Anelastic acoustic impedance and the correspondence principle

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ABSTRACT

A general definition of seismic wave impedance is proposed as a matrix differential operator transforming the displacement boundary conditions into traction ones. This impedance is proportional to the standard acoustic impedance at all incidence angles and allows extensions to attenuative media and to the full elastic case. In all cases, reflection amplitudes at the contact of two media are uniquely described by the ratios of their impedances. Here, the anelastic acoustic impedance is studied in detail and attenuation contrasts are shown to produce phase-shifted reflections. Notably, the correspondence principle (i.e., the approach based on complex-valued elastic modules in the frequency domain) leads to incorrect phase shifts of the impedance due to attenuation and consequently to wrong waveforms reflected from attenuation contrasts. Boundary conditions and the Lagrange formulation of elastodynamics suggest that elastic constants should remain real in the presence of attenuation and the various types of energy dissipation should be described by their specific mechanisms. The correspondence principle and complex-valued elastic moduli appear to be applicable only to homogeneous media and therefore they should be used with caution when applied to heterogeneous cases.

Key words: Attenuation, P-wave, Theory, Visco-elasticity.

INTRODUCTION

Acoustic impedance is the key quantity used for characterizing the reflection and transmission of seismic waves. Since the 1970s, it has become the primary tool used in seismic reflectivity inversion and interpretation (Lindseth 1979). However, despite its canonical character and widespread use, the theoretical background of this concept is still not entirely clear and requires some attention. In particular, modifications of the acoustic impedance due to anelasticity are still not well understood and in fact they may be subtle and strongly debatable. Acoustic impedance is often interpreted heuristically as, for example, the product of density and velocity or generator of reflectivity time series. Nevertheless, as shown below from the basic principles of wave mechanics, acoustic impedance can be rigorously defined as a property of field boundary conditions. This definition allows extending this concept to full visco-elastic cases.

Wave propagation in anelastic media has been extensively studied (e.g., Lockett 1962; Anderson and Archambeau 1964; Cooper and Reiss 1966; Richards 1984; Borcherdt and Wennerberg 1985; Carcione 2007; Borcherdt 2009) but almost exclusively by using the theory of visco-elasticity and some form of the correspondence principle. The essence of this approach is in describing the attenuation as a modification of elasticity, with real-valued elastic moduli replaced with their complex-valued counterparts in the frequency domain (Bland 1960; Carcione 2007). In particular, in a medium with attenuation quality factor $Q$, the shear modulus $\mu$ is assigned a negative complex argument of $[-\arctan(Q^{-1})]$, which leads to the well-known negative imaginary shift in wave velocities and consequently in the acoustic impedance (e.g., Anderson and Archambeau 1964). However, this last effect is incorrect,
because the imaginary part of acoustic impedance should be positive, as was recently demonstrated by explicit calculations of reflection amplitudes from attenuation contrasts (Lines, Vasheghani and Treitel 2008; Morozov 2009b).

An example of such phase-shifted reflectivity resulting from complex-moduli elastodynamics is shown in Fig. 1(a–d). Here, a reflection from the bottom of a layer at 1-km depth with a 10% increase in Re $\mathbb{Z}$ across the boundary is modelled by using the ‘reflectivity’ approach (Fuchs and Müller 1971), in which attenuation effects are incorporated by using complex-valued medium velocities. The overburden is uniform and lossless ($V_p = 2$ km/s, $V_S = 1$ km/s, density $\rho = 2$ g/cm$^3$ and $Q_{P}^{-1} = Q_{S}^{-1} = 0$). In the presence of attenuation increase to $Q_{P}^{-1} = Q_{S}^{-1} = 0.2$ below the reflector, the reflection becomes time-advanced (Fig. 1b) relative to the attenuation-free case (see the reference time pick indicated by the dashed grey line in Fig. 1a). Its phase shift equals $45^\circ$ and after its compensation (Fig. 1c), the reflection becomes similar to the attenuation-free one (Fig. 1a). In the absence of a contrast in $\rho V$, the reflection from a jump in $Q^{-1}$ becomes phase-advanced by $90^\circ$ (Fig. 1d), compared to the attenuation-free case.

Phase-advanced reflections from a positive attenuation contrast (Fig. 1b,d) result from interpreting viscous energy dissipation as time-retarded elasticity (and consequently the Lamé moduli are generalized to ‘relaxation-rate’ functions). Such descriptions lead to correct equations of harmonic-wave propagation in uniform media; however in the presence of a discontinuity, it is not obvious that such retarded forces are also applied to the boundary. The boundary acts as a ‘visco-elastic’ membrane itself. By contrast, as shown below, by using the standard (instantaneous) Hooke’s law and considering attenuation as caused by external (non-elastic) causes, reflection phase shifts (Fig. 1e,f) become opposite to those arising from visco-elasticity (Fig. 1b,d, respectively).

The literature on visco-elasticity is vast and covers almost every aspect of wave mechanics (for a recent detailed overview, see Carcione 2007). In the time domain, complex-valued elastic moduli correspond to several types of time-delayed stress-strain relations that are often used to explain the dependence of seismic $Q$ on frequency. Examples of such models include Maxwell, Kelvin-Voigt and Zener (the standard linear solid) models and superpositions of such models were used to explain the Earth’s absorption band (Liu, Anderson and Kanamori 1976). The impact of such models on seismologists’ understanding of attenuation is so strong that they may have even biased the observations toward pervasive

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**Figure 1** Numerical models of a P-wave reflection from 1-km depth for a range of source-receiver offsets by using the reflectivity method (Fuchs and Müller 1971). Top row: a) with no attenuation contrast; b) with positive velocity/density and attenuation contrasts; c) same, with an additional $-45^\circ$ phase rotation to compensate the effect of $\delta(Q^{-1})$; d) with a pure attenuation contrast. Bottom row: results of the present approach: e) reflection from a velocity-density contrast (as in c); f) pure attenuation contrast (as in d). Velocity and attenuation contrasts across the boundary are indicated in the labels. Dashed grey line indicates the position of the near-offset peak in plot a.)
The anelastic acoustic impedance problem provides a simple example to study the combined effects of wave propagation and boundary conditions. Below, I review several existing acoustic impedance definitions, offer its new generalized form, and derive its relation to the quality parameter, $Q$, of the medium. In addition, I show that unlike the traditional acoustic impedance, $\tilde{Z}$ turns out to be ‘reactive’ (imaginary) for lossless medium, which also makes it a more straightforward analogue to the electromagnetic impedance. Although providing convenient mathematical solutions to the wave equations in uniform media, these solutions remain phenomenological. Equivalent and visco-elastic models complicate the understanding of such a fundamental quantity as the elastic force, which becomes combined with the force of friction. However, the elastic force is directly measurable and is critical when considering field boundary conditions and therefore its behaviour should be carefully understood. Development of the anelastic impedance concept represents one approach to developing such an understanding.

The anelastic acoustic impedance problem provides a simple example to study the combined effects of wave propagation and boundary conditions. Below, I review several existing acoustic impedance definitions, offer its new generalized form, denoted $\tilde{Z}$ and derive its relation to the quality parameter, $Q$, of the medium. In addition, I show that unlike the traditional acoustic impedance, $\tilde{Z}$ turns out to be ‘reactive’ (imaginary) for lossless medium, which also makes it a more straightforward analogue to the electromagnetic impedance. Although $\tilde{Z}$ is applicable to the general cases of oblique incidence and P- and S-wave mode conversions, most of the discussion below uses normal-incidence acoustic examples. Further, I describe the problem of the complex phase of the impedance and discuss the general applicability of the correspondence principle to solving wave problems in heterogeneous media. By contrast to the broadly accepted view, I argue that the Lamé moduli should remain real-valued in the presence of attenuation. I also briefly illustrate an alternate (and also well-known) description of anelasticity based on Lagrange’s formalism. Finally, the above statements are illustrated by considering two simple mechanical systems.

**ACOUSTIC IMPEDANCE**

To begin, let us consider the following several definitions of the acoustic impedance. For P-waves at normal incidence, acoustic impedance is broadly known to equal the product of medium density, $\rho$ and wave velocity, $V$:

$$Z = \rho V. \tag{1}$$

However, this is still not a true definition but only the final formula for an isotropic attenuation-free medium. More fundamentally, acoustic impedance is defined as the ratio of pressure, $p$, to particle velocity component normal to the boundary, $u_c$ (Brekhovskikh 1980)

$$Z = \frac{p}{u_c}. \tag{2}$$

Here and below, dots indicate the time derivatives. Definition (2) allows extending impedance (1) to oblique incidence at angle $\theta$:

$$Z = \frac{\rho V}{\cos \theta}. \tag{3}$$

In more complex cases, such as the elastic wavefield with attenuation, a yet more general definition for $Z$ from first principles is required. Such a definition can be derived from the role acoustic impedance plays in elasto- and electro-dynamics, in which $Z$ is constructed so that the reflection coefficient $r_{12}$ from a boundary of two media is solely determined by the ratio of their impedances $Z_1$ and $Z_2$:

$$r_{12} = \frac{Z_2 - 1}{Z_2 + 1}. \tag{4}$$

Note that the well-known identity for the transmission coefficient $t_{12} = 1 - r_{12}$ also follows from the displacement-continuity boundary condition and does not depend on parameters of the two media. Equation (4) also indicates the fundamental scale ambiguity of $Z$, which can be multiplied by an arbitrary complex factor $c$:

$$Z \rightarrow Z' = cZ \tag{5}$$

without altering the observed reflectivity.

Based purely on expression (4), several extensions of the normal-incidence acoustic impedance (1) to oblique incidence were proposed and called elastic impedances (e.g., Connolly 1999; Whitcombe et al. 2002; Ma 2003; Santos and Tygel 2004; VerWest 2004; Martins 2006). However, all of these definitions were based on the scaling ambiguity of impedance (5) and heuristic integrations of the reflectivity time series and did not rigorously represent properties of the medium (for more on this, see Morozov 2010). Such definitions will therefore not be considered here.

Seeking a rigorous definition of the acoustic impedance that would incorporate both oblique incidence and attenuation, we need to look into wave boundary conditions. Note that...
equation (4) follows from solving the boundary conditions on the reflecting interface. In the acoustic case, these conditions require the continuity of normal displacement, $u_z$, and stress, $\sigma_{zz}$, across the interface:

$$
\begin{align*}
\left. u_z \right|_1 &= \left. u_z \right|_2 \\
\left. \sigma_{zz} \right|_1 &= \left. \sigma_{zz} \right|_2,
\end{align*}
$$

where subscripts 1 and 2 indicate the propagating media. From Hooke's law, stress is always proportional to the spatial derivatives of displacement:

$$
\begin{pmatrix}
\sigma_{zz} \\
\sigma_{zx}
\end{pmatrix} = -\tilde{Z}
\begin{pmatrix}
u_z \\
u_x
\end{pmatrix},
$$

where $\tilde{Z}$ is a linear matrix differential operator. For example, for an elastic field, this operator equals

$$
\tilde{Z} = -\begin{pmatrix}
\lambda + 2\mu & \lambda \partial_x \\
\mu \partial_x & \mu \partial_z
\end{pmatrix},
$$

where $\lambda$ and $\mu$ are the Lamé constants of the medium. Expressions (7) and (8) give apparently the most general definition of impedance.

In the acoustic case and for a planar harmonic P-wave, the displacements can be expressed through the component orthogonal to the boundary, $u_z$:

$$
\begin{pmatrix}
u_z \\
u_x
\end{pmatrix} = u_z \begin{pmatrix} 1 \\ \tan \theta \end{pmatrix},
$$

and therefore a scalar impedance relating the normal stress to displacement can be derived from the general matrix form (7)

$$
\tilde{Z}_A = \frac{\sigma_{zz}}{u_z} = \left[-\tilde{Z} \begin{pmatrix} 1 \\ \tan \theta \end{pmatrix}\right]_z.
$$

Since $\sigma_{zz} = -\tilde{Z}_A u_z$ in each of the two media, the reflection amplitude becomes determined entirely by the values of $\tilde{Z}_A|_1$ and $\tilde{Z}_A|_2$. This impedance corresponds (by relation (4)) to the reflection coefficient of the following form:

$$
r_{12} = \frac{u_{z\text{reflected}}}{u_{z\text{incident}}},
$$

which is evaluated in medium 1 (incident).

Thus, for harmonic waves, the impedance can be uniquely defined as a matrix relating the stress and displacement boundary conditions in the incident, reflected, or transmitted waves (equations (7)–(10)). Note that in the acoustic case, definition (2) has practically the same meaning, except that the velocity, $u_z$, is used instead of the displacement, $u_x$, leading to an additional factor $c = -i\omega$, where $\omega$ is the angular frequency. Indeed, in the acoustic case ($\mu = 0$), $Z_A$ in equation (3) is proportional to $\tilde{Z}_A$ in (10) for all angles:

$$
Z_A = \frac{\tilde{Z}_A}{-i\omega}.
$$

Because of scaling invariance (see expression (5)), factor $-i\omega$ in equation (12) is insignificant for describing the reflectivity but it may become important when considering non-stationary waves. However, in the following, we will consider harmonic fields and mostly use the conventional $Z_A$ defined in equations (3) and (12).

**ACOUSTIC IMPEDANCE IN THE PRESENCE OF ATTENUATION**

By solving the wave boundary conditions on a welded contact of two elastic media, Lines et al. (2008) noted that attenuation contrasts cause phase-rotated reflections. These authors used the conventional definition of $Z$ (2) and concluded that contrasts in $Q^{-1}$ can cause reflections in the absence of impedance contrasts. However, Morozov (2009b) pointed out that impedance should depend on attenuation and therefore such reflections could be accounted for by incorporating the $Q$ factor in $Z$:

$$
Z = \rho V \left(1 + \frac{i}{2Q}\right),
$$

and using formula (4).

However, a problem arises when comparing this expression to the correspondence principle, which describes the effect of attenuation by a negative complex argument of the medium velocity (Aki and Richards 2002):

$$
V(\Omega) = V \left(1 - i\frac{\Omega}{2Q}\right).
$$

When used to extend the acoustic impedance relation (1), this formula gives an opposite sign of Im$Z$ compared to the exact expression (13). Thus, the correspondence principle should be used with caution in this type of problem. A detailed discussion of the reason for such a discrepancy and its general solution are given later in this paper.

To derive the correct expression (13) following Lines et al. (2008), consider a plane harmonic P-wave normally-incident on a welded contact of two media (Fig. 2a). In the presence of attenuation, its potential, $\psi(r,t)$, can be written as

$$
\psi(r,t) = A \exp(-i\theta - ikr - \alpha r) \equiv A \exp(-i\theta - ik'r).
$$

where $\omega$ is the circular frequency, $r$ is the coordinate vector, $k$ is the wavenumber vector and $\alpha$ is the attenuation vector. Complex-valued vector $k' = k + i\alpha$ is the effective
wavenumber including both the propagation and attenuation effects. For simplicity, consider a homogeneous wave with parallel \( k \) and \( \alpha \). The spatial attenuation coefficient \( \alpha \) can then be related to \( k \) through the spatial attenuation quality factor \( Q \) as

\[
\alpha = \frac{k}{2Q}.
\]

From the expressions for the P-wave potential (15) in each of the two media, we obtain the corresponding displacements

\[
u_i(r,t) = \partial_t \phi_i(r,t) = ik_i \phi_i(r,t),
\]

velocities

\[
u_i(r,t) = \omega k_i \phi_i(r,t),
\]

strains

\[
\epsilon_{ij}(r,t) = \partial_i \partial_j \phi(r,t) = -k_i k_j \phi(r,t)
\]

and stresses

\[
\sigma_{ij}(r,t) = -\left(\lambda \delta_i j + \mu k_i k_j \right) \phi(r,t),
\]

where \( \lambda \) and \( \mu \) are the Lamé constants, \( \partial_i \) denotes \( \partial / \partial x_i \), and \( \delta_{ij} \) is the Kronecker symbol (unit matrix element). Now let us only consider the normal-incidence case. From expressions (18) and (20), we can relate the traction and velocity boundary conditions by a single factor \( Z \):

\[
\sigma_{zz}(r,t) = -Z \nu_z(r,t),
\]

where \( Z \) becomes the conventional impedance (equation (12))

\[
Z = \frac{(\lambda + 2\mu)k'}{\rho \omega} = \rho V \left( 1 + \frac{i}{2Q} \right),
\]

as in equation (13).

Equation (22) shows that \( Z \) varies with \( Q \) as the wavenumber and not as the complex-valued phase velocity (equation (14)) and hence \( \text{Im}Z \) is positive. In the presence of a contrast in \( Q \), \( r_{12} \) becomes complex-valued and corresponds to phase-shifted reflectivity (Lines et al. 2008). For example, for small contrasts in \( \rho V \) and \( Q^{-1} \), equation (4) gives

\[
r_{12} \approx \frac{\delta \ln Z}{2} \approx \frac{\delta \ln (\rho V)}{2} + i \frac{\delta (Q^{-1})}{4},
\]

showing that reflections from positive attenuation contrasts are delayed in phase by

\[
\delta \phi = \arg(r_{12}) \approx \arctan(\delta (Q^{-1}) / \delta \ln (\rho V) / 2).
\]

Notably, there seem to exist no computer codes for simulating energy dissipation by using the first principles described above and all of the existing codes (known to the author) are based on the visco-elastic model. The popularity of this model can be explained by its simplicity, which arises from describing the energy-dissipation property of the medium by only two parameters (P- and S-wave \( Q^{-1} \)), which are further attributed to the extended elastic moduli or complex-valued velocities. However, physically, this picture cannot generally be true. Many factors control the energy dissipation within the Earth, such as fracturing, fluid content and saturation, viscosity, porosity, permeability, tortuosity, properties of ‘dry’ friction on grain boundaries and faults and distributions of scatterers. Most of these factors are only remotely (if at all) related to the velocity and elastic constants and therefore lumping them all in \( Q_\rho^{-1} \) and \( Q_\omega^{-1} \) (i.e., phase factors of the shear and bulk moduli) should generally be impossible. To fully describe seismic wave attenuation, multi-phase models based on Lagrangian or microscopic mechanics are required, such as Biot’s (1962) theory of saturated porous medium. Note that in this approach, the very notion of \( Q \) as a meaningful and unique medium parameter may become invalid and unnecessary.

Despite the lack of rigorous numerical simulators, the difference between the reflectivities arising from above and
visco-elastic approaches, an approximate solution (equation (23)) can be derived by changing the sign of $\text{Im} r_{12}$ in the corresponding visco-elastic solution. For a single-boundary case, this can be achieved by shifting the phases of the resulting seismograms by $(-2\delta \phi)$. By using this approach, visco-elastic ‘reflectivity’ synthetics in Fig. 1(b,d) were transformed into those shown in Fig. 1(e,f), respectively.

Because the effects of $Q$ on the impedance and reflectivity are relatively weak, they only become pronounced for low $Q$. For example, a contrast from $Q = \infty$ to $Q = 5$ causes normal-incidence reflections of the same amplitudes as a 10% velocity-density contrast (Fig. 1). However, visco-elastic solutions yield clearly phase-advanced reflections in this case (Fig. 1b), whereas our solutions – phase-delayed (Fig. 1e). In particular, reflections from pure attenuation contrasts change polarities when switching from the correspondence-principle to explicit formulation (Fig. 1d,f, respectively).

Note that although the reflections in some cases appear advanced in time (Fig. 1b,d), their causality is violated by neither positive nor negative phase shifts in expressions (22) and (23). For opposite-side reflections (for example, from low-$Q$ to high-$Q$ media) the signs of these phase delays become opposite. Because the phase shifts upon reflection are frequency-independent, the corresponding group velocity delay is $d(\delta \phi/d\omega) = 0$ in both visco-elastic and our theories. Nevertheless, correct signs of phase delays are important for determining the shapes of wavelets reflected from the boundary and more generally, for modelling wave propagation in heterogeneous attenuative media.

**REACTIVE/RESISTIVE PROPERTIES OF ACOUSTIC IMPEDANCE**

As mentioned above, although generalized to the case of attenuative medium, the acoustic impedance in expression (13) still remains somewhat unsatisfactory in the sense that its $\rho V$ part (describing the reflectivity) is real-valued but $\rho V/Q$ (corresponding to energy dissipation) – imaginary. In the electrical circuit theory, from which the concept of impedance has originated, the impedance is usually decomposed in an opposite manner:

$$Z = R + iX,$$

where $R \geq 0$ is the resistance (i.e., energy dissipation) and $X$ is the reactive (energy-conserving) part. A closer analogy to electromagnetic impedance can be obtained by using $\tilde{Z}$ (equation (8)), which relates stress to the displacement. For a P-wave at normal incidence, matrix $\tilde{Z}$ (equation (8)), reduces to a single factor

$$Z = -i(\lambda + 2\mu)k' = -i\omega \rho V_p \left(1 + \frac{i}{2Q_p}\right),$$

where $V_p$ and $Q_p$ are the P-wave velocity and attenuation factor, respectively. Similarly, for an S-wave, the same matrix yields

$$Z = -i\mu k' = -i\omega \rho V_s \left(1 + \frac{i}{2Q_s}\right),$$

with the corresponding $V_s$ and $Q_s$. Such impedances can be interpreted as the effective ‘spring constants’ of the waves acting on the boundary (compare relation (7) to Hooke’s law) and they can also be described as the ‘complex moduli’ of the waves (Bland 1960). Figure 2(b) illustrates this analogy, which is, however, still loose because both the displacements and forces must be considered as additive in the parallel spring arrangement in medium 1. Also, the principal part of this spring constant is imaginary because the phase of traction is shifted by $90^\circ$ relative to the displacement in an elastic wave.

Equation (4) holds with impedance $\tilde{Z}$ similarly to the traditional acoustic impedance. The resistive part of $\tilde{Z}$, $\Re \tilde{Z} = \frac{\omega \rho}{2\mu}$, is positive and corresponds to energy dissipation, similarly to electric resistance. The elastic energy density (Aki and Richards 2002) in our example is

$$E(r, t) = \frac{1}{2} \Re \left[ \sigma^\prime_{rz}(r, t) \sigma^\prime_{rz}(r, t) \right] = \frac{\ln \left[k \tilde{Z}^2\right]}{2} |\mu_r(r, t)|^2,$$

where equations (17), (19) and (20) were used. Note that the energy density is principally controlled by the imaginary (reactive) part of $\tilde{Z}$ and decreases with distance $z$ according to the dissipation law

$$\frac{\partial \ln E(r, t)}{\partial z} = 2 \frac{\partial \ln |\mu_r(r, t)|}{\partial z} = -2\alpha = -\frac{k}{Q},$$

Thus, the quality factor corresponds to a phase shift of $\tilde{Z}$ from a purely reactive impedance: $Q^{-1} = 2 \arg(i\tilde{Z})$ and $Q^{-1}$ measures the relative energy dissipation per one wavelength of the wave travelpath.

In respect to its inversion from reflectivity data, $\tilde{Z}$ is close to the traditional acoustic impedance (equation (1)), which is heavily rooted in seismology and acoustics and embedded in numerous practical applications. Because the frequency dependence of $\tilde{Z}$ does not affect the reflectivity, it should cause no changes in acoustic-impedance inversion approaches. The additional $\arctan(Q^{-1}/2)$ phase factor can be incorporated in acoustic impedance inversion, providing a more accurate treatment of the effects of attenuation. At the same time, as Morozov and Ma (2009) argued, strictly speaking, impedance-type attributes are never inverted for but rather
constructed from the reflectivities by exploiting the scaling ambiguity (equation (5)) and utilizing non-seismic information. Such inversion can be carried out for any attribute satisfying equation (4), including either $\bar{Z}$ or $Z_A$.

**WAVE ATTENUATION AND THE COMPLEX-VALUED ELASTIC MODULI**

The reason for the impedance discrepancy produced by the correspondence principle is that in reality, quantity $V(Q)$ in equation (14) represents the phase velocity and not the material wave speed $V_m$, which is merely a combination of elastic parameters (such as $V_m = \sqrt{(\lambda + 2\mu)/\rho}$ for P-waves). Note that the material and phase velocities equal each other only in a uniform, isotropic, boundless and attenuation-free medium. Phase velocity is related to the complex wavenumber, $k'$, as $V = \omega k'$ and consequently its imaginary part can be viewed as negative when $\text{Im}k' > 0$. A complex-valued material $V_m$ suggests that $\lambda$ and/or $\mu$ also attain negative imaginary parts and indeed, the concept of complex-valued $\mu$ is often used for explaining attenuation (e.g., Anderson and Archambeau 1964; Aki and Richards 2002). However, in heterogeneous structures, a single phase velocity usually corresponds to a distribution of $V_m$ and analytical relations between them can hardly be justified in the general case. Assumptions of such analytical relations may produce elegant but inaccurate or erroneous solutions and inversion methods. For example, surface-wave energy dissipation derived by using the correspondence principle (Anderson, Ben-Menahem and Archambeau 1965) exceeds the cumulative dissipation within the medium by 10–20%, thereby violating the conservation of energy.

In wave mechanics, the description of the displacement field consists of two components: 1) the dynamic principle, which may be given in the form of the Lagrange-Hamilton variational principle or the corresponding wave equations and 2) boundary or radiation conditions. The wave equations are correctly transformed and greatly simplified by the correspondence principle, which was used in most studies of attenuation (e.g., Anderson and Archambeau 1964; Cooper and Reiss 1966; Richards 1984; Borchert and Wennerberg 1985; Aki and Richards 2002). However, when applied to boundary conditions (equation (6)), this transformation makes $\lambda$ and/or $\mu$ in equation (22) complex-valued and the stress,

$$\sigma_{ij} = \lambda \varepsilon_{ikl} \delta_{ij} + 2\mu \varepsilon_{ij},$$

(29)

acquires an additional phase delay $[-2\arctan(Q^{-1/2})]$ relative to the elastic case (as can be seen, for example, from the relation $\lambda + 2\mu = \rho V^2(Q)$ for P-waves). This delay overwhelms the attenuation-related $\arctan(Q^{-1/2})$ phase advance of $\varepsilon_{ij}$ in respect to $u_i$, reverses the phase of acoustic impedance in equation (13) and leads to incorrect phases of reflections from attenuation contrasts. The implied assumption “…it is understood that the boundary conditions for the two problems are identical…” (Bland 1960, p. 96) appears to be unviable with complex-valued elastic moduli.

Thus, boundary conditions show that elastic parameters $\lambda$ and $\mu$, as well as density $\rho$, should remain real-valued in the presence of attenuation. This can also be seen from the Lagrangian of the elastic field (see Bourbié, Coussy and Zinsiger 1987), which remains real-valued when extended to complex-valued $u_i$:

$$L \{u, \dot{u}\} = \int \left( \frac{\rho}{2} \dot{u}_i \dot{u}_i^* - \frac{\lambda}{2} \varepsilon_{ij} \varepsilon_{ij}^* - \mu \varepsilon_{ij} \varepsilon_{ij}^* \right) d^3 r$$

(30)

Here, ‘*’ denotes complex conjugation. Note that the existence of real-valued kinetic and elastic energies represents the most fundamental principle describing the dynamics of any physical system.

Rigorously, the attenuation is described not by modifying $\lambda$ and $\mu$ but by adding an external dissipation force to the right-hand side of Lagrange’s equations (Bourbié et al. 1987; Razavy 2005)

$$\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{u}_i} \right) = \frac{\delta L}{\delta u_i} - \frac{\delta D}{\delta u_i}.$$  

(31)

Here, $D$ is a (usually) second-order functional describing energy dissipation (not to be confused with the ‘specific dissipation function’ $Q^{-1}$ used by Bland (1960), Anderson and Archambeau (1964) and others). Several realistic examples of dissipation functionals for porous fluid-filled rock were given by Bourbié et al. (1987). For a simple example, consider the following hypothetical form of $D$ constructed similarly to the elastic energy but using velocities:

$$D \{\ddot{u}\} = -\int \xi \left( \frac{\lambda}{2} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}^* + \mu \varepsilon_{ij} \varepsilon_{ij}^* \right) d^3 r,$$  

(32)

where $\xi$ is a small attenuation factor measured in time units. With such $D$, the P- and S-waves are not coupled by attenuation and the homogeneous equation of motion (equation (31)) for a P-wave in a uniform medium becomes

$$\ddot{u}_i - V_m^2 \dddot{u}_i - \xi V_m^2 \dot{u}_i = 0.$$  

(33)

From its plane-wave solution propagating in the direction of axis $z$

$$u_i(z, t) = A \exp(-i\omega t + ikz).$$  

(34)
the dispersion relation gives a positive \( \text{Im} k' \), as expected:

\[
k' = \frac{\omega}{V_m \sqrt{1 - i \omega \xi}} \approx \frac{\omega}{V_m} \left( 1 + \frac{i \omega \xi}{2} \right).
\]  

(35)

Comparing this equation to equation (16), we see that in this example, the \( Q \) factor is inversely frequency-dependent:

\[
Q = \frac{1}{\xi \omega}.
\]  

(36)

as in Biot’s theory of saturated porous rock (for an excellent review, see Bourbié et al. 1987). The wave phase velocity can be viewed as complex-valued but different from \( V_m \):

\[
V_{\text{phase}} = \frac{\omega}{k'} = V_m \sqrt{1 - \frac{1}{Q}}.
\]  

(37)

In agreement with the correspondence principle, the last two terms in wave equation (33) can indeed be combined in a complex-valued squared material \( V_m' \):

\[
V_m^2 \to (V_m')^2 = V_m^2 (1 - i \xi \omega),
\]  

(38)

with the corresponding modifications (i.e., added negative imaginary parts) to \( \lambda \) and \( \mu \). However, although this transformation makes \( V_{\text{phase}} \) equal \( V_m' \), it does not extend to the boundary conditions and should not be interpreted literally. As one can see, the correspondence-principle transformation (equation (38)) relates to the properties of the plane-wave solution rather than of the propagating-medium itself.

The sensitivity of the boundary conditions to the correspondence-principle transformation does not only affect the reflection/transmission and acoustic impedance problems considered in this paper. For example, in modelling surface waves in a layered Earth, this transformation applied to layers with contrasting \( Q \) shifts the phases of the stress-related boundary-value equations (see equation (4) in Anderson and Archambeau 1964). Consequently, the eigenfunctions of the wavefield modify similarly to the reflection coefficients in the acoustic impedance problem above. The resulting distributions of the kinetic and potential energies also change, leading to predictions of \( Q^{-1} \) different from those given, for example, for surface waves in Anderson et al. (1965). Thus, it appears that the correspondence principle may only apply to wave-equation solutions in boundless uniform media. Unfortunately, such cases are of little practical utility in seismology.

To conclude this section, another note on physical terminology and semantics is appropriate. The dissipation functional \( D \) contains time derivatives of \( u_t \) and is functionally closer to the kinetic energy than to the elastic-energy part of the Lagrangian (equation (30)). This once again shows that energy dissipation is closer to the kinetic energy than to modifications of the elastic moduli. For example, viscosity forces are proportional to pore-fluid filtration velocities (Biot 1962), similarly to the forces applied to dashpots in equivalent mechanical models (Bland 1960). The elastic energy does not convert to heat directly but only by causing relative particle motions. Dissipation is therefore kinetic by its nature and perhaps could be formally associated with an imaginary part of mass density; however, this analogy is commonly interpreted as ‘imperfect gravity’ and discarded (Anderson and Archambeau 1964). Therefore, the main premise of the currently established attenuation model, namely that the strain energy is dissipated by means of ‘imperfect elasticity’ (e.g., Aki and Richards 2002) appears unfounded. Moreover, \( D \) is added as an external dissipative force in the equations of motion (equation (31)), it is not included in the Lagrangian. Therefore, wave attenuation should also not be viewed as a modification of either the kinetic or potential energies. Complex-valued elastic moduli may be convenient for describing the similarities between solutions to elastic and anelastic wave equations but their rigorous physical meaning still remains to be established and they do always guarantee correct solutions for \( Q^{-1} \) in heterogeneous media.

### MECHANICAL ANALOGUES

For sceptics who may think that the above critique of the complex-moduli concept is based on the definition of acoustic impedance or subtleties of wave processes, let us see how this concept also fails in ordinary mechanics. Consider a linear oscillator with viscous friction proportional to the velocity (Fig. 3a):

\[
\dot{m} \ddot{x} = -m \omega_0^2 x - \xi m \omega_0 \dot{x},
\]  

(39)

where \( x \) is the displacement, \( m \) is the mass, \( \omega_0 \) is the natural frequency and \( \xi \ll 1 \) is the dimensionless dissipation constant, which equals \( 1/Q \) of the oscillator. The spring constant equals \( k = m \omega_0^2 \). The second term in the right-hand side of this equation is the viscous dissipation force.

The general solution to equation (39) is \( x(t) = \text{Re}(A \exp(-i \omega t)) \), where

\[
\omega_0' \approx \omega_0 \left( 1 - \frac{i \xi}{2} \right)
\]  

(40)

is the ‘complex frequency’ and \( A \) is an arbitrary complex-valued amplitude. One can introduce a complex-valued spring constant (or also mass)

\[
k' = m \omega_0'^2 \approx k(1 - i \xi),
\]  

(41)
and equation (39) takes the form of a free-oscillator equation
\[ m\ddot{x} = -m\omega_0^2 x. \] (42)

If an external force \( f(t) \) is added to the right-hand side of equation (39), the solution can also be derived from its Fourier transform \( f(\omega) \) by using the complex natural frequency only:
\[ A(\omega) = \frac{f(\omega)}{m(\omega_0^2 - \omega^2)}. \] (43)

Thus, in the mathematical world of complex-valued solutions to the homogeneous equation (39) (i.e., the ‘relaxation spectrum’), the correspondence principle describes the solutions very effectively.

However, considering the expressions for the force in the real world, the limitations of this description become apparent. First, equation (41) attributes a property of the dashpot \( \xi \) to the spring. This is possible only in retrospect, because of the simplicity of the particular homogenous solution (equation (40)) and only for an equivalent model for which we have exactly one dashpot per spring. The elastic force of the spring is mixed with the attenuation and cannot be obtained from the equivalent equation (42). Second, the elastic force equals \((-kx)\), i.e., it is always at a 180° phase relative to \( x(t) \) and not simply related to the ‘complex force’ \( F = -k' Ae^{-i\omega t} \). The complex force has an additional phase advance of about \( \xi \) in respect to \( x(t) \). Nevertheless, the force is an easily measurable physical quantity (e.g., by measuring the extension of the spring) and represents the key element of most boundary conditions. Visco-elastic boundary conditions are based on equating the complex forces (equation (42)), which in fact represent the inertial forces, instead of the elastic forces. This substitution may lead to phase-shifted solutions that have no counterparts in the real world.

For more complex systems, such as consisting of, for example, two dashpots with three springs (Fig. 3b), it is unclear how to distribute the attenuation parameters among the spring constants. As shown in the Appendix, by solving the dynamic equations, one can derive the normal oscillation modes corresponding to the two degrees of freedom and the corresponding natural frequencies can again be modified to incorporate the attenuation. However, these parameters only describe the oscillatory solutions rather than real mechanical parameters of the system and do not simply extend the spring parameters \( k_1, k_2 \) and \( k_3 \) into the complex plane. This situation is analogous to the complex-valued phase velocities differing from the real-valued material velocities in heterogeneous media.

Finally, as in the wave examples above, a rigorous formulation of attenuation can be obtained without the complex-valued \( k' \), by using the following Rayleigh dissipation function in Lagrange’s equations (Razavy 2005):
\[ D = \xi \frac{m\omega_0}{2} x^2. \] (44)

Note that the dissipation is associated with the single dashpot and not the whole oscillator (Fig. 3a) and it has no relation to \( k \). This expression can be naturally extended to any mechanical systems, such as shown in Fig. 3(b).

**CONCLUSIONS**

Three groups of conclusions can be drawn from the above analysis. First, unlike often assumed, the acoustic impedance depends on the attenuation, although its relative effect, (which equals \( i\omega Q \)), is small in typical Earth media. Similarly to the acoustics and electrical circuit theory, reflection coefficients can always be derived exclusively from impedance contrasts. A generalized form of the anelastic acoustic impedance, denoted \( \tilde{Z}_A \), is proposed, which represents a fundamental relation between the displacement and traction boundary conditions. The classic acoustic impedance corresponds to the scaled and phase-rotated new impedance: \( Z_A = i\tilde{Z}_A/\omega \). In many practical applications, \( \tilde{Z}_A \) and \( Z_A \) can be used...
interchangeably. However, the advantage of $\tilde{Z}_A$ is in its rigorous form and extendibility to general cases of attenuation and P/S mode conversions, in which case the impedance becomes of matrix form, $\tilde{Z}$ (Morozov 2010). In the presence of attenuation, the imaginary part of $Z_A$ is positive, which explains the phase-shifted reflections from attenuation contrasts pointed out by Lines et al. (2008) but contradicts the correspondence principle.

Second, the correspondence principle is rigorously applicable only to the cases of boundless uniform media. This principle implicitly substitutes properties of a wave, such as its logarithmic amplitude decays or $Q$, for the properties of the medium. Such substitution should generally be invalid in a heterogeneous medium. When used to infer the acoustic impedance in the presence of attenuation, the correspondence principle leads to erroneous signs of $\arg(Z)$ and phases of reflections. Therefore, the recommended approach is to use the Lagrangian mechanics or explicit equations of motion and boundary conditions while keeping the density and elastic constants real. In a potentially useful analogy, the new impedance $\tilde{Z}$ can be interpreted as the effective ‘spring constant’ of the wave at the boundary. With such interpretation, the process of wave reflection and transmission can be visualized as the state of equilibrium of three welded springs.

Third, the alternate definition of $\tilde{Z}_A$ leads to an improved analogy with electromagnetism and wave mechanics than the conventional acoustic impedance while retaining its familiar relation to reflectivity. $\tilde{Z}_A$ associates wave impedance with inductance or capacitance rather than with resistance and removes the awkward association of $Q$ with reactive response.

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APPENDIX A. NORMAL MODES OF THE MECHANICAL MODEL IN FIG. 3(b)

In linear elasticity, viscous forces are attributed to the dashpots in Fig. 3(b) and the equation of motion for the two masses are:

\[-m_1 \ddot{x}_1 - k_1 x_1 + \xi_1 (\dot{x}_{1,D} - \dot{x}_1) = 0, \quad (A1a)\]
\[-m_2 \ddot{x}_2 - k_2 (x_2 - x_{1,D}) - \xi_2 \dot{x}_2 - k_3 x_2 = 0, \quad (A1b)\]

where \(x_{1,D}\) denotes the coordinate of the cup of dashpot 1. Because the cup is considered massless, its viscous force is compensated by the elastic force of the second spring:

\[\xi_1 (\dot{x}_{1,D} - \dot{x}_1) = k_2 (x_2 - x_{1,D}). \quad (A1c)\]

For harmonic oscillations with frequency \(\omega\) and \(\omega \xi_{1,2} \ll 1\), equation (A1c) gives

\[x_{1,D} \approx x_2 \left(1 + \frac{i \omega \xi_1}{k_2}\right) - \frac{i \omega \xi_1}{k_2} x_1, \quad (A2)\]

and the system of homogeneous equation (A1) becomes:

\[x_1 (m_1 \omega^2 - k_1 + i \omega \xi_1) + x_2 (-i \omega \xi_1) = 0, \quad (A3a)\]

\[x_1 (-i \omega \xi_1) + x_2 (m_2 \omega^2 - k_3 + 2 i \omega \xi_2) = 0. \quad (A3b)\]

If \(\xi_1 = \xi_2 = \tilde{\xi}\), the two masses are decoupled (Fig. 3b) and the corresponding normal frequencies from equation (3) are \(\omega_{0,1} = \sqrt{k_1/m_1}\) and \(\omega_{0,2} = \sqrt{k_3/m_2}\), respectively. In the presence of weak dissipation, by perturbing the frequency by \(\delta \omega, \omega \to \omega + \delta \omega\) in the determinant of the linear system (equation (A3)) and keeping only linear terms in \(\xi\), we obtain:

\[(i \omega \xi_1 + 2 m_1 \omega \delta \omega) (m_2 \omega^2 - k_3) + (m_1 \omega^2 - k_1) (2 i \omega \xi_2 + 2 m_2 \omega \delta \omega) = 0. \quad (A4)\]

From this equation, the shifts of each of the two normal frequencies \(\omega_{0,n} (n = 1, 2)\) are

\[\delta \omega_n = -\frac{i}{2} \left[\frac{\xi_1 (m_2 \omega_{0,n}^2 - k_3) + 2 \xi_2 (m_1 \omega_{0,n}^2 - k_1)}{m_1 (m_2 \omega_{0,n}^2 - k_3) + m_2 (m_1 \omega_{0,n}^2 - k_1)}\right]. \quad (A5)\]

As expected, these frequency shifts are negative and imaginary, in agreement with the \(\exp([\text{Im} \omega]t)\) attenuation law. However, they are clearly associated only with the normal oscillation modes and can hardly be meaningfully attributed to the three spring constants.