# Modelling travel times, rays and wavefronts

- Travel-time field
- Ray-tracing
  - For travel times
  - For amplitudes
- WKBJ (high-frequency) approximation
- Eikonal equation
- Practical travel-time modelling methods

- Reading:
  - Shearer, 4 and 6

# Travel-time field, rays, and wavefronts

- Rays and wavefronts represent key attributes of "the travel time field" $t(\mathbf{x})$
- If $t(\mathbf{x})$ is the time at which certain wave reaches point $\mathbf{x}$, then:
  - Wavefronts are surfaces $t(\mathbf{x}) = \text{const}$
  - Rays are streamlines of the gradient of $t(\mathbf{x})$
  - Gradient of the time field is the ray parameter vector:

$$\mathbf{p} = \nabla\left[t\left(\mathbf{x}\right)\right] \quad \text{or} \quad p_i = \frac{\partial t\left(\mathbf{x}\right)}{\partial x_i}$$

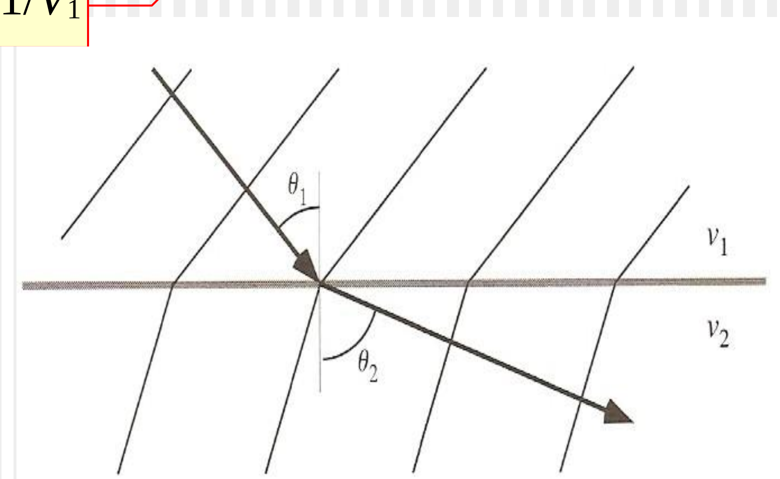
- Ray theory corresponds to the high-frequency limit of wave fields:
  - wavefronts and rays are smooth but bend sharply on discontinuities

# Snell's law

- The key property of the travel-time field is that it is continuous across any velocity contrasts
- Therefore, the wavefronts and rays are also continuous, although they change shapes across boundaries
- If we look at the ray-parameter equation $p_i = \frac{\partial t(\mathbf{x})}{\partial x_i}$ along the direction of a boundary, we will see that the ray parameter on both sides of the boundary must be the same
- That is, the wave slowness along the boundary is the same on both sides. This is the Snell's law:

$$p = s_1 \sin\theta_1 = s_2 \sin\theta_2$$

Slowness = $1/V_1$



# WKBJ (high-frequency) approximation of wave equations

- Originates from Liouville and Green (~1837)
- Named after Wentzel, Kramers, Brillouin, and Jeffreys (~1923-26)
- Gives approximate solutions of a general differential equation with small parameter $\varepsilon << 1$ in the leading derivative:

$$\varepsilon \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0$$

- The solution is sought in the form:

$$y(x) = A \exp\left[\frac{1}{\delta} \sum_{k=0}^{\infty} \delta^k S_k(x)\right]$$

where small variable $\delta << 1$ is related to $\varepsilon$

- I am only trying to give the general spirit of this approach; don’t worry about the detail

# WKBJ approximation of wave equation

- Consider the wave equation for a harmonic wave (Helmholtz equation) in variable wave speed $c(x)$:

$$\frac{1}{c(x)^2}\frac{\partial^2 u}{\partial^2 t}-\frac{\partial^2 u}{\partial^2 x}=\frac{-\omega^2}{c(x)^2}u-\frac{\partial^2 u}{\partial^2 x}=0$$

that is:

Denote this coefficient $\varepsilon^2$

$$\left(\frac{c_0}{\omega}\right)^2\frac{\partial^2 u}{\partial^2 x}=-\left(\frac{c_0}{c}\right)^2 u$$

and $c_0$ is some characteristic value of $c(x)$

- The coefficient in the left-hand side is small (frequency is high), and so let us denote it by $\delta^2$:

$$\left(\frac{c_0}{\omega}\right)^2=\delta^2$$

- …and look for a solution like this:

$$u(x)=\exp\left[\frac{1}{\delta}\sum_{k=0}^{\infty}\delta^k S_k(x)\right]$$

# WKBJ approximation of wave equation

- The wave equation becomes:

$$\varepsilon^2\left[\frac{1}{\delta^2}\left(\sum_{k=0}^{\infty}\delta^k S_k'(x)\right)^2+\frac{1}{\delta}\sum_{k=0}^{\infty}\delta^k S_k''(x)\right]=-\left(\frac{c_0}{c}\right)^2$$

- To the leading order, with $\delta\to 0$:

This is the "eikonal equation"

$$\frac{1}{\delta^2}\left(\frac{c_0}{\omega}\right)^2 S_0'^2=-\left(\frac{c_0}{c}\right)^2$$

- Thus, $\delta$ is proportional to $\in$, and we can take:

$$\delta=\frac{c_0}{\omega},\qquad S_0(x)=\pm i\int\frac{c_0dx'}{c(x')}+const$$

and the WKBJ solution becomes:

$$u(x)\approx A\exp\left[\pm i\omega\int_{x_0}^{x}\frac{dx'}{c(x')}\right]$$

Ray travel time from $x_0$ to $x$

# WKBJ approximation of wave equation (end)

- The above solution:

$$u(x) \approx A \exp\left[ \pm i\omega \int_{x_0}^{x} \frac{dx'}{c(x')} \right]$$

only gives the ray-theoretical phase of the wave

  - It is equivalent to the solution of the eikonal equation (discussed later)

- The amplitude can be estimated by the second-order WKBJ approximation
  - This is called the (energy) transport equation

# Eikonal equation

- From German *Eikonal*, which comes from Greek $\varepsilon\iota\kappa\omega\nu$, image (that is, "icon")
- Provides the link between the wave and geometrical optics (and acoustics)
- If $t(\mathbf{x})$ is the time at which certain wave reaches point $\mathbf{x}$, then in the geometrical (high-frequency) limit, it must satisfy:

$$\left|\vec{\nabla} t(\mathbf{x})\right| = \frac{1}{c(\mathbf{x})}$$

- This is called the eikonal equation for the seismic travel-time field $t(\mathbf{x})$
- This equation is very broadly used in fast 2-D and 3-D wavefront- and ray-tracing algorithms

# Travel-time modelling methods

- Ray tracing (shooting)
  - $\tau$-p methods (in layered media)

We will not discuss them here
See Shearer

- Eikonal-equation based wavefront propagation
- Ray bending
- Shortest-time ray methods

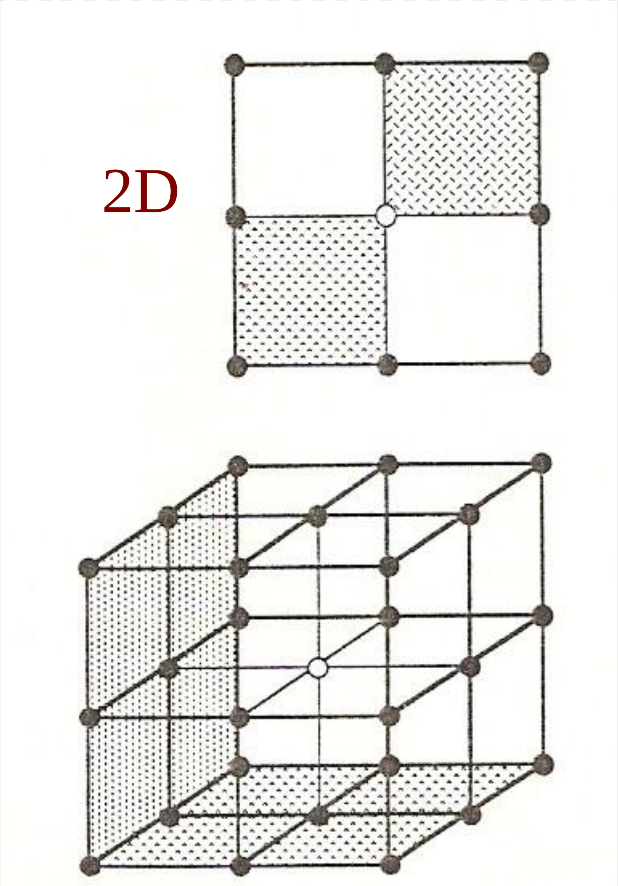
# Ray shooting
(A simple approach in 2D)

- Velocity model is split into triangular cells
- In each cell, the velocity has a constant gradient
- In a constant velocity gradient, the ray is always a circular arc (we will see this later)
- Starting from the source the ray is constructed by combining such arcs
- Accurate, but complex method
  - Computationally intensive when many rays are needed
  - May have problems in complex structures

# Eikonal first-arrival travel-time calculation

- Initialize the near-source times
- At each iteration, try timing each node by using the already timed adjacent nodes
  - Use waves from adjacent points, pairs of points (linear sources), or triplets (planar sources)
- In plots on the right, the white node is being timed at the current step
- To time this node, various combinations of adjacent timed nodes (black) are tried
  - Among them, the earliest time arriving at the white node is selected

2D



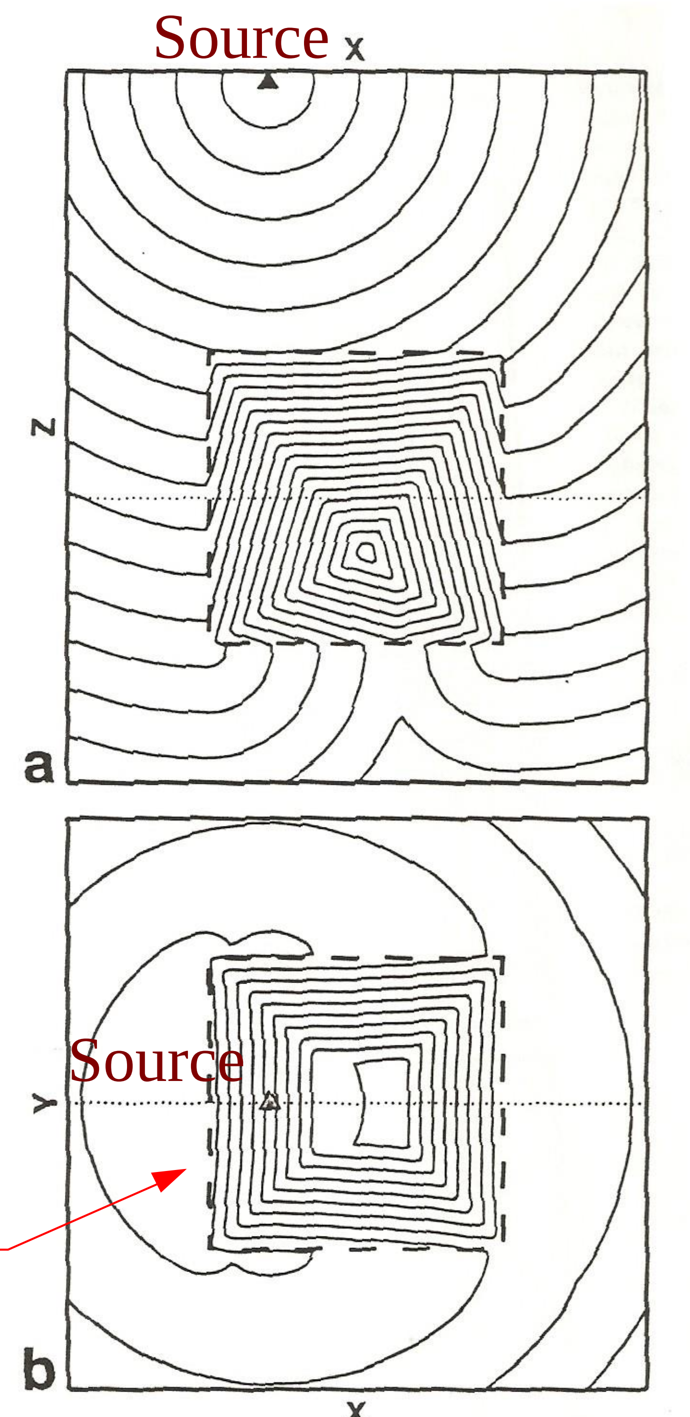
3D

*Podvin and Leconte, 1991*

# Example

## First-arrival travel times in 3D

- Eikonal travel-time calculation
- Rays to every point can be obtained by tracing $t(\mathbf{x})$ gradients back to the source




Low-velocity cube

*Zelt, 1993?*

# Ray bending

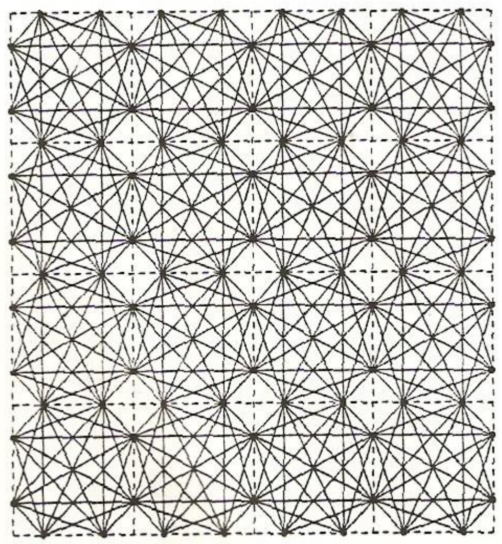
- Directly employing the Fermat principle
- Connecting the source and receiver by a smallest-time ray
- Pros:
  - Accurate and stable
  - By sampling rays with times within half of the dominant wave period from the smallest-time ray, “banana”-shaped volumes can be obtained. These volumes represent the area affecting the recorded wave arrival
- Cons:
  - Works only for selected, fixed source-receiver pairs
  - Computationally intensive

# Shortest-path ray tracing

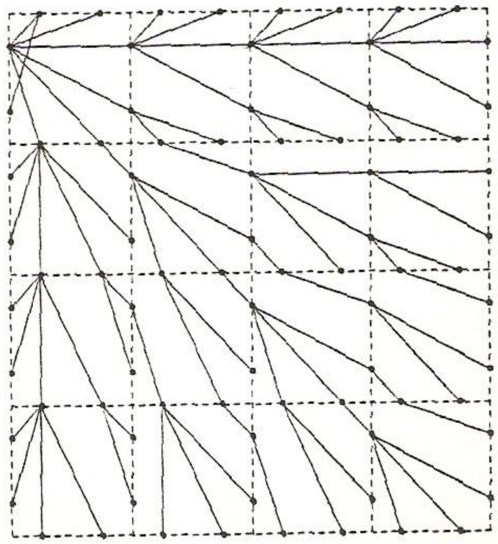
- A suitable grid of possible ray paths is created
  - Including reasonable dips and structures
- Starting from the source, shortest-time paths are identified
- Fast and stable method
  - Good for quick general assessment of time field
  - Can be followed by ray bending for accuracy

All paths considered

Shortest paths

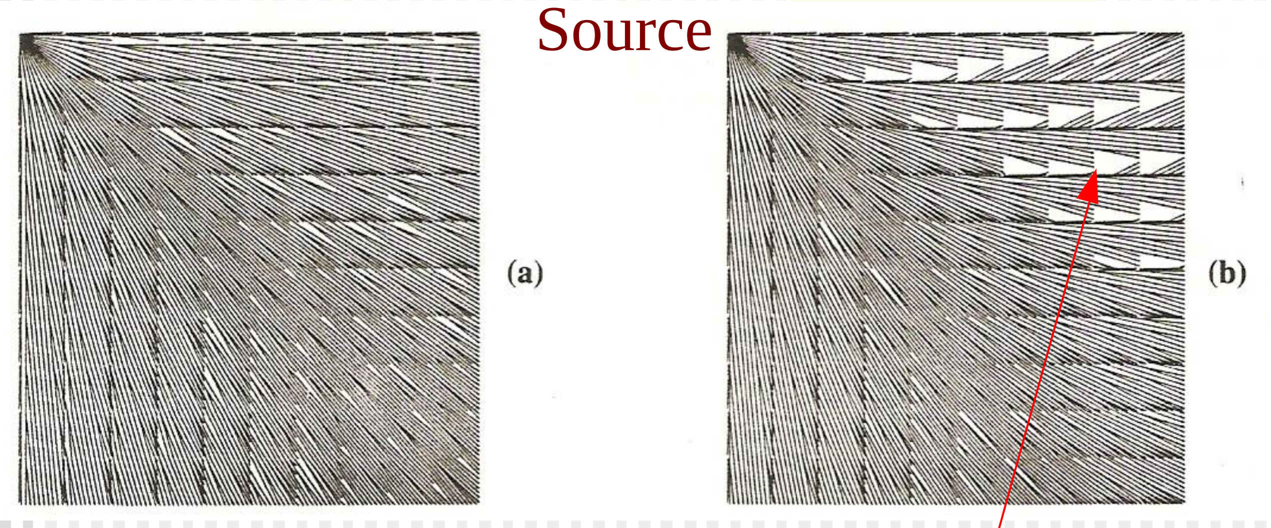


Source



*Moser, 1991*

# Example:
## Shortest-time paths




*Moser, 1991*

# Ray-based amplitudes

- Amplitudes can be estimated from ray-flux tubes
  - For example, the *Geometrical spreading* is often modelled in this way
- If energy flux remains constant:

$$E_{flux} = c\left(\frac{\rho}{2}A^2\omega^2\right)$$

Kinetic energy density

then amplitude varies as:

$$\frac{A_2}{A_1} = \sqrt{\frac{dS_1}{dS_2}}\sqrt{\frac{\rho_1 c_1}{\rho_2 c_2}}$$

Geometrical spreading

Ratio of impedances