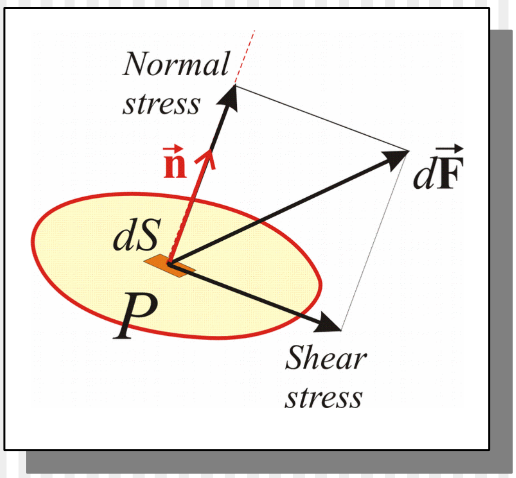
Elements of Rock Mechanics

- Stress and strain
- Creep
- Constitutive equation
  - Hooke's law
- Empirical relations
- Effects of porosity and fluids
- Anelasticity and viscoelasticity

- Reading:
  - Shearer, 3

Stress

- Consider the interior of a deformed body:



At point P, force $d\mathbf{F}$ acts on any infinitesimal area dS. $d\mathbf{F}$ is a projection of *stress tensor*, σ, onto $\mathbf{n}$:

$$dF_i = \sigma_{ij} n_j dS$$

- Stress σ_{ij} is measured in [*Newton/m²*], or Pascal (unit of pressure).
- $d\mathbf{F}$ can be decomposed into two components relative to the orientation of the surface, $\mathbf{n}$:
  - Parallel (normal stress)

$$(dF_n)_i = n_i \cdot (projection\ of\ F\ onto\ n) = n_i \sigma_{kj} n_k n_j dS$$

Note summation over k and j

  - Tangential (shear stress, *traction*)

$$d\vec{F}_\tau = d\vec{F} - d\vec{F}_n$$

Forces acting on a small cube

- Consider a small parallelepiped ($dx \times dy \times dz = dV$) within the elastic body.
- Exercise 1: show that the *force* applied to the parallelepiped from the outside is:

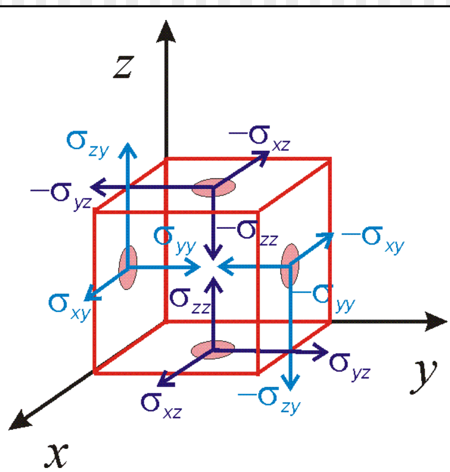
$$F_i = -\partial_j \sigma_{ij} dV$$

Keep in mind implied summations over repeated indices

(This is simply minus divergence ("convergence") of stress!)

- Exercise 2: Show that *torque* applied to the cube from the outside is:

$$L_i = -\epsilon_{ijk} \sigma_{jk} dV$$



$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Symmetry of stress tensor

- Thus, L is <u>proportional to dV:</u> $L = O(dV)$

Big "O"

- The *moment of inertia* for any of the axes is proportional to $dV \cdot length^2$:

$$I_x = \int_{dV} (y^2 + z^2)\,\rho\, dV$$

Little "o"

and so it tends to 0 faster than dV: $I = o(dV)$.

- Angular acceleration: $\theta = L/I$, must be <u>*finite*</u> as $dV \to 0$, and therefore:

$$L_i / dV = -\epsilon_{ijk}\,\sigma_{jk} = 0.$$

- Consequently, the stress tensor is *symmetric:*

$\sigma_{ij} = \sigma_{ji}$

- σ_{ji} has only 6 independent parameters out of 9:

$$\begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = dS \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_x \\ n_x \end{pmatrix},$$

Normal stress components

$$\sigma_{xy} = \sigma_{yx},$$
$$\sigma_{xz} = \sigma_{zx},$$
$$\sigma_{yz} = \sigma_{zy}$$

Shear stress components are symmetric

Principal stresses

- The symmetric stress matrix can always be *diagonalized* by properly selecting the (X, Y, Z) directions (*principal axes*)
  - For these directions, the stress force **F** is orthogonal to dS (that is, parallel to directional vectors **n**)
  - With this choice of coordinate axes, the stress tensor is *diagonal*:

$$\boldsymbol{\sigma}_{\text{principal}} = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

Negative values mean *pressure*, positive - *tension*

- For a given $\boldsymbol{\sigma}$, principal axes and stresses can be found by solving for *eigenvectors* of matrix $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma}\, \boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i$$

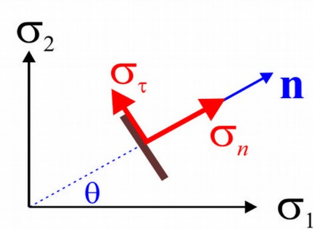
Principal direction vector

Principal stress

Mohr's circle

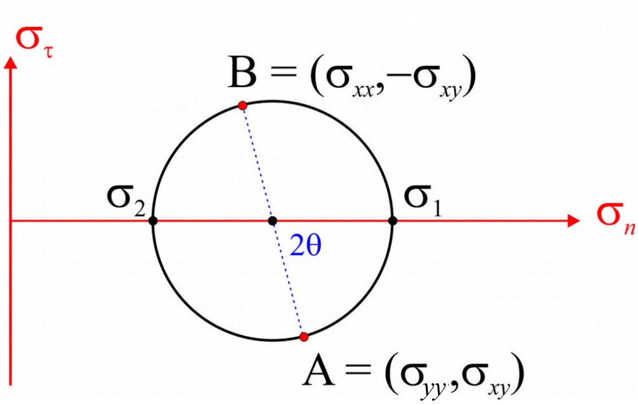
- It is easy to show that in 2D, when the two principal stresses equal σ_1 and σ_2, the normal and tangential (shear) stresses on surface oriented at angle θ equal:

$$\begin{cases} \sigma_n = \dfrac{\sigma_1 + \sigma_2}{2} + \dfrac{\sigma_1 - \sigma_2}{2}\cos 2\theta, \\ \sigma_\tau = -\dfrac{\sigma_1 - \sigma_2}{2}\sin 2\theta. \end{cases}$$



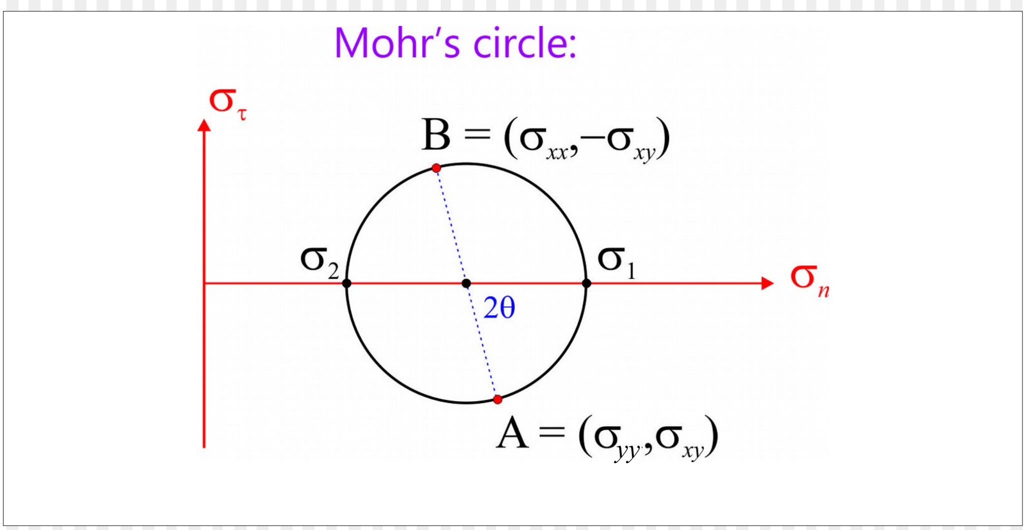
- Mohr (1914) illustrated these formulas with a diagram:

Mohr’s circle:



Mohr's circle (cont.)

- Two ways to use:
  1) When knowing the principal stresses and angle θ, start from points σ_1, σ_2, and find σ_n and σ_τ.
  - When knowing the stress tensor (σ_{xx}, σ_{xy}, and σ_{yy}), start from points A and B and find σ_1, σ_2, and the direction of principal direction σ_1 (θ).

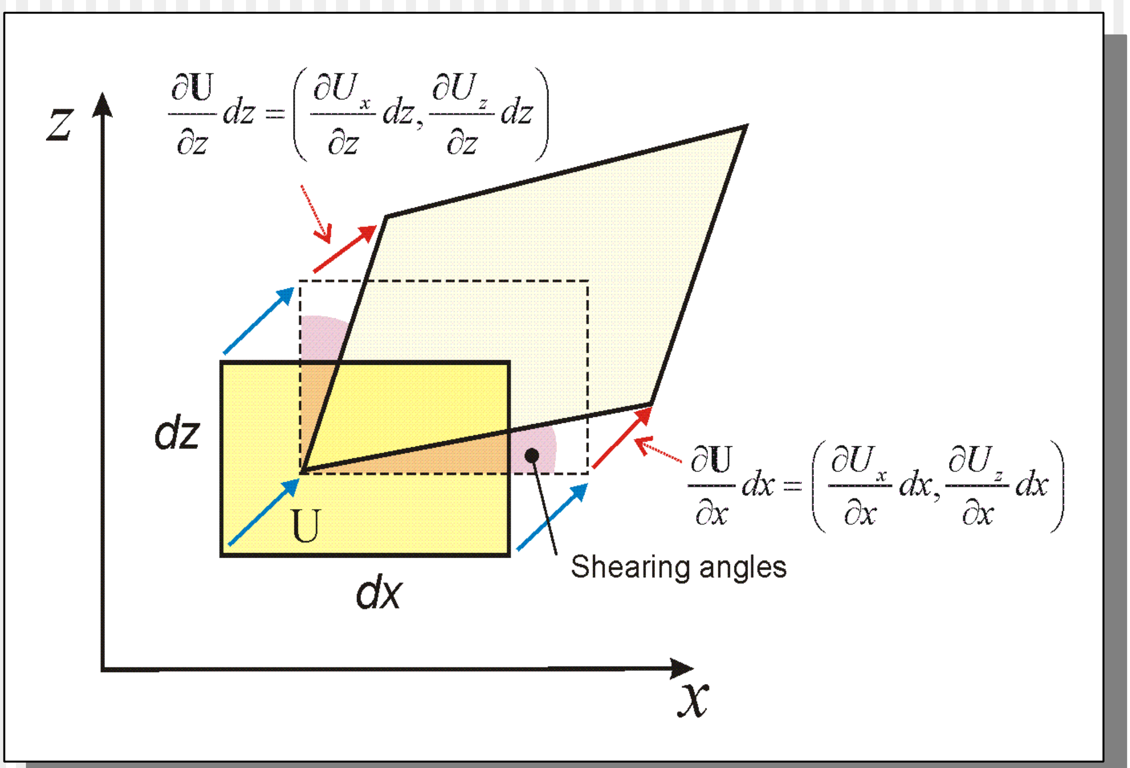


Strain

- Strain is a measure of deformation, i.e., *variation of relative displacement* as associated with a *particular direction* within the body
- It is, therefore, also a *tensor*
  - Represented by a matrix
  - Like stress, it is decomposed into *normal* and *shear* components
- Seismic waves yield strains of 10^{-10}-10^{-6}
  - So we can rely on *infinitesimal* strain theory

Elementary Strain

- When a body is deformed, *displacements* (**U**) of its points are dependent on (x,y,z), and consist of:
  - Translation (blue arrows below)
  - Deformation (red arrows)
- Elementary strain is: $e_{ij} = \frac{\partial U_i}{\partial x_j}$



Stretching and Rotation

- Exercise 1: Derive the elementary strain associated with a uniform stretching of the body:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1+\gamma & 0 \\ 0 & 1+\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Exercise 2: Derive the elementary strain associated with rotation by a small angle α:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  - What is characteristic about this strain matrix?

Strain Components

- *Anti-symmetric* combinations of e_{ij} yield rotations of the body without changing its shape:
  - *e.g.*, $\frac{1}{2}\left(\frac{\partial U_z}{\partial x}-\frac{\partial U_x}{\partial z}\right)$ yields rotation about the 'y' axis.
  - So, the case of $\frac{\partial U_z}{\partial x}=\frac{\partial U_x}{\partial z}$ is called *pure shear* (no rotation).
- To characterize *deformation*, only the *symmetric* component of the elementary strain is used:

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right),$$

$$\varepsilon_{ij}=\varepsilon_{ji},\ \text{where}\ \mathrm{i, j}=x, \mathrm{y, or\ z}$$

$$\varepsilon=\begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{1}{2}\left(\frac{\partial U_x}{\partial y}+\frac{\partial U_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial U_x}{\partial z}+\frac{\partial U_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U_y}{\partial x}+\frac{\partial U_x}{\partial y}\right) & \frac{\partial U_y}{\partial y} & \frac{1}{2}\left(\frac{\partial U_y}{\partial z}+\frac{\partial U_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial U_z}{\partial x}+\frac{\partial U_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial U_z}{\partial y}+\frac{\partial U_y}{\partial z}\right) & \frac{\partial U_z}{\partial z} \end{pmatrix}$$

Dilatational Strain (relative volume change during deformation)

- Original volume: $V=\delta x \delta y \delta z$
- Deformed volume: $V+\delta V=(1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\delta x \delta y \delta z$
- Dilatational strain:

$$\Delta=\frac{\delta V}{V}=(1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})-1\approx\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}$$

$$\Delta=\varepsilon_{ii}=\partial_i U_i=\vec{\nabla}\vec{U}=div\,\vec{U}$$

- Note that *shearing (deviatoric) strain does not change the volume*.

Deviatoric Strain (pure shear)

- Strain without change of volume:

$$\tilde{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{\Delta}{3}\delta_{ij}$$

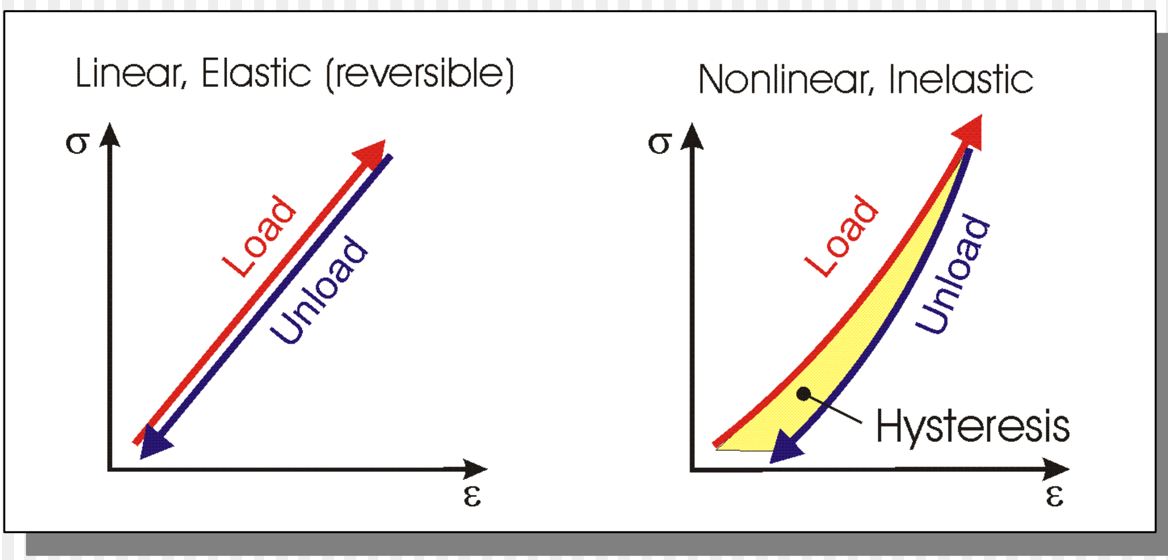
$$Trace\left(\tilde{\varepsilon}_{ij}\right) \equiv \tilde{\varepsilon}_{ii} = 0$$

Constitutive equation

- The “constitutive equation” describes the stress developed in a deformed body:

  F = -k**x** for an ordinary spring (1-D)

  $\sigma \sim \varepsilon$ (in some sense) for a '*linear*', '*elastic*' 3-D solid. This is what these terms mean:



- For a general (*anisotropic*) medium, there are 36 coefficients of proportionality between six independent σ_{ij} and six ε_{ij}:

$$\sigma_{ij} = \Lambda_{ij,kl}\varepsilon_{kl}.$$

Hooke's Law (isotropic medium)

- For isotropic medium, the instantaneous strain/stress relation is described by just 2 constants:

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\varepsilon_{ij}$$

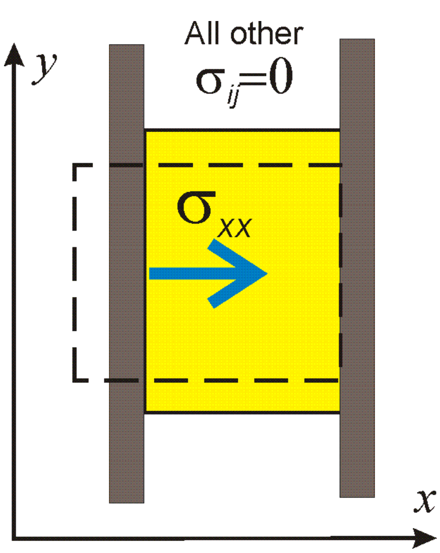
  - δ_{ij} is the "Kronecker symbol" (unit tensor) equal 1 for $i = j$ and 0 otherwise;
  - λ and μ are called the *Lamé constants*.
- Question: what are the units for λ and μ?

Elastic moduli

- Although λ and μ provide a natural mathematical parametrization for $\sigma(\varepsilon)$, they are typically intermixed in experiment environments
  - Their combinations, called "*elastic moduli*" are typically directly measured.
  - For example, *P*-wave speed is sensitive to $M = \lambda + 2\mu$, which is called the "*P*-wave modulus"
- Two important pairs of elastic moduli are:
  - Young's and Poisson's
  - Bulk and shear

Young's and Poisson's moduli

- Depending on boundary conditions (i.e., experimental setup), different combinations of λ' and μ may be convenient. These combinations are called *elastic constants*, or moduli:
  - Young's modulus and Poisson's ratio:
    - Consider a cylindrical sample uniformly squeezed along axis X:



$$\sigma_{xx} = \lambda'\Delta + 2\mu\varepsilon_{xx},$$

$$\sigma_{yy} = \lambda'\Delta + 2\mu\varepsilon_{yy} = 0,$$

$$\sigma_{zz} = \lambda'\Delta + 2\mu\varepsilon_{zz} = 0 \Rightarrow \varepsilon_{yy} = \varepsilon_{zz} = \frac{-\lambda'\Delta}{2\mu}.$$

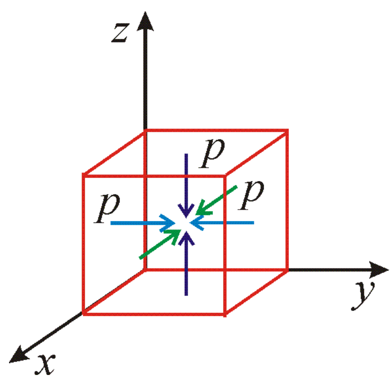
Young's modulus: $E = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$

Poisson's ratio: $\nu = \frac{\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda}{2(\lambda + \mu)}$

- Note: The Poisson's ratio is more often denoted σ
- It measures the ratio of λ and μ: $\frac{\mu}{\lambda} = \frac{1}{2\sigma} - 1$

Bulk and Shear Moduli

- Bulk modulus, K
  - Consider a cube subjected to hydrostatic pressure



$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p,$$

$$-3p = 3\lambda'\Delta + 2\mu\Delta$$

Bulk modulus: $K = \frac{-p}{\Delta} = \lambda' + \frac{2}{3}\mu$

- The Lame constant μ complements K in describing the shear rigidity of the medium. Thus, μ is also called the '*rigidity modulus*'
- For rocks:
  - Generally, $10 \text{ Gpa} < \mu < K < E < 200 \text{ Gpa}$
  - $0 < \nu < ½$ always; for rocks, $0.05 < \nu < 0.45$, for most "hard" rocks, ν is near 0.25.
- For fluids, $\nu = ½$ and $\mu = 0$ (no shear resistance)

Empirical relations: $K, \lambda, \mu(\rho, V_P)$

- From expressions for wave velocities, we can estimate the elastic moduli:

$$\mu = \rho V_S^2$$

$$\lambda = \rho \left(V_P^2 - 2 V_S^2 \right)$$

$$K = \lambda + \frac{2}{3}\mu = \rho \left(V_P^2 - \frac{4}{3} V_S^2 \right)$$

Empirical relations: *density*(V_P)

(See the plot is in the following slide)

- Nafe-Drake curve for a wide variety of sedimentary and crystalline rocks (Ludwig, 1970):

$$\rho\,[g/cm^3] = 1.6612\,V_P - 0.4721\,V_P^2 + 0.0671\,V_P^3 - 0.0043\,V_P^4 + 0.000106\,V_P^5$$

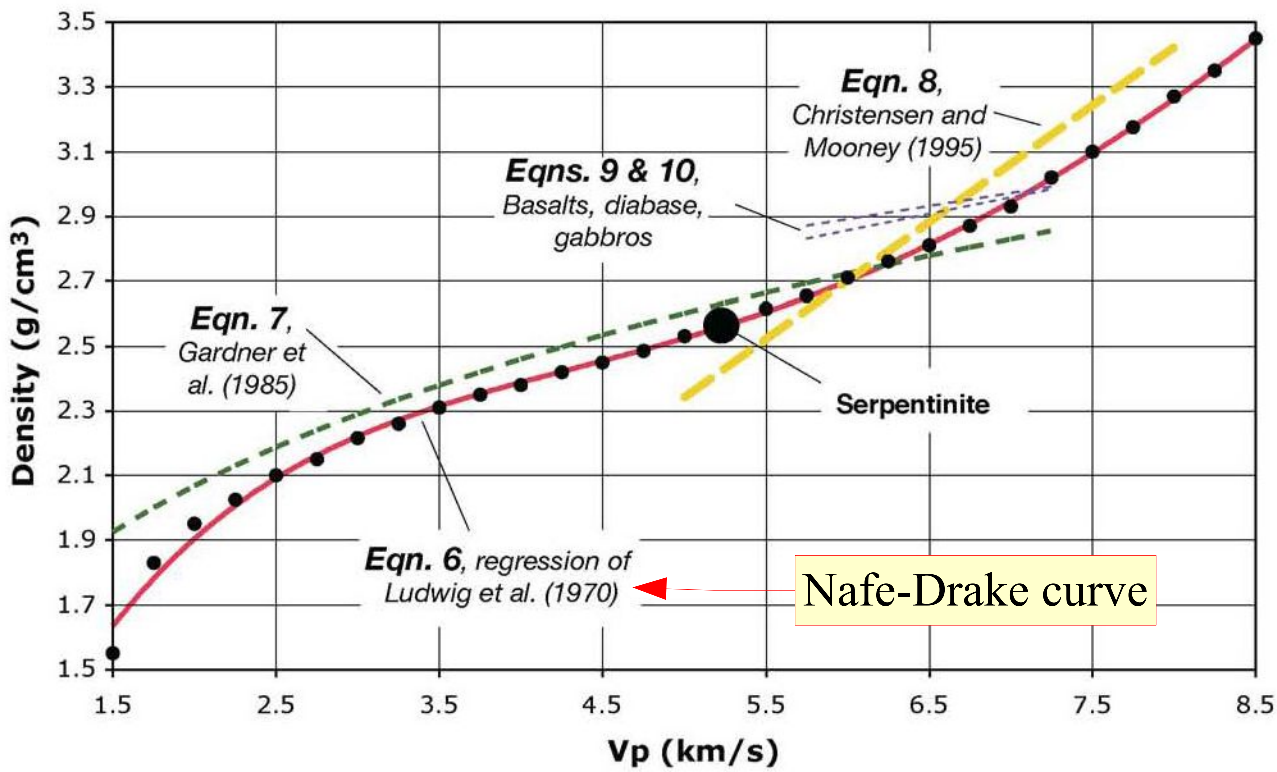
- Gardner's rule for sedimentary rocks and $1.5 < V_P < 6.1$ km/s (Gardner et al., 1984):

$$\rho\,[g/cm^3] = 1.74\,V_P^{0.25}$$

- For crystalline rocks rocks $5.5 < V_P < 7.5$ km/s (Christensen and Mooney., 1995):

$$\rho\,[g/cm^3] = 0.541 + 0.3601\,V_P$$

Empirical relations: *density*(V_P)



From T. Brocher, USGS Open File Report 05-1317, 2005

Empirical relations: $V_S(V_P)$

(See the plot is in the following slide)

- "Mudline" for clay-rich sedimentary rocks (Castagna et al., 1985)

$$V_S[km/s]=(V_P-1.36)/1.16$$

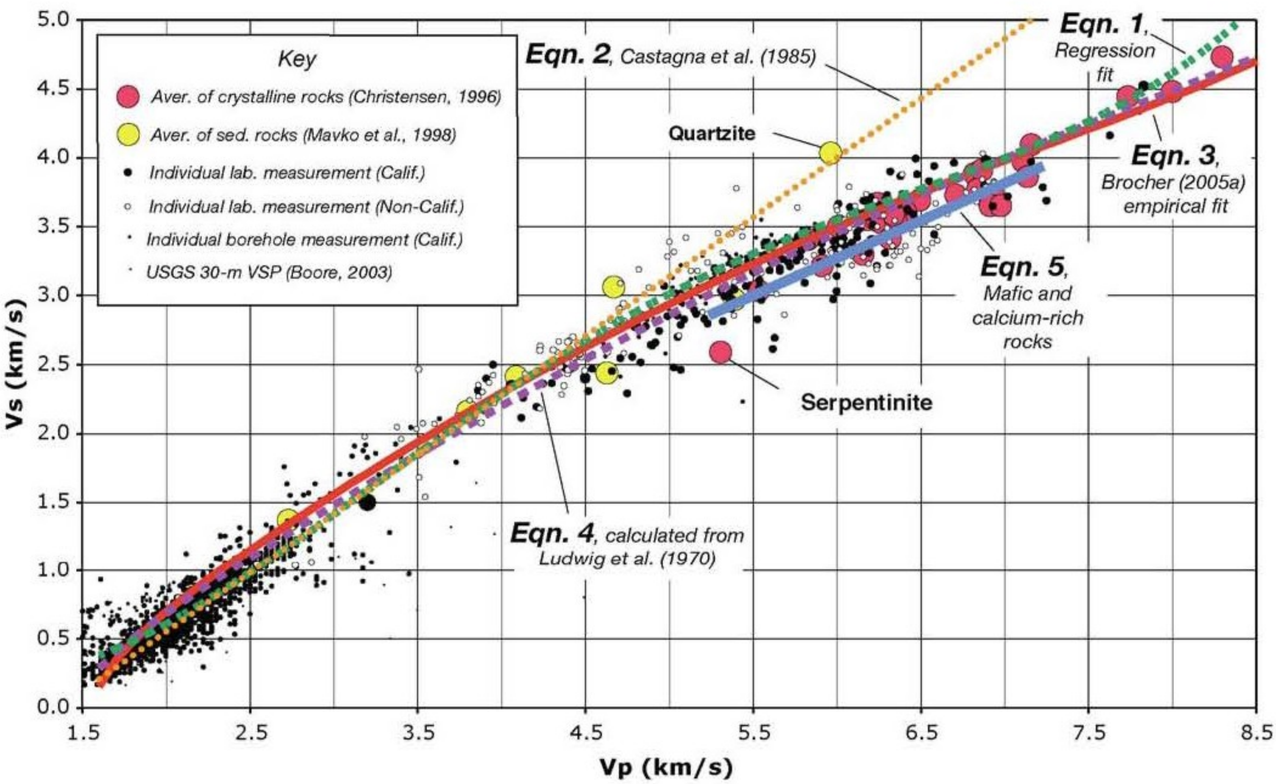
- Extension for higher velocity crustal rocks (1.5 < V_P < 8 km/s; California, Brocher, 2005):

$$V_S[km/s]=0.7858-1.2344\,V_P+0.7949\,V_P^2-0.1238\,V_P^3+0.0064\,V_P^4$$

- "Mafic line" for calcium-rich rocks (dolomites), mafic rocks, and gabbros (Brocher, 2005):

$$V_S[km/s]=2.88+0.52\left(V_P-5.25\right)$$

Empirical relations: $V_S(V_P)$



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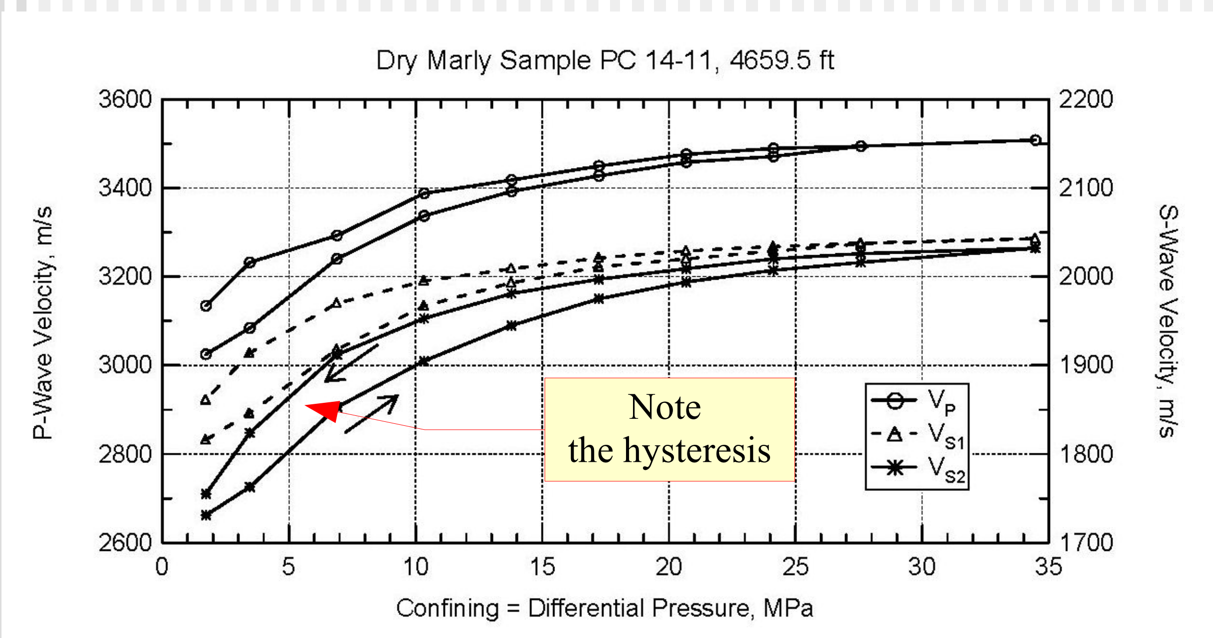
Effect of pressure

- Differential pressure is the difference of confining and pore pressures (related to pressure within rock matrix)
- Differential pressure closes cracks and generally increases the moduli, especially *K*:



Dry carbonate sample from Weyburn reservoir

Effects of porosity

- Pores in rock
  - Reduce average density, ρ
  - Reduce the total elastic energy stored, and thus reduce the *K* and μ
- Fractures have similar effects, but these effects are always anisotropic

Effects of pore fluids

- Fluid in rock-matrix pores increases the bulk modulus
  - Gassmann's model (next page)
  - It has no effect on shear modulus
- Relative to rock-matrix ρ, density effectively decreases:

$$\rho \rightarrow \rho + \rho_f (1 - a)$$

  where ρ_f is fluid density, and a > 1 is the "tortuosity" of the pores
- In consequence, V_P increases, V_S decreases, and the Poisson's ratio and V_P/V_S increase

Gassmann's equation
("fluid substitution")

- Relates the elastic moduli of fluid-saturated rock (K_s, μ_s) to those of dry porous rock (K_d, μ_d):

$$K_s = K_d + \frac{K_0 \left(1 - K_d / K_0\right)^2}{1 - K_d / K_0 - \phi \left(1 - K_0 / K_f\right)}$$

$$\mu_s = \mu_d$$

  where K_f is fluid bulk modulus, and K_0 is the bulk modulus of the matrix

  - Note: $K_f < K_d \leqslant K_s \leqslant K_0$
- Assumptions:
  - Isotropic, homogeneous, elastic, monomineralic medium;
  - Pore space is well-connected and in pressure equilibrium;
  - Closed system with no fluid movement across boundaries;
  - No chemical reactions.

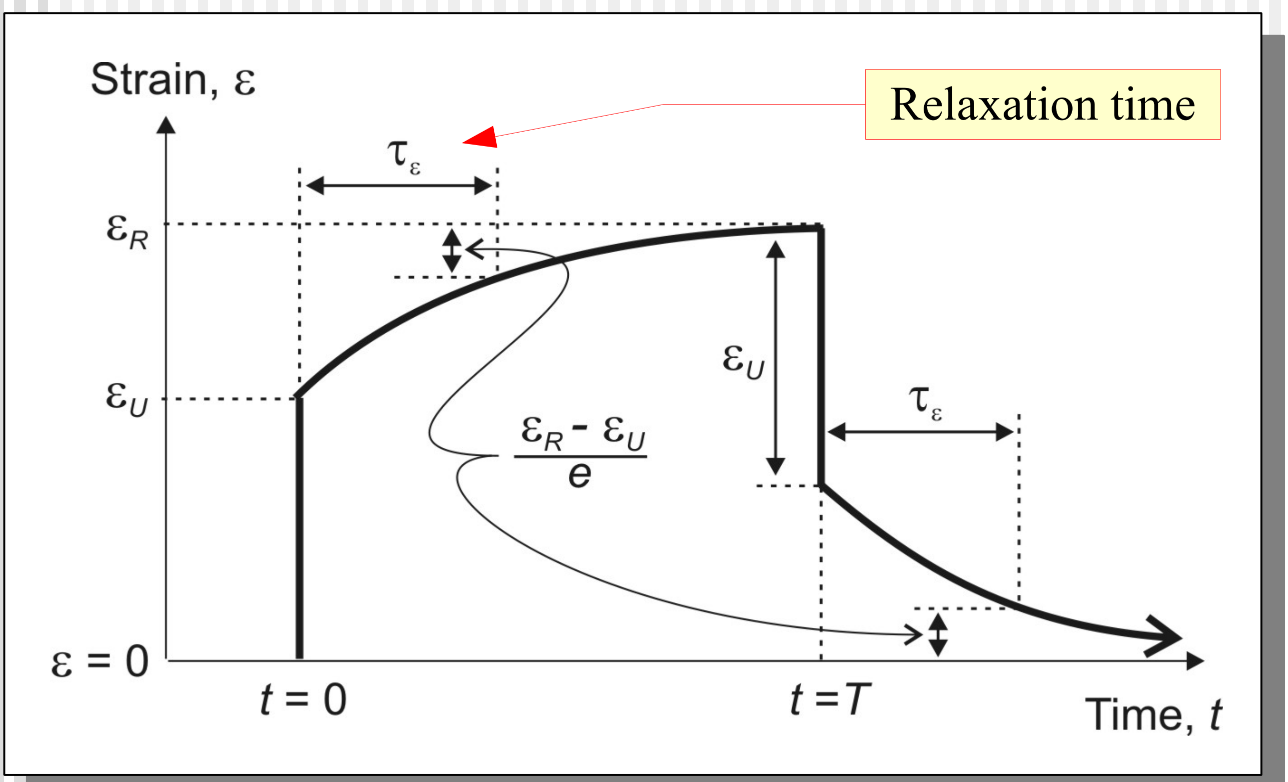
Creep

- When step-function stress $\sigma(t) = \sigma_0\theta(t)$ is applied to a solid, it exhibits *creep*:

$$\varepsilon(t) = \frac{\sigma_0}{M_U}\left[1 + \phi(t)\right]$$

"Creep function"

"Unrelaxed modulus"



Viscoelasticity

- It is thought that creep-like processes also explain:
  - Attenuation of seismic waves (0.002 – 100 Hz)
  - Attenuation of Earth's free oscillations (periods ~1 hour)
  - Chandler wobble (period ~433 days)
- General *viscoelastic* model: stress depends on the *history of strain rate*

$$\sigma(t) = \int_{-\infty}^{t} M(t-\tau)\,\dot{\varepsilon}(\tau)\,d\tau$$

Viscoelastic modulus

- Constitutive equation for the *"standard linear solid"* (Zener, 1949):

$$\sigma + \tau_{\sigma}\dot{\sigma} = M\left(\varepsilon + \tau_{\varepsilon}\dot{\varepsilon}\right)$$

Elastic Energy Density

- Mechanical work is required to deform an elastic body; as a result, elastic energy is accumulated in the strain/stress field
- When released, this energy gives rise to earthquakes and seismic waves
- For a loaded spring (1-D elastic body), $E = \frac{1}{2}kx^2 = \frac{1}{2}Fx$
- Similarly, for a deformed elastic medium, the *elastic energy density* is:

$$E_{elastic} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

Energy Flux in a Wave

- Later, we will see that in a wave, the kinetic energy density equals the elastic energy:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij}=\frac{1}{2}\rho\dot{u}^2$$

- and so the total energy density:

$$E=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}\rho\dot{u}^2=\rho\dot{u}^2$$

- The energy propagates with wave speed *V*, and so the average energy flux equals:

$$J=VE=\rho V\langle\dot{u}^2\rangle=\frac{1}{2}ZA_v^2$$

where Z = ρV is the impedance, and A_v is the particle-velocity amplitude

Lagrangian mechanics

- Instead of equations of motion, modern (*i.e.*, 18th century!) "analytical mechanics" is described in terms of energy functions of generalized coordinates x and velocities $\dot{x}$:
  - Kinetic: (for example) $E_k = \frac{1}{2} m \dot{x}^2$
  - Potential: $E_p = \frac{1}{2} k x^2$
- These are combined in the Lagrangian function:

$$L(x, \dot{x}) = E_k - E_p$$

- Equations of motion become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{dL}{dx} = 0$$

Lagrangian mechanics of elastic medium

- Lagrangian of isotropic elastic field:

$$L(\boldsymbol{u}, \dot{\boldsymbol{u}}) = \int dV \left[\frac{1}{2} \rho \dot{u}_i \dot{u}_i - \left(\frac{1}{2} \lambda \varepsilon_{ii}^{\ 2} + \mu \varepsilon_{ij} \varepsilon_{ij} \right) \right]$$

These are the only two second-order combinations of ε that are scalar and invariant with respect to rotations

- This shows the true meanings of *Lamé parameters*
  - They correspond to the contributions of two different types of deformation (compression and shear) to the potential energy

Lagrangian mechanics

- Exercise: use the Hooke's law to show that

$$E_{elastic} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

is indeed equivalent to:

$$E_{elastic} = \frac{1}{2} \lambda \varepsilon_{ii}^{2} + \mu \varepsilon_{ij} \varepsilon_{ij}$$