Seismic waves

- Recap of theory which you have seen in Geol335:
  - Equations of motion
  - Wave equations
  - P- and S-waves
- New topics:
  - Seismic impedance
  - Wave potentials
  - Energy of a seismic wave

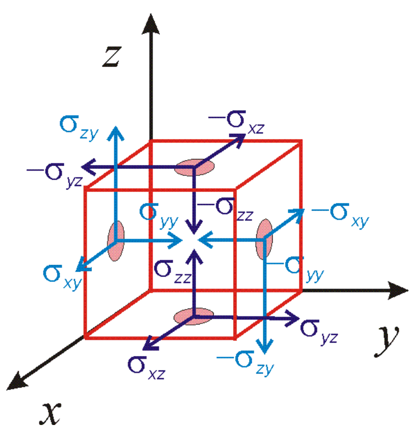
- Reading:
  - Telford *et a*l., Section 4.2
  - Shearer, 3
  - Sheriff and Geldart, Sections 2.1-4

Forces acting on a small cube

- Consider a small volume ($dx \times dy \times dz = dV$) within the elastic body
- *Force* applied to the parallelepiped from all directions is due to the variation (gradient) of stress σ in space
  - The net force is proportional to the volume of the small body:

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j} dV$$

(Summation over 'j' implied as usual)



$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Equations of Motion

(Motion of the elastic body with time)

- Uncompensated net force will result in *acceleration* (second Newton's law):

$$\rho\delta V\frac{\partial^2 U_i}{\partial t^2}=F_i$$

$$\rho\frac{\partial^2 U_i}{\partial t^2}=\left(\frac{\partial\sigma_{ix}}{\partial x}+\frac{\partial\sigma_{iy}}{\partial y}+\frac{\partial\sigma_{iz}}{\partial z}\right)$$

$$\rho\frac{\partial^2 U_x}{\partial t^2}=\frac{\partial}{\partial x}\left(\lambda'\Delta+2\mu\frac{\partial U_x}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial U_x}{\partial y}+\frac{\partial U_y}{\partial x}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial U_x}{\partial z}+\frac{\partial U_z}{\partial x}\right)$$

$$=\lambda'\frac{\partial\Delta}{\partial x}+\mu\frac{\partial}{\partial x}\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)+\mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right)$$

$$=(\lambda'+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 U_x$$

- Therefore, the *equations of motion* for the components of **U**:

$$\rho\frac{\partial^2 U_i}{\partial t^2}=(\lambda+\mu)\frac{\partial\Delta}{\partial x_i}+\mu\nabla^2 U_i$$

Reminder of notation "nabla" – it is a vector of differentiation operators (giving gradient of '*f*') :

$$\nabla_i f\equiv\frac{\partial f}{\partial x_i}$$

Nabla squared is the "Laplacian" –sum of second derivatives of a function:

$$\nabla^2 f\equiv\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

Wave potentials

Compressional and Shear waves

- The above equations describe two types of waves
- These waves can be separated by noting a general property of vector fields ("Lamé theorem"):
  - An arbitrary vector field can be represented by a sum of a gradient of some scalar field ϕ and a curl of some vector field $\boldsymbol{\psi}$

$$\mathbf{U} = \nabla\phi + \nabla\times\boldsymbol{\psi} \qquad \left(\text{or} \quad U_i = \partial_i\phi + \varepsilon_{ijk}\partial_j\psi_k \right)$$

$$\nabla\cdot\boldsymbol{\psi} = 0$$

Because there are 4 components in ψ and ϕ only 3 in **U,** we need to constrain ψ.

- Exercise: substitute the above ϕ and $\boldsymbol{\psi}$ into the equation of motion from preceding slide:

$$\rho\frac{\partial^2 U_i}{\partial t^2} = (\lambda+\mu)\frac{\partial\Delta}{\partial x_i} + \mu\nabla^2 U_i$$

and show this:

$$\rho\frac{\partial^2\phi}{\partial t^2} = (\lambda+2\mu)\nabla^2\phi$$

P-wave (scalar) potential.

$$\rho\frac{\partial^2\psi_i}{\partial t^2} = \mu\nabla^2\psi_i$$

S-wave (vector) potential.

Wave velocities

Compressional and Shear waves

- These are wave equations; compare to the general form of equation describing wave processes:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] f(x,y,z,t) = 0$$

- *Compressional* (*P*) wave velocity:

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

- *Shear* (*S*) wave velocity:

$$V_S = \sqrt{\frac{\mu}{\rho}}$$

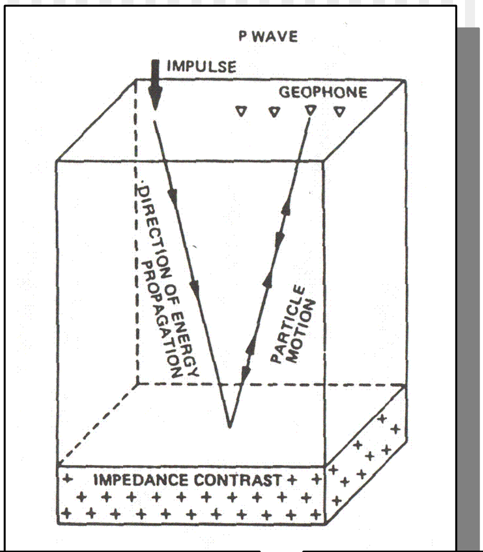
  - $V_S < V_P$
  - for $\nu = 0.25$: $V_P / V_S = \sqrt{3}$
- Note that the V_P/V_S depends on the Poisson's ratio alone:

$$\frac{V_S}{V_P} = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1/2 - \nu}{1 - \nu}}$$

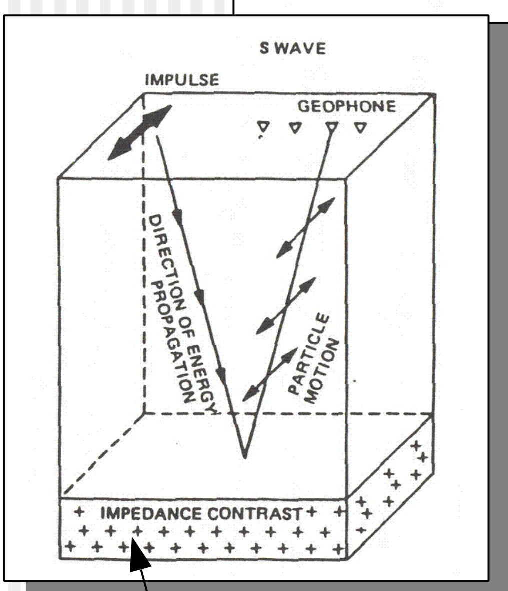
Wave Polarization

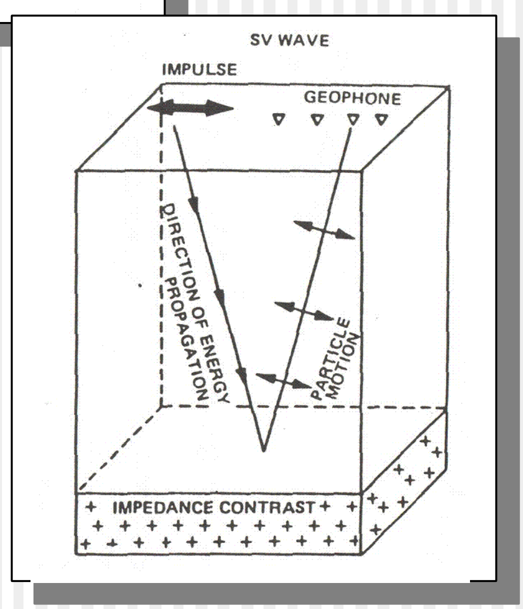
- Elastic solid supports two types of *body waves* (arrows show particle motions within the wave):

P



S





Note that this is still an **ISOTROPIC** reflector.
In general, reflection will intermix
the S-wave polarization modes,
and P-wave will convert into SV upon reflection.

Notes on the use of potentials

- Wave potentials are very useful for solving elastic wave problems
- Just take ϕ or ψ satisfying the wave equation, e.g.:

$$\phi(\mathbf{r},t) = Ae^{i\omega\left(t-\frac{\mathbf{rn}}{V_P}\right)}$$ (plane wave)

...and use the equations for potentials to derive the displacements:

$$\mathbf{U} = \nabla\phi + \nabla\times\boldsymbol{\psi}$$

...and stress from Hooke's law:

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\varepsilon_{ij}$$

- Units for wave potentials are

  Displacement × Distance

- For a harmonic wave, if you find the potential, you can obtain:

  Displacement amplitude = ω×(potential amplitude)/V

Example:
Compressional (*P*) wave

- Scalar potential for *plane harmonic* wave:

$$\phi(\mathbf{r},t) = Ae^{i\omega\left(t-\frac{\mathbf{rn}}{V_P}\right)}$$

- Displacement:

$$u_i(\mathbf{r},t) = \partial_i\phi(\mathbf{r},t) = \frac{-i\omega n_i}{V_P}Ae^{i\omega\left(t-\frac{\mathbf{rn}}{V_P}\right)}$$

note that the displacement is always <u>along **n**</u>

- Strain:

$$\varepsilon_{ij}(\mathbf{r},t) = \partial_i u_j(\mathbf{r},t) = \frac{-\omega^2 n_i n_j}{V_P^2}Ae^{i\omega\left(t-\frac{\mathbf{rn}}{V_P}\right)}$$

- Dilatational strain:

$$\Delta = \varepsilon_{ii}(\mathbf{r},t) = \frac{-\omega^2}{V_P^2}e^{i\omega\left(t-\frac{\mathbf{rn}}{V_P}\right)} = \frac{-\omega^2}{V_P^2}\phi(\mathbf{r},t).$$

- Stress:

$$\sigma_{ij}(\mathbf{r},t) = \frac{-\omega^2}{V_P^2}\left(\lambda\delta_{ij} + 2\mu n_i n_j\right)\phi(\mathbf{r},t)$$

Impedance

- In general, *the acoustic Impedance, Z,* is a measure of the "amount of resistance to particle motion"
  - What does this mean? This is not so easy to say
- Rigorously, in the theory of elasticity, wave impedance is the *ratio of stress to particle velocity*
  - Thus, for a given applied stress, particle velocity is inversely proportional to impedance
  - From the preceding page, For *P* wave, in the direction of its propagation, the impedance is:

$$Z(\mathbf{r},t) = \frac{\sigma_{nn}(\mathbf{r},t)}{\dot{u}_n(\mathbf{r},t)} = \frac{\lambda + 2\mu}{V_P} = \rho V_P$$

    - This is your familiar formula: **impedance equals the product of density and wave velocity**
    - impedance does not depend on frequency but *depends on the wave type and propagation direction*

This is what it means!

Elastic Energy Density

- Recall that for a deformed elastic medium, the *energy density* is:

$$E = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

Elastic Energy Density *in a wave*

- For a plane wave:

$$u_i = u_i\left(t - \mathbf{p}\cdot\mathbf{x}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = -\frac{1}{2}\left(\dot{u}_i p_j + \dot{u}_j p_i\right)$$

...and therefore:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left[\left(\lambda+\mu\right)\left(\mathbf{p}\cdot\dot{\mathbf{u}}\right)^2 + \mu\left(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\right)\left(\mathbf{p}\cdot\mathbf{p}\right)\right]$$

- For *P*- and *S*-waves, this gives:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\lambda+2\mu\right)\mathbf{p}^2\dot{\mathbf{u}}^2 = \frac{1}{2}\rho\dot{\mathbf{u}}^2$$ *P*-wave

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\mu\mathbf{p}^2\dot{\mathbf{u}}^2 = \frac{1}{2}\rho\dot{\mathbf{u}}^2$$ *S*-wave

- Thus, at any point within a wave, *strain energy always equals the kinetic energy*

Unlike in an oscillation of a pendulum,
mechanical energy is ***NOT*** conserved locally in a wave!

- *Energy travels at the same speed* as the wave pulse