Elements of Rock Mechanics

- Stress and strain
- Principal directions of stresses
  - Mohr’s circle
- Constitutive equations
  - Hooke's law
- Elastic moduli

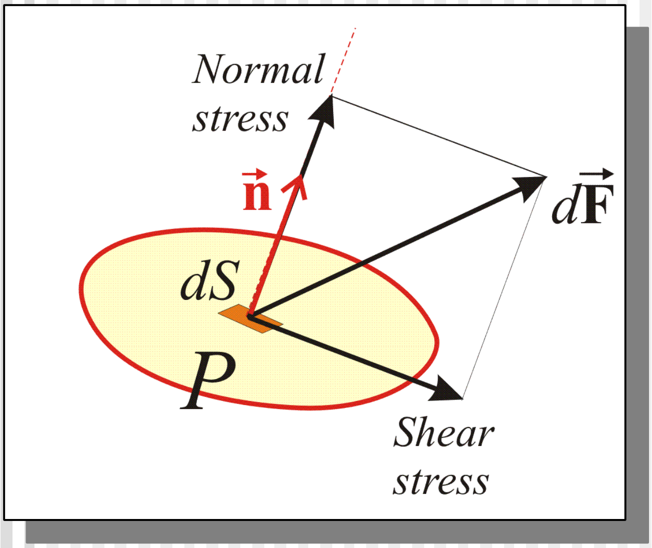
- Reading:
  - Shearer, 3

Rock Mechanics

- To describe rock, or any other mechanical system, we need to discuss:
  - Measures of deformation (strain)
  - Measures of forces (stress)
  - Relation between them (constitutive equation, Hooke’s law)
- We have already looked into these topics in Geol335, and here, we start by reviewing them again

Stress

- Consider the interior of a deformed body:



At point *P*, force $d\mathbf{F}$ acts on any infinitesimal area dS. $d\mathbf{F}$ is a projection of *stress tensor*, $\boldsymbol{\sigma}$, onto $\mathbf{n}$:

$$dF_i = \sigma_{ij} n_j dS$$

- Stress σ_{ij} is measured in [*Newton/m*^{2}], or Pascal (unit of pressure).
- $d\mathbf{F}$ can be decomposed into two components relative to the orientation of the surface, $\mathbf{n}$:
  - Parallel (normal stress)

$$\left(dF_n\right)_i = n_i \cdot (\text{projection of } \mathbf{F} \text{ onto } \mathbf{n}) = n_i \sigma_{kj} n_k n_j dS$$

Note summation over k and j

  - Tangential (shear stress, *traction*)

$$d\mathbf{F}_\tau = d\mathbf{F} - d\mathbf{F}_n$$

Forces acting on a small cube

- Consider a small parallelepiped ($dx \times dy \times dz = dV$) within the elastic body
- Exercise 1: show that the *force* applied to the parallelepiped from the outside is:

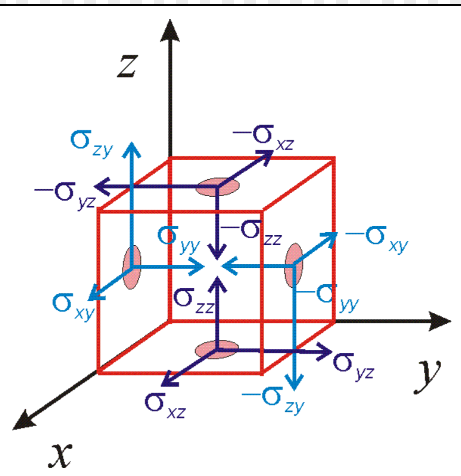
$$F_i = -\partial_j \sigma_{ij} dV$$

Keep in mind implied summations over repeated indices

(This is simply minus divergence ("convergence") of stress!)

- Exercise 2: Show that *torque* applied to the cube from the outside is:

$$L_i = -\varepsilon_{ijk} \sigma_{jk} dV$$



$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Symmetry of stress tensor

- Thus, L is <u>proportional to *dV*: </u>$L = O(dV)$ ← Big "O"
- The *moment of inertia* for any of the axes is proportional to $dV \cdot length^2$:

$$I_x = \int_{dV} \left(y^2 + z^2\right) \rho dV$$

  and so it tends to 0 faster than *dV*: $I = o(dV)$. ← Little "o"

- Angular acceleration: $\theta = L/I$, must be <u>*finite*</u> as $dV \to 0$. Therefore, the torque must be zero:

$$L_i / dV = -\varepsilon_{ijk} \sigma_{jk} = 0$$

- Consequently, the stress tensor is *symmetric:* $\sigma_{ij} = \sigma_{ji}$
- σ_{ji} has only 6 independent parameters out of 9:

$$\begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = dS \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_x \\ n_x \end{pmatrix},$$

Normal stress components

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy}$$

Shear stress components are symmetric

Principal stresses

- The symmetric stress matrix can always be *diagonalized* by properly selecting the (*X*, *Y*, *Z*) directions (*principal axes*)
  - For these directions, the stress force **F** is orthogonal to dS (that is, parallel to directional vectors $\mathbf{n}$)
  - With this choice of coordinate axes, the stress tensor is *diagonal*:

$$\boldsymbol{\sigma}_{\text{principal}} = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

Negative values mean *pressure*, positive - *tension*

- For a given stress tensor $\boldsymbol{\sigma}$, the principal axes and stresses can be found by solving for *eigenvectors* of matrix $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma}\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

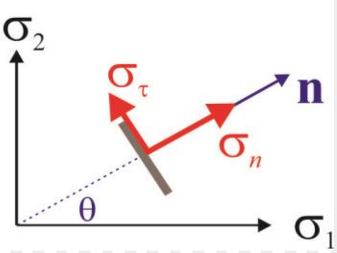
Principal direction vector

Principal stress

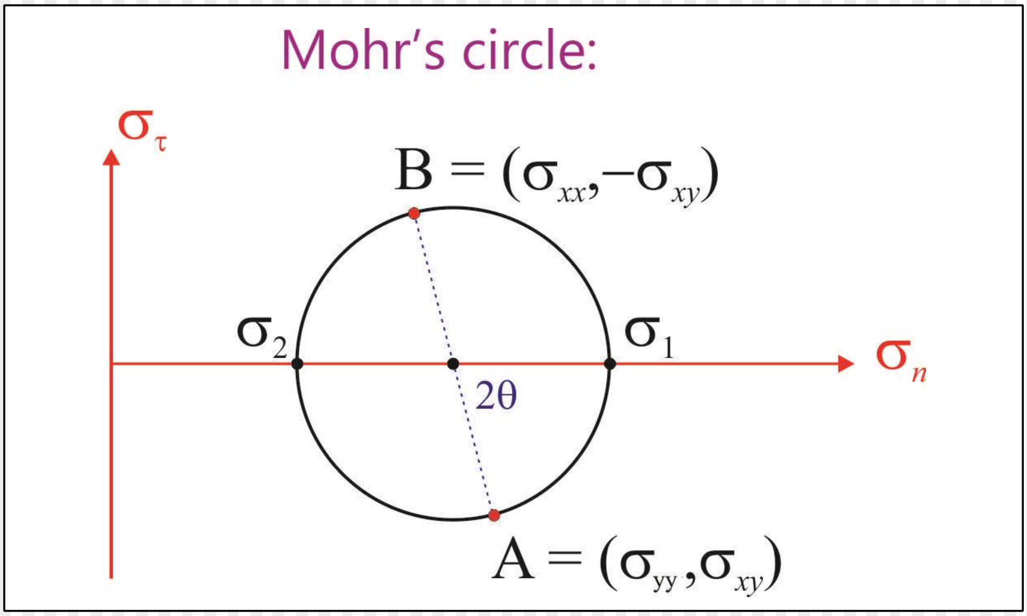
Mohr's circle

- It is easy to show that in 2D, when the two principal stresses equal σ_1 and σ_2, the normal and tangential (shear) stresses on a surface oriented at angle θ equal:

$$\begin{cases} \sigma_n = \dfrac{\sigma_1 + \sigma_2}{2} + \dfrac{\sigma_1 - \sigma_2}{2} \cos 2\theta, \\ \sigma_\tau = -\dfrac{\sigma_1 - \sigma_2}{2} \sin 2\theta. \end{cases}$$

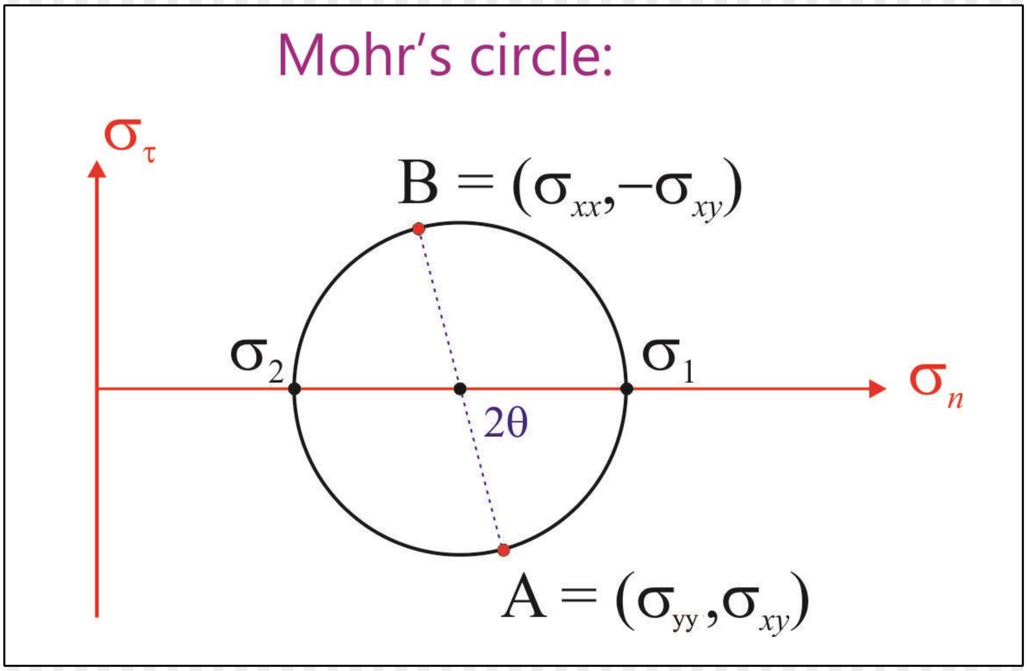


- Mohr (1914) gave a diagram to evaluate these formulas graphically:



Mohr's circle (cont.)

- Two ways to use the Mohr's circle:
  1) If knowing the principal stresses and angle θ, start by drawing points σ_1, σ_2, and find σ_n and σ_τ.
  2) If knowing the stress tensor (σ_{xx}, σ_{xy}, and σ_{yy}), start from points A and B, and find σ_1, σ_2, and the angle θ of the principal direction σ_1.

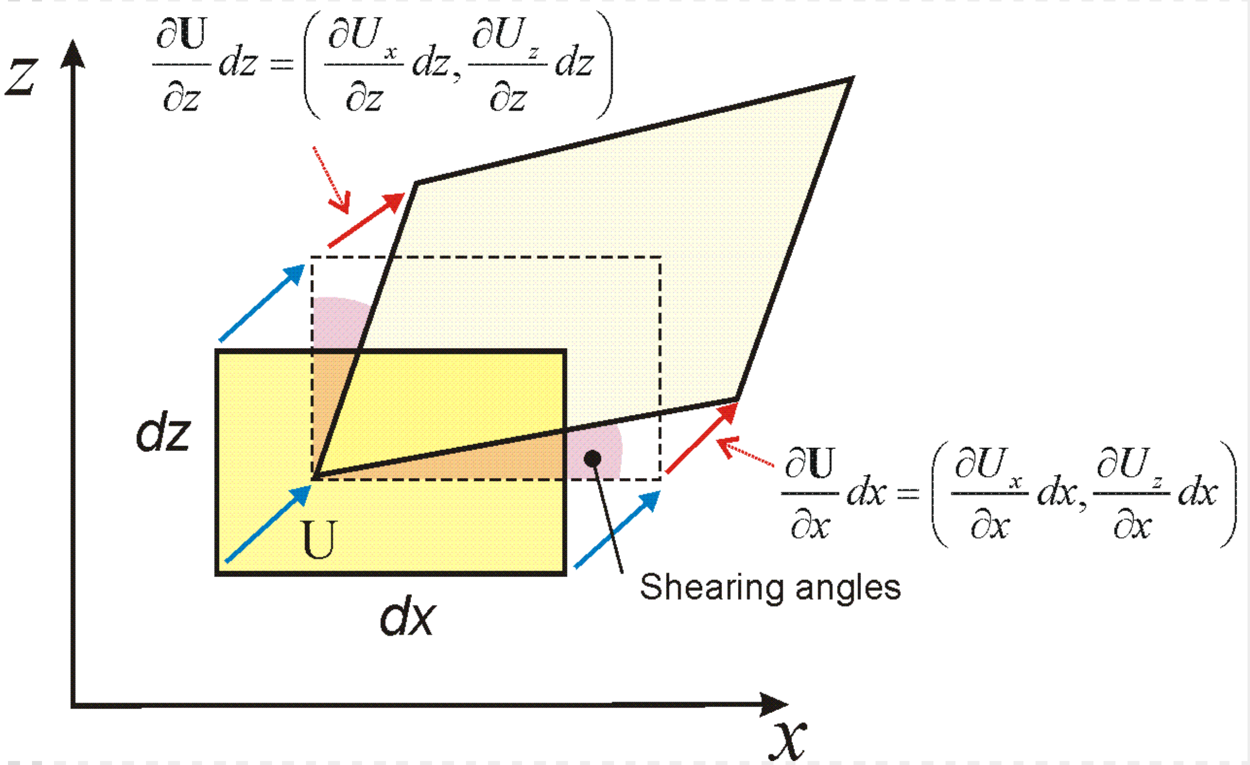


Strain

- Strain is a measure of deformation of a body, i.e., *variation of relative displacement* as associated with a *particular direction* within the body
- Therefore, strain is also a *tensor*
  - Represented by a matrix
  - Like stress, it is decomposed into *normal* and *shear* components
- Seismic waves yield strains of 10^{-10} to 10^{-6}
  - So we can rely on *infinitesimal* strain theory

Elementary Strain

- When a body is deformed, *displacements* (**U**) of its points depend on coordinates (x,y,z), and consist of:
  - Translation (blue arrows below)
  - Deformation (red arrows)
- Elementary strain is: $e_{ij} = \frac{\partial U_i}{\partial x_j}$



Stretching and Rotation

- Exercise 1: Derive the elementary strain associated with a uniform stretching of the body:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1+\gamma & 0 \\ 0 & 1+\gamma \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Exercise 2: Derive the elementary strain associated with rotation by a small angle α:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  - Note that the off-diagonal part of this strain matrix is anti-symmetric (has opposite signs of equal-magnitude values)

Strain Components

- *Anti-symmetric* e_{ij} yield rotations of the body without changing its shape:
  - For example, deformations in which $\frac{\partial U_z}{\partial x} = -\frac{\partial U_x}{\partial z}$ represent pure rotations about the 'Y' axis
  - The opposite case $\frac{\partial U_z}{\partial x} = \frac{\partial U_x}{\partial z}$ is called *pure shear* (no rotation of the elementary volume)
- To characterize *deformation*, only the *symmetric* part of the elementary strain is used:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right),$$

$$\varepsilon_{ij} = \varepsilon_{ji}, \text{ where } \mathrm{i, j} = x, \mathrm{y}, \text{or } \mathrm{z}$$

$$\varepsilon = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{1}{2}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y}\right) & \frac{\partial U_y}{\partial y} & \frac{1}{2}\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial U_z}{\partial y} + \frac{\partial U_y}{\partial z}\right) & \frac{\partial U_z}{\partial z} \end{pmatrix}$$

Dilatational Strain (relative volume change during deformation)

- Original volume: $V=\delta x \delta y \delta z$
- Deformed volume: $V+\delta V=(1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\delta x \delta y \delta z$
- Thus, we have several equivalent formulas for the dilatational strain, denoted Δ:

$$\Delta = \frac{\delta V}{V} = \left(1+\varepsilon_{xx}\right)\left(1+\varepsilon_{yy}\right)\left(1+\varepsilon_{zz}\right)-1 \approx \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}$$

$$\Delta = \varepsilon_{ii} = \partial_i U_i = div\mathbf{U} = \nabla\mathbf{U}$$

- Note that *shearing (deviatoric) strain does not change the volume*.

Deviatoric Strain (pure shear)

- Strain without change of volume:

$$\tilde{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{\Delta}{3}\delta_{ij}$$

$$\mathrm{Trace}\left(\tilde{\varepsilon}_{ij}\right) = \tilde{\varepsilon}_{kk} = \Delta - \frac{\Delta}{3}\mathrm{Trace}\left(\delta_{ij}\right) = 0$$

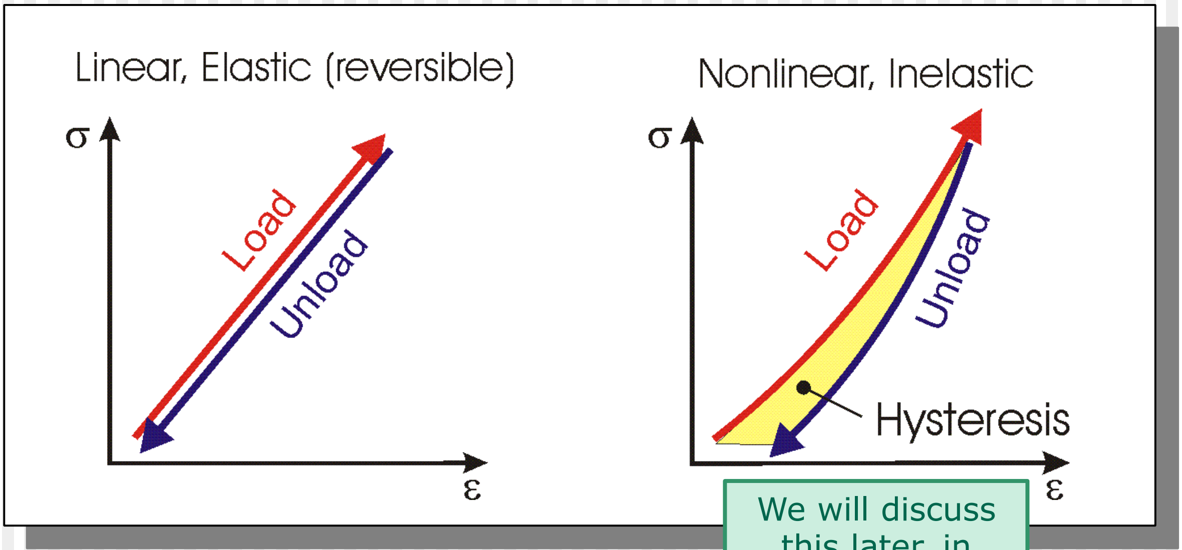
- Can you confirm this relation? What is the trace of δ_{ij} (identity matrix)?

Constitutive equation

- The "constitutive equation" describes the relation of stress to strain:

  F = -k**x** for an ordinary spring (1-D)

  σ ~ ε (in some sense) for a '*linear*' and '*elastic*' 3-D solid. This is what these terms mean:



We will discuss this later, in "viscoelasticity"

- For a general (*anisotropic*) medium, there are 36 coefficients of proportionality between six independent σ_{ij} and six ε_{ij}:

$$\sigma_{ij} = \Lambda_{ij,kl}\varepsilon_{kl}$$

Hooke's Law (isotropic medium)

- For isotropic medium, the instantaneous strain/stress relation is described by just two constants:

$$\sigma_{ij} = \lambda\Delta\delta_{ij} + 2\mu\varepsilon_{ij}$$

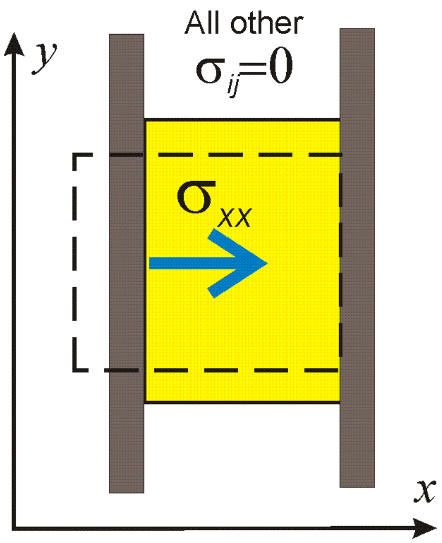
  - δ_{ij} is the “Kronecker symbol” (unit tensor) equal 1 for *i* =*j* and 0 otherwise;
  - λ and μ are elastic material properties called the *Lamé constants* (or moduli).
- Question: what are the units for λ and μ?

Elastic moduli

- Although λ and μ provide a natural mathematical parametrization for $\sigma(\varepsilon)$, they are typically intermixed in practical applications
    - Their combinations, called "*elastic moduli*" are typically measured or affect seismic waves
    - For example, *P*-wave speed is sensitive to $M = \lambda + 2\mu$, which is called the "*P*-wave modulus"
- Two important practical elastic moduli are:
    - Young's modulus and Poisson's ratio
    - Bulk and shear
    - "P-wave modulus" M

Young’s modulus and Poisson’s ratio

- Young’s modulus and Poisson’s ratio occur in an experiment with unidirectional compression or tension
  - Consider a cylindrical rock sample uniformly compressed along axis X:



$$\sigma_{xx} = \lambda'\Delta + 2\mu\varepsilon_{xx},$$

$$\sigma_{yy} = \lambda'\Delta + 2\mu\varepsilon_{yy} = 0,$$

$$\sigma_{zz} = \lambda'\Delta + 2\mu\varepsilon_{zz} = 0 \Rightarrow \varepsilon_{yy} = \varepsilon_{zz} = \frac{-\lambda'\Delta}{2\mu}.$$

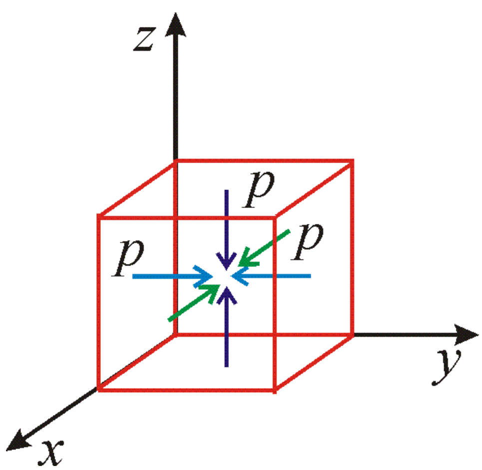
Young's modulus: $E = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$

Poisson's ratio: $\nu = \frac{-\varepsilon_{zz}}{\varepsilon_{xx}} = \frac{\lambda}{2(\lambda + \mu)}$

- Note: The Poisson's ratio is also often denoted σ
- It measures the ratio of λ and μ: $\frac{\mu}{\lambda} = \frac{1}{2\nu} - 1$

Bulk and Shear Moduli

- To obtain the bulk modulus, *K*, consider a cube subjected to hydrostatic pressure



$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p,$$

$$-3p = 3\lambda'\Delta + 2\mu\Delta$$

Bulk modulus: $K = \frac{-p}{\Delta} = \lambda' + \frac{2}{3}\mu$

- The Lame constant μ complements K in describing the shear rigidity of the medium. Thus, μ is also called the '*rigidity modulus*'
- For rocks:
  - Generally, $10 \text{ GPa} < \mu < K < E < 200 \text{ GPa}$
  - $0 < \nu < ½$ always; for rocks, $0.05 < \nu < 0.45$, for most "hard" rocks, ν is near 0.25.
  - For wet sedimentary rock, ν is above 0.3
- For fluids, $\nu = ½$ and $\mu = 0$ (no shear resistance)

P-wave Modulus

- As we will see later (and may recall from Geol335), velocities of P waves are determined by a combination of λ and μ called the “P-wave modulus”:

$$M = \lambda + 2\mu = K + \frac{4}{3}\mu$$