Elements of Rock Mechanics
(advanced topics)

Also not for exam but interesting topics:

- Creep
- Anelasticity and viscoelasticity
- Lagrangian mechanics (used in Lab #3)

- Reading:
  - Shearer, 3

Creep

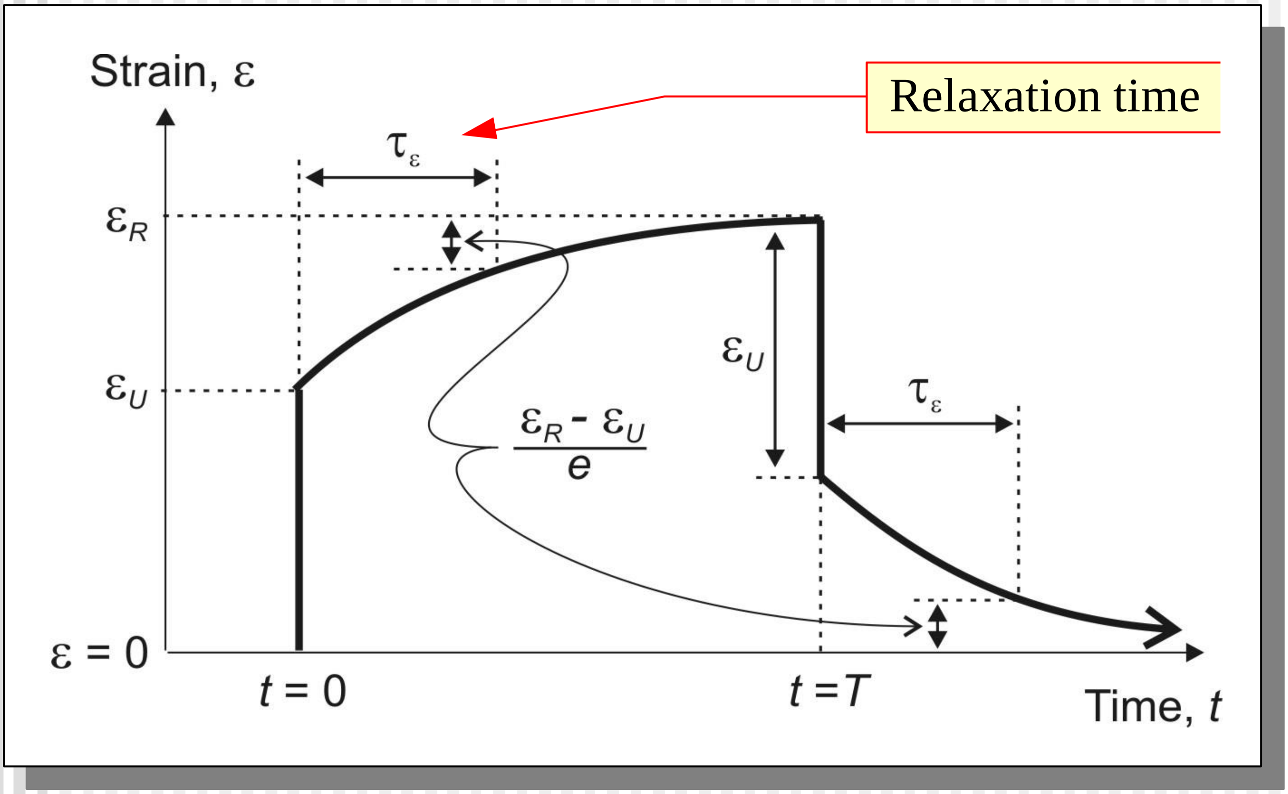
$\theta(t)$ here is the "step function" here: $\theta(\mathrm{t}) = 0$ for $t < 0$, $\theta(\mathrm{t}) = 1$ for $t \geq 0$

- When step-function stress $\sigma(t) = \sigma_0 \theta(t)$ is applied to a solid, it does nit deform instantaneously but exhibits *creep* (delayed, flow-like behaviour):

"Creep function"

$$\varepsilon(t) = \frac{\sigma_0}{M_U}\left[1 + \psi(t)\right]$$

"Unrelaxed modulus"



Viscoelasticity

- It is thought that creep-like (time-dependent) processes also explain:
  - Attenuation of seismic waves (at frequencies 0.002 – 100 Hz)
  - Attenuation of Earth's free oscillations (periods ~1 hour)
  - Chandler wobble (oscillation of the rotation axis of the Earth with period of ~433 days)
- The general *viscoelastic* model states that stress depends on the time *history of strain rate*

$$\sigma(t) = \int_{-\infty}^{t} M(t-\tau)\,\dot{\varepsilon}(\tau)\,d\tau$$

Viscoelastic modulus

- Instead of the Hooke's law relating strain to stress as $\sigma = M\varepsilon$, the viscoelastic constitutive equation for the *"standard linear solid"* (Zener, 1949) relates strain, stress, and also their rates of change with time:

$$\sigma + \tau_\sigma \dot{\sigma} = M\left(\varepsilon + \tau_\varepsilon \dot{\varepsilon}\right)$$

Elastic Energy Density

- Mechanical work is required to deform an elastic body. As a result, elastic energy is accumulated in the strain/stress field
- When released, this energy gives rise to earthquakes and seismic waves
- For a loaded spring (1-D elastic body), $E = ½kx^2 = ½Fx$
- Similarly, for a deformed elastic medium, the *elastic energy density* is:

$$E_{elastic} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

Energy Flux in a Wave

- Later, we will see that in a wave, the kinetic energy density equals the elastic energy:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\rho\dot{u}^2$$

- and so the total energy density:

$$E = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + \frac{1}{2}\rho\dot{u}^2 = \rho\dot{u}^2$$

- The energy propagates with wave speed *V*, and so the average energy flux equals:

$$J = VE = \rho V\left\langle \dot{u}^2 \right\rangle = \frac{1}{2}ZA_v^2$$

where $Z = \rho V$ is called the impedance, and A_v is the particle-velocity amplitude

Lagrangian mechanics

- Instead of equations of motion, modern (*i.e.*, 18th century!) "analytical mechanics" is described in terms of energy functions of generalized coordinates x and velocities $\dot{x}$:
    - Kinetic: (for example) $E_k = \frac{1}{2} m\dot{x}^2$
    - Potential: $E_p = \frac{1}{2} kx^2$
- These are combined in the Lagrangian function:

$$L(x, \dot{x}) = E_k - E_p$$

- Equations of motion in all cases become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

- Exercise: Show that with the above expressions for the kinetic and elastic energies, this equation gives the usual second Newton's law:

$$m\frac{d}{dt}(\dot{x}) = -kx$$

(that is: "mass times acceleration equals applied elastic force")

Lagrangian mechanics of elastic medium

- Lagrangian of isotropic elastic field:

$$L(u,\dot{u}) = \int dV \left[\frac{1}{2}\rho \dot{u}_i \dot{u}_i - \left(\frac{1}{2}\lambda \varepsilon_{ii}{}^2 + \mu \varepsilon_{ij}\varepsilon_{ij} \right) \right]$$

These are the only two second-order combinations of ε that are scalar and invariant with respect to rotations

- This shows the true meanings of *Lamé parameters*
  - They correspond to the contributions of two different types of deformation (compression and shear) to the potential energy

- Exercise: use the Hooke's law to show that

$$E_{elastic} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

is indeed equivalent to:

$$E_{elastic} = \frac{1}{2}\lambda \varepsilon_{ii}{}^2 + \mu \varepsilon_{ij}\varepsilon_{ij}$$