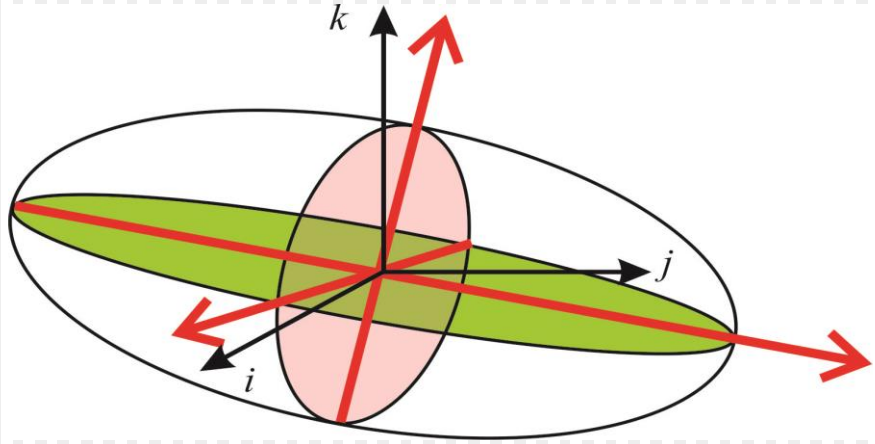
Quadratic form

- Tensor **T** can also be represented by its **quadratic form** Φ (function of an arbitrary vector **x**):

$$\Phi\left(\mathbf{x}\right) = x_i T_{ij} x_j \equiv \mathbf{x}^T \mathbf{T}\mathbf{x}$$

Dot product of **x** and **Tx**

- This is a scalar quantity – independent of rotations of coordinate systems
- Surface of $\Phi(\mathbf{x}) = \text{const}$ describes the general properties of this form
  - Ellipsoidal shape (finite dimensions)
  - Hyperboloidal (infinite)
  - Conical (intermediate)
  - Principal axes (axes and planes of symmetry)



Principal directions

- Principal directions are obtained as eigenvectors $\mathbf{e}_i$ of the tensor matrix:

$$\mathbf{T}\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

Usually take $|\mathbf{e}_i| = 1$

- Eigenvalues λ_i are solved for from the following determinant vanishing:

$$\det\left(\mathbf{T} - \lambda_i \mathbf{I}\right) = 0$$

- Because for stress and strain tensors, the matrix is real and symmetric, all three eigenvalues are real
- The corresponding $\mathbf{e}_i$ give the principal directions (of stress or strain)
  - $\lambda_i < 0$ – compression, $\lambda_i > 0$ – tension
  - When rotated to the directions of $\mathbf{e}_i$, the tensor becomes diagonal (zero shear stress or strain)

Waves

- In seismology, *WAVES* are stable spatial field patterns, which may be:
  - Standing:

$$u = \cos(\omega_n t) f_n(\mathbf{r})$$

These are commonly harmonic, with specific ω_n and f_n for wavemode n

  - Propagating with time:

$$u = f(\mathbf{rn} \pm ct)$$

Plane wave propagating along direction vector **n.**

$$u = \frac{1}{|\mathbf{r}|} f(|\mathbf{r}| \pm ct)$$

Spherical wave

$$u = \frac{1}{\sqrt{\rho}} f(\rho \pm ct)$$

Cylindrical wave (ρ is the distance from axis)

The argument of $f()$ is called *phase*

$f()$ is the waveform, at time t, its zero is at $x = ct$

Wave equation and the principle of superposition

- Wave equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = source\left(\vec{r},t\right) \qquad \text{Scalar}$$

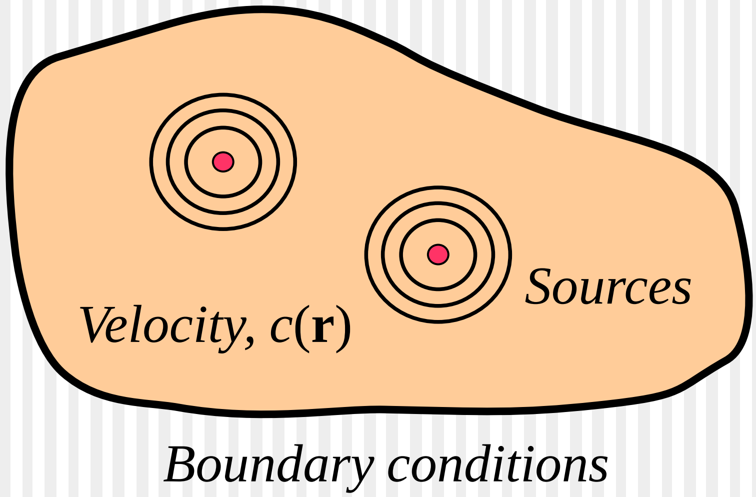
$$\frac{1}{c^2}\frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla^2 \mathbf{u} = \overrightarrow{source}\left(\mathbf{r},t\right). \qquad \text{Vector}$$

- Note that the wave equation is *linear*: if $u_1(\mathbf{r},t)$ and $u_2(\mathbf{r},t)$ are its solutions then $u_1(\mathbf{r},t) + u_2(\mathbf{r},t)$ is also a solution.
  - This property is known as the *principle of superposition.*
  - Because of it, the total wavefield can always be *decomposed* into field generated by elementary sources:
    - Point sources – spherical waves;
    - Linear sources – cylindrical waves;
    - Planar sources – plane waves.

In a homogeneous velocity field

Boundary conditions

- Boundaries (sharp contrasts) in the velocity field $c(\mathbf{r})$ result in *secondary sources* that produce reflected, converted, or scattered waves.
- The amplitudes of these sources and waves are determined through the appropriate *boundary conditions*
  - *e.g.*, *zero displacement* at a rigid boundary (*kinematic* boundary condition);
  - ...or *zero force* at a free boundary (*dynamic* boundary condition).



Three factors determining the wave field