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Exact Elastic *P*/*SV* impedance

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Abstract

Several extensions of the concept of Acoustic Impedance to oblique incidence exist and are known as Elastic Impedances (EI). These quantities are constructed by heuristic integrations of reflectivity series but still involve approximations and do not represent a unique medium property. Nevertheless, for unambiguous interpretation, it is desirable to have an EI that would: 1) be a mechanical property of the medium and 2) yield exact reflection coefficients at all angles of incidence. Here, such a definition is given for *P* and/or *SV* wave propagation in an arbitrary isotropic medium. The exact elastic *P*/*SV* impedance is a matrix quantity and represents the differential operator relating the stress and strain boundary conditions. With the use of the matrix form of the reflectivity problem, no approximations are required for accurate modeling of reflection (*P*/*P* and *SV*/*SV*) and mode-conversion (*P*/*SV* and *SV*/*P*) coefficients at all angles and for any contrasts in elastic properties. The matrix EI can be computed from real well logs and inverted from ray-parameter dependent seismic reflectivity. Known limiting cases of *P*- and *S*-wave acoustic impedances are accurately reproduced, and the approach also allows the extension of the concept of impedance to attenuative medium. The matrix impedance readily lends itself for inversion, with uncertainties typical of the standard acoustic-impedance inversion problem.

Introduction

Impedance is the key quantity used for characterizing reflection and transmission of seismic waves. For *P* waves at normal incidence, the Acoustic Impedance (AI) is usually taken as the product of density (ρ) and velocity (V) of the medium

$$AI = \rho V \,. \tag{1}$$

Several extensions of this formula to oblique incidence were proposed. In these extensions, the impedance was invariably constructed as a quantity Z such that the reflection coefficient $r_{1,2}$ from a boundary of two media was solely determined by the ratio of their impedances, $z_{2,1} = Z_2/Z_1$

$$r_{1,2} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{z_{2,1} - 1}{z_{2,1} + 1} \,. \tag{2}$$

Consequently, impedance is always defined with an uncertainty of an arbitrary scalar, which is usually selected so that Z reduces to AI at normal incidence.

All impedance measures proposed to date attempt achieving two goals: 1) inverting the series of reflectivity values (equation 2) for a Z-series and 2) giving Z the sense of an intrinsic elastic property of the propagation medium. The first of these goals is easily achieved by using one of the following approximations for impedance contrast

$$z_{i+1,i} = \frac{1+r_{i,i+1}}{1-r_{i,i+1}} \approx 1 + 2r_{i,i+1} \approx \exp\left(2r_{i,i+1}\right), \tag{3}$$

and recursive multiplication

$$Z_i = Z_0 \prod_{k=1}^{i} z_{k,k-1} \,. \tag{4}$$

However, the second goal is unachievable in such inversion, because elastic *P*/*SV* wave reflection and conversion cannot be rigorously described by a single scalar parameter. In addition, the reflectivity-series integration (equations 3 and 4) may be unstable (e.g., Francis, 1997) and often requires regularization. Reflectivities only constrain the ratios of impedances across each interface, and cumulative values of *Z* resulting from multiplying hundreds of factors in equation 4 may drift in complex ways depending on the approximations, data and numerical noise, and also the algorithms employed. This drift may include variations with depth and incidence angles, which may be difficult to control. The variety of the proposed forms of Elastic Impedance (EI) outlined in the following section illustrates this uncertainty.

The debate of the differences between the various EI approximations may lie largely within the uncertainties of such *ad hoc* integration of reflectivities. All of the existing EI expressions rely on assumed parameter correlations and do not correspond well to real observation environments, in which deviations from background trends are typically of primary interest. Thus, the two fundamental problems of such approaches are: 1) deriving the impedance from reflectivities, whereas it should actually arise from the boundary conditions and only *lead* to reflectivities (e.g., VerWest et al., 2000), and 2) in treating the EI as a scalar quantity, whereas for the welded-interface boundary conditions, the impedance should be a matrix incorporating both *P* and *SV* waves simultaneously.

Nevertheless, in this paper, I show that reflectivity at any incidence angle can be readily presented in a rigorous impedance form by emphasizing the two physical principles above. Although the new expression for EI and the reflectivity values (equation 2) are of matrix form, they still can be easily numerically evaluated and inverted, similarly to the existing heuristic EI's and AI. Because of its rigorous nature, the new matrix EI is free from any assumptions and approximations, leads to accurate reflection/conversion equations at all incidence angles, and is unambiguously related to mechanical properties of the medium.

A striking general consequence from the realization that EI should be an elastic property of the rock is that along with the ray parameter, it should depend on only two elastic parameters (for example, products λk and μk of the Lamé parameters λ and μ and the vertical wavenumber k) and not three (V_P, V_S, and ρ) as in all existing EI definitions. The density (ρ) is a part of neither the Hooke's law nor the boundary conditions. Clearly, finding a two-parameter dependence from the data should be a much easier task, and the results should also be more reliable and portable. Below, after a brief discussion of what appears to be the common weaknesses of the existing EI measures, I present a new, "substantive" EI definition, in the form of a matrix function of *AI* and the V_S/V_P ratio, and compare it to several known impedance solutions.

Existing EI measures

Connolly (1999) introduced the initial concept of EI which is currently in broad use. His EI describes the dependence of *P*-wave reflectivity on the incidence angle, θ

$$EI\left(\rho, V_P, V_S \mid \theta\right) = \rho V_P \rho^{-4K\sin^2\theta} V_P^{\tan^2\theta} V_S^{-8K\sin^2\theta} \,, \tag{5}$$

where V_P and V_S are the *P*- and *S*-wave velocities, respectively, ρ is the density, and $K = (V_S/V_P)^2$, which is assumed to be constant across the reflecting boundary. However, this K = const assumption is difficult to justify in cases of practical interest. Another serious problem with EI is that its dimensionality does not correspond to impedance and varies with θ. Variation ranges of EI with θ also strongly depend on the measurement units (e.g., VerWest et al., 2000), and therefore it cannot be considered a constitutive property of the medium.

To work around the dimensionality problem, Whitcombe (2002) proposed a normalized EI, which is the EI divided by itself measured at some reference level and scaled by AI

$$EI_N(\rho, V_P, V_S \mid \theta) = \rho V_P \left(\frac{\rho}{\rho_0}\right)^{-4K\sin^2\theta} \left(\frac{V_P}{V_{P,0}}\right)^{\tan^2\theta} \left(\frac{V_S}{V_{S,0}}\right)^{-8K\sin^2\theta}. \tag{6}$$

Martins (2006) further extended this concept to the cases of weak anisotropy, and Mallick (2007) studied its stability in respect to tuning and the effects of multiples. Whitcombe et al. (2002) also proposed an extended EI by modifying certain functional dependencies in two-term EI in expression 6 and designing empirical rules for highlight the contributions of fluids and lithology in it. However, despite its many useful applications, EI still hardly corresponds to a mechanical property of the medium, first because of is relative character and also because the integration of reflectivity is carried out along non-physical paths of constant θ.

To relax the above limitations, Ma (2003) proposed the Ray-path EI (RI), which was constructed by integrating the reflectivity along an actual reflection path with ray parameter (moveout) p

$$RI(\rho, V_P, V_S \mid p) = \frac{\rho V_P}{\sqrt{1 - V_P^2 p^2}} (1 - V_S^2 p^2)^{2(\gamma+2)}. \tag{7}$$

In this expression, the V_S/V_P ratio no longer needed to be constant; however, the density was taken to be functionally related to V_S, $\rho = \rho_0 V_S^{\gamma}$ (Potter et al., 1998), and small conversion angles were also assumed, $pV_S << 1$. Santos and Tygel (2004) gave another form of RI

$$RI(\rho, V_P, V_S \mid p) = \frac{\rho V_P}{\sqrt{1 - V_P^2 p^2}} \exp\left[-2(\gamma+2) V_S^2 p^2\right]. \tag{8}$$

At $pV_S << 1$, these approximations are equivalent and correspond to Bortfeld's (1961) formula, but they are differently extrapolated to greater p values. VerWest et al. (2000) and VerWest (2004) also proposed two similar RI forms, such as

$$RI(\rho, V_P, V_S \mid p) = \frac{\rho V_P}{\sqrt{1 - V_P^2 p^2}} \exp\left[-4 V_S^2 p^2 \frac{\rho}{\rho_0}\right]. \tag{9}$$

Note that although closer than EI to describing the reflectivity along an actual reflection path, the RI in forms 7-9 still falls short of representing an intrinsic rock property. Indefinite integrals included in the derivation of RI (e.g., Santos and Tygel, 2004; VerWest, 2004) contain arbitrary integration constants, which may be ray-parameter-dependent. Therefore, similarly to the normalized EI in equation 6, expressions 7-9 only approximate the impedance ratios between different depth levels, for example:

$$\frac{RI(\rho_1, V_{P,1}, V_{S,1} \mid p)}{RI(\rho 2, V_{P,2}, V_{S,2} \mid p)} = \frac{\rho_1 V_{P,1}\sqrt{1 - V_{P,2}^2 p^2}}{\rho_2 V_{P,2}\sqrt{1 - V_{P,1}^2 p^2}} \exp\left[-2(k+2)\left(V_{S,1}^2 - V_{S,2}^2\right)p^2\right], \tag{10}$$

in which a path with $\gamma =$ const is expected to be followed. Without such a trend provided by at least one reference state, parameter γ in EI/RI equations 6-10 is undefined.

In summary, the existing EI and RI definitions provide good (particularly RI) descriptions of relative P-wave reflectivity properties between different depth levels. However, these descriptions are limited to stringent assumptions about correlated V_P, V_S, and ρ variations which may be difficult to justify. Thus, it appears that it would be highly desirable to find a truly intrinsic rock property that could accurately describe both P- and S-wave reflectivities and could be invertible and interpretable in the same way as impedance normally is. Such a property is proposed in the following sections.

P/SV impedance matrix

To derive the general P/SV impedance, first consider an elastic wavefield in the vicinity of a planar boundary separating two half-spaces with uniform elastic properties (Figure 1). From two components of displacement, the non-zero components of strain can be expressed as

$$\begin{pmatrix} \varepsilon_{zz} \\ \varepsilon_{zx} \\ \varepsilon_{xx} \end{pmatrix} = \begin{pmatrix} \partial_z & 0 \\ \frac{1}{2}\partial_x & \frac{1}{2}\partial_z \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} u_z \\ u_x \end{pmatrix}. \tag{11}$$

From the Hooke's law, the stress tensor is given by $\sigma_{ij} = \lambda\varepsilon_{\kappa\kappa}\delta_{ij} + 2\mu\varepsilon_{ij}$, where λ and μ are the Lamé constants, and δ_{ij} is the Kronecker symbol. Therefore, the two components of traction on the boundary between the two half-spaces are

$$\begin{pmatrix} \sigma_{zz} \\ \sigma_{zx} \end{pmatrix} = \begin{pmatrix} (\lambda + 2\mu)\partial_z & \lambda\partial_x \\ \mu\partial_x & \mu\partial_z \end{pmatrix} \begin{pmatrix} u_z \\ u_x \end{pmatrix}. \tag{12}$$

From this expression, the impedance operator relating the displacement components to stress can be defined as

$$\tilde{\mathbf{Z}} = -\begin{pmatrix} (\lambda + 2\mu)\partial_z & \lambda\partial_x \\ \mu\partial_x & \mu\partial_z \end{pmatrix}. \tag{13}$$

As shown below, the impedance matrix $\tilde{\mathbf{Z}}$ allows solving both the stress and displacement boundary conditions in terms of $\tilde{\mathbf{Z}}$ alone, similarly to equation 2. In the absence of attenuation, this impedance is reactive (i.e., imaginary), which makes it a correct analog to the electric-line impedance. However, in acoustics, the conventional definition of impedance is a real-valued scalar quantity Z, which is the ratio of normal stress to particle velocity (Aki and Richards, 2002). Such impedance could be described as "resistive," which is a somewhat poor analogy for propagating-wave impedance. Nevertheless, in a harmonic wave [$u_{z,x} \propto \exp(-i\omega t)$], the velocity is proportional to the displacement, and therefore the conventional impedance matrix $\mathbf{Z}$ can be defined as a frequency-scaled "reactive" $\tilde{\mathbf{Z}}$

$$\mathbf{Z} = \frac{i\tilde{\mathbf{Z}}}{\omega}. \tag{14}$$

This operator reduces to the standard impedance formulas in several known limiting cases given below.

To examine some examples, consider a plane wave incident on a boundary between two elastic media, with wave displacement $u_{z,x} \propto \exp(-i\omega t + ik\cos\theta z + ik\sin\theta x)$, where k is the wavenumber , and θ is the incidence angle (Figure 1). In this case, the impedance operator reduces to matrix multiplication

$$\mathbf{Z} = \frac{\lambda k}{\omega}\begin{pmatrix} (1+2\mu')\cos\theta & \sin\theta \\ \mu'\sin\theta & \mu'\cos\theta \end{pmatrix}, \tag{15}$$

where $\mu' = \mu/\lambda$. Note that density (ρ) is implicitly present in this equation through the dependence of k on ω. Also, from Snell's law and stationarity of the wave equation, parameters $\zeta = k\sin\theta$ and ω are common to all converted and reflected wave modes.

For a P-wave at normal incidence ($\theta = 0$), the impedance given by equation 15 becomes

$$\mathbf{Z} = \frac{\lambda k}{\omega}\begin{pmatrix} 1+2\mu' & 0 \\ 0 & \mu' \end{pmatrix} = \rho V_P \begin{pmatrix} 1 & 0 \\ 0 & \frac{V_S^2}{V_P^2} \end{pmatrix}, \tag{16}$$

whose first element corresponds to the conventional P-wave impedance. If attenuation is present (i.e., $u_{z,x} \propto \exp(-i\omega t + ikz - kz/2Q_P)$), equation 13 yields (Morozov, 2009)

$$\mathbf{Z} = \rho V_P \begin{pmatrix} \left(1+\frac{i}{2Q_P}\right) & 0 \\ 0 & \frac{V_S^2}{V_P^2}\left(1+\frac{i}{2Q_P}\right) \end{pmatrix}. \tag{17}$$

In the acoustic limit ($\mu = 0$) with no attenuation but at an arbitrary incidence angle θ

$$\mathbf{Z} = \begin{pmatrix} \rho V_P\cos\theta & \rho V_P\sin\theta \\ 0 & 0 \end{pmatrix}. \tag{18}$$

Since in this case, $u_x/u_z = \tan\theta$, the conventional (scalar) acoustic impedance represents the ratio of pressure to velocity perpendicular to the boundary

$$Z_{AI}(\theta) = \frac{\sigma_{zz}}{i\omega u_z} = \left[\mathbf{Z}\begin{pmatrix} 1 \\ \tan\theta \end{pmatrix}\right]_z = \frac{\rho V_P}{\cos\theta}, \tag{19}$$

which is the well-known AI expression valid at all angles (Brekhovskikh, 1980).

The simple matrix impedance formulas 13 and 14 rigorously cover all cases of P/SV wave propagation at any incidence angles and even for non-planar waves. Therefore, approximations 5-9 may be unnecessary; however, in order to utilize the matrix forms of **Z**, we need to see how the reflectivity formula 2 changes in the case of oblique, P/SV wave incidence.

P/SV reflectivity in matrix form

Here, I use the scattering-matrix approach by Aki and Richards (2002) to show that all formulas for reflection and transmission coefficients can be expressed in terms of impedance matrices of the type given by equation 13. Consider an incident, reflected, and transmitted *P* and/or *SV*-wave fields (subscripts '*inc*', '*r*' and '*t*', respectively), with the following components of displacements

$$\begin{pmatrix} u_z \\ u_x \end{pmatrix}_{inc} = \mathbf{U}_{1,+}\begin{pmatrix} A_{inc} \\ B_{inc} \end{pmatrix}, \quad \begin{pmatrix} u_z \\ u_x \end{pmatrix}_{r} = \mathbf{U}_{1,-}\begin{pmatrix} A_{r} \\ B_{r} \end{pmatrix}, \quad \begin{pmatrix} u_z \\ u_x \end{pmatrix}_{t} = \mathbf{U}_{2,+}\begin{pmatrix} A_{t} \\ B_{t} \end{pmatrix}, \tag{20}$$

where subscripts m = 1 and 2, indicate the corresponding medium (Figure 1). In these expressions, the downward and upward "propagator" matrices $\mathbf{U}_{m,+}(z,x,t)$ and $\mathbf{U}_{m,-}(z,x,t)$ for these plane waves are

$$\mathbf{U}_{m,\pm}(z,x,t) = e^{-i\omega t + i\varsigma x}\begin{pmatrix} \pm\cos\theta_{P,m} e^{\pm ik_{P,m}\cos\theta_{P,m} z} & \mp\sin\theta_{S,m} e^{\pm ik_{S,m}\cos\theta_{S,m} z} \\ \sin\theta_{P,m} e^{\pm ik_{P,m}\cos\theta_{P,m} z} & \cos\theta_{S,m} e^{\pm ik_{S,m}\cos\theta_{S,m} z} \end{pmatrix}, \tag{21}$$

according to the standard *P*- and *S*-wave polarity convention (Aki and Richards, 2002). Here, $k_{P,S}$ denote the *P*- and *S*-wave wavenumbers, respectively. From equations 12 and 13, the corresponding stress components for each of these waves (labeled '*w*') become

$$\begin{pmatrix} \sigma_{zz} \\ \sigma_{zx} \end{pmatrix}_{w} = -\widetilde{\mathbf{Z}}\begin{pmatrix} u_z \\ u_x \end{pmatrix}_{w} = -\widetilde{\mathbf{Z}}\mathbf{U}\begin{pmatrix} A_{w} \\ B_{w} \end{pmatrix}. \tag{22}$$

On the welded boundary of the two media, both components of the displacement and stress are continuous

$$\left.\begin{pmatrix} u_z \\ u_x \\ \sigma_{zz} \\ \sigma_{zx} \end{pmatrix}\right|_1 = \left.\begin{pmatrix} u_z \\ u_x \\ \sigma_{zz} \\ \sigma_{zx} \end{pmatrix}\right|_2, \tag{23}$$

which can be written as

$$\begin{cases} \mathbf{U}_{1,+}\begin{pmatrix} A_{inc} \\ B_{inc} \end{pmatrix} + \mathbf{U}_{1,-}\begin{pmatrix} A_{r} \\ B_{r} \end{pmatrix} = \mathbf{U}_{2,+}\begin{pmatrix} A_{t} \\ B_{t} \end{pmatrix}, \\ \mathbf{S}_{1,+}\begin{pmatrix} A_{inc} \\ B_{inc} \end{pmatrix} + \mathbf{S}_{1,-}\begin{pmatrix} A_{r} \\ B_{r} \end{pmatrix} = \mathbf{S}_{2,+}\begin{pmatrix} A_{t} \\ B_{t} \end{pmatrix}, \end{cases} \tag{24}$$

with

$$\mathbf{S}_{m,\pm} = \widetilde{\mathbf{Z}}_m \mathbf{U}_{m,\pm}\Big|_{x=0} = -ie^{i\varsigma x}\begin{pmatrix} k_{P,m}\left(\lambda_m + 2\mu_m\cos^2\theta_{P,m}\right) & -\mu_m k_{S,m}\sin 2\theta_{S,m} \\ \pm\mu_m k_{P,m}\sin 2\theta_{P,m} & \pm\mu_m k_{S,m}\cos 2\theta_{S,m} \end{pmatrix}, \tag{25}$$

where all matrices are taken at z=0, and subscripts m = 1 and 2 indicate the elastic parameters in the corresponding media. The solution to equations 24 is

$$\begin{cases} \begin{pmatrix} A_r \\ B_r \end{pmatrix} = \mathbf{R} \begin{pmatrix} A_{inc} \\ B_{inc} \end{pmatrix}, \\ \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \mathbf{T} \begin{pmatrix} A_{inc} \\ B_{inc} \end{pmatrix}. \end{cases} \tag{26}$$

where the reflectivity and transmission matrices are, respectively

$$\mathbf{R} = -\left[\mathbf{S}_{2,+}\mathbf{U}_{2,+}^{-1}\mathbf{U}_{1,-} - \mathbf{S}_{1,-}\right]^{-1}\left(\mathbf{S}_{2,+}\mathbf{U}_{2,+}^{-1}\mathbf{U}_{1,+} - \mathbf{S}_{1,+}\right), \text{ and} \tag{27}$$

$$\mathbf{T} = \mathbf{U}_{2,+}^{-1}\left(\mathbf{U}_{1,+} + \mathbf{U}_{1,-}\mathbf{R}\right). \tag{28}$$

These formulas give the "propagator" form (e.g., Aki and Richards, 2002) for all reflection, transmission, and mode conversion amplitudes for both *P*- and *S*- incident waves from medium 1.

Considering that matrices $\mathbf{S}_{m,\pm}$ are proportional to the corresponding $\mathbf{Z}_{m,\pm}$, note the similarity of equations 27-28 to the normal-incidence reflectivity by equation 2. This expression is independent of the ambiguity of impedance scaling, and it is the same for both "reactive" and "resistive" impedance definitions in equations 13 and 14.

Despite their matrix form, reflectivity formulas 26 are only slightly more difficult to evaluate numerically (for example, by using Matlab) than expression 2. At the same time, the effort invested in 2×2 matrix computations pays off by removal of all assumptions and approximations, differences between the various EI models, and the relative and path-dependent character of EI. Approximate solutions may still arise from insufficient data when, for example, *S*-wave or density information is not available. However, in such cases, the approximations can still be made explicit, such as the use of the Gardners' formula for density or assuming a depth-variant V_P/V_S, as in equations 5-9. Such approximations can be analyzed and justified separately, and not embedded in the EI definition.

Inversion

As a complete description of the medium transmission/reflection properties, the impedance matrix is a function of all three of its elastic parameters: λ, μ, and ρ (equation 15, with ρ entering through the k/ω ratio). If we have a matrix **R** combining all three recorded reflected and converted amplitudes from both *P*- and *S*- incident waves, equation 27 can be inverted for $\mathbf{S}_{2,+}$ (in terms of its parameters λ_2, μ_2, and ρ_2)

$$\mathbf{S}_{2,+} = \left[\mathbf{U}_{2,+}^{-1}\mathbf{U}_{1,-} + \mathbf{U}_{2,+}^{-1}\mathbf{U}_{1,+}\mathbf{R}\right]^{-1}\left(\mathbf{S}_{1,-}\mathbf{R} + \mathbf{S}_{1,+}\right), \tag{29}$$

yielding a recursive matrix inversion formula for $\tilde{\mathbf{Z}}$ that extends the scalar equation 4.

However, all *P*, *S*, and *P*/*S* wave reflection coefficients are rarely available, and therefore the formal matrix inverse 29 at oblique incidence may be impractical. A practical inversion procedure for **Z** consists of solving the forward equations 27 given the recorded *P*/*P*, *SV*/*SV*, and/or *P*/*SV* reflection amplitudes at several values of ray parameter for the elastic parameters of the medium. Therefore, generally, **Z** can be derived by (λ, μ, ρ) (LMR) inversion.

Three-parameter LMR curve fitting may be prone of instabilities (e.g., Mallick, 2007), and the impedance property **Z** is particularly valuable for indicating the stable combinations of these parameters. Thus, equation 15 shows that **Z** is a function of λk, μ', and θ. The sensitivity of *P/P* reflection amplitudes to the values of θ in the second medium is quite weak below ~35°

incidence, and for *P/SV* amplitudes – below ~25° (Figure 2). Note that using the correct Snell's law for the values of $\theta_{P,\,S,\,2}$ (Figure 1) is not important in the expressions for **Z** at such angles. Therefore, **Z** is principally a function of parameters λk and μ', or equivalently, of *AI* and μ'. For the medium beneath the reflecting boundary, these parameters can be estimated by fitting the reflection amplitudes at a range of incidence angles. For example, considering the two amplitudes shown by gray dots in Figure 2a as *P/P* reflectivity data, we can reconstruct the *AI* and μ' values of the underlying layer (Figure 3), and thereby reproduce the entire **Z** matrix. Notably, this matrix would also describe the effects of *P*/*SV*, *SV*/*SV*, and *SV*/*P* reflections and transmissions. Also, from the shapes of data misfit contours in Figure 3, note that a ~1% uncertainty in AI corresponds to ~10% uncertainty in μ', with a moderate positive correlation of these uncertainties.

Thus, similarly to the AI, matrix *P*/*SV* impedance is invertible from seismic data but requires AVO information. Similarly to AI, the single-boundary inversion step shown in Figure 3 should be continued recursively, producing a matrix EI column from AVO or *P*/*S* reflectivity data. Similarly to AI, this recursive inverse should suffer from instabilities caused by the lack of low-frequency information, noise, and also by the fundamental scale ambiguity of the impedance. Note that the inversion is conducted synchronously for all ray parameter values, i.e., the entire reflection/conversion AVO curves are inverted for **Z** at each depth level. To correlate the results to a well log, one would need to estimate its *AI* and μ' values and compute the components of **Z** (equation 15) instead of using the approximate formulas 5-9. More detail and examples of such inversion using real data will be given elsewhere.

Elastic impedance

In this section, we compare the EI in matrix forms 13-15 to the approximate forms 5-9. Matrix **Z** in equation 15 depends on the velocity/density structure and on the direction of wave propagation, but its effects on different types of waves are different. For a *P*-wave with wavenumber $k = \omega/V_P$ propagating at incidence angle θ, **Z** becomes (equations 15, 21)

$$\mathbf{Z}_P = \rho V_P \begin{pmatrix} \cos\theta & \left(1 - \dfrac{2V_S^2}{V_P^2}\right)\sin\theta \\ \dfrac{V_S^2}{V_P^2}\sin\theta & \dfrac{V_S^2}{V_P^2}\cos\theta \end{pmatrix}, \tag{30}$$

and for an incident *S* wave ($k = \omega/V_S$)

$$\mathbf{Z}_S = \rho V_S \begin{pmatrix} \dfrac{V_P^2}{V_S^2}\cos\theta & \left(\dfrac{V_P^2}{V_S^2} - 2\right)\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{31}$$

Compared to the approximate *P*-wave RI (equations 5-9), the dependencies of the resulting matrix EI components on θ are quite simple (see Figure 4). Note that the vertical-component part $Z_{zz,z}$ is generally close to RI at all angles (Figure 4), although all components of **Z** are needed to accurately describe the reflection AVO effects. Comparison to EI attributes (equations 5 and 6) cannot be made, because they do not represent single-medium properties. Figure 5 shows logs of the components of matrix **Z** (equation 30) computed in a real well from CREWES Blackfoot project (CREWES, 2004). Note the similarity of the first component ($Z_{zz,z}$) to other impedance measures, and the differences between the four components of **Z**.

Another interesting scalar impedance-type property can be constructed from matrix **Z**

(see equation 19)

$$Z_P(\theta) = \left[\mathbf{Z} \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \right]_z = \frac{\rho V_P}{\cos\theta} \left[1 - \frac{2V_S^2}{V_P^2} \sin^2\theta \right] = \frac{\rho V_P}{\sqrt{1-(pV_P)^2}} \left[1 - 2(pV_S)^2 \right] \quad . \quad (32)$$

Its meaning is the ratio of the normal traction to normal displacement in the incident *P* wave alone, i.e. what could be called the "*P*-wave-alone" impedance, i.e., the response of the boundary to the incident *P* wave. This quantity corresponds to the acoustic impedance at all incidence angles. As one can see from Figure 4, in the typical velocity ratio $V_S/V_p = 0.59$, Z_P it only weakly varies with incidence angles, and the faster reduction of RI's with angles is due to the effects of *P*/*SV* wave conversions. However, note that this impedance still should not be used for generation of reflectivity series as in equations 5-8. These series can be accurately reproduced by using the complete matrix **Z**.

It is important to realize that the meaning of the matrix EI (**Z**) proposed here is different from those suggested previously (equations 5-9). The approximate impedances in formulas 5-9 resulted from heuristic summations of *P*/*P* reflectivity series derived from approximations of the reflectivity equations (Aki and Richards, 2002) and slightly different approximations for time integrals. Transmission losses were ignored, mode conversions only included through *P*/*P* AVO effects, small-amplitude and in some cases small-angle limits were considered, and stringent constraints were imposed in order to perform the integrations. However, integration of reflectivity series (equations 3 and 4) is well-known for its instability (e.g., Francis, 1997), and therefore the differences between the existing EI forms may reflect the biases accumulated during the reflectivity discretization and integration. VerWest et al. (2000) also noted the differences from using the Aki and Richards (2002) and Bortfeld (1961) approximations during such integration.

By contrast, the matrix impedance (equation 30) arises from considerations of the boundary conditions in a single medium, and not from integrating a reflectivity series. Elements of this matrix arise from a simple differential operator relating the stress in the wavefield to its *P*- and *S*-wave displacements (or velocities, equation 13), and this operator is closely related to the elastic parameters of the medium. It depends on neither any specific wave type nor its propagation history through other layers. Due to its matrix form, it incorporates all effects of *P*/*SV* reflections and conversions, and yields the correct acoustic limit at all incidence angles. Therefore, the matrix EI above should provide the most unambiguous information for interpreting elastic rock properties from seismic inversion.

Conclusions

P/*SV* wave impedance of an elastic medium can be uniquely described as a matrix quantity depending on its mechanical parameters and angle of incidence. Reflection, transmission, and conversion coefficients at arbitrary angles are related to matrix-impedance contrasts similarly to the acoustic cases. When using the matrix form of the reflectivity problem, no approximations are required for accurate modeling of reflection coefficients at all angles. Known cases of *P*- and *S*-wave acoustic impedance are accurately reproduced at all incidence angles, and the approach also allows the extension of the concept of impedance to attenuative medium. The matrix impedance readily lends itself for inversion, with uncertainties typical of the standard acoustic-impedance inversion problem.

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Figures

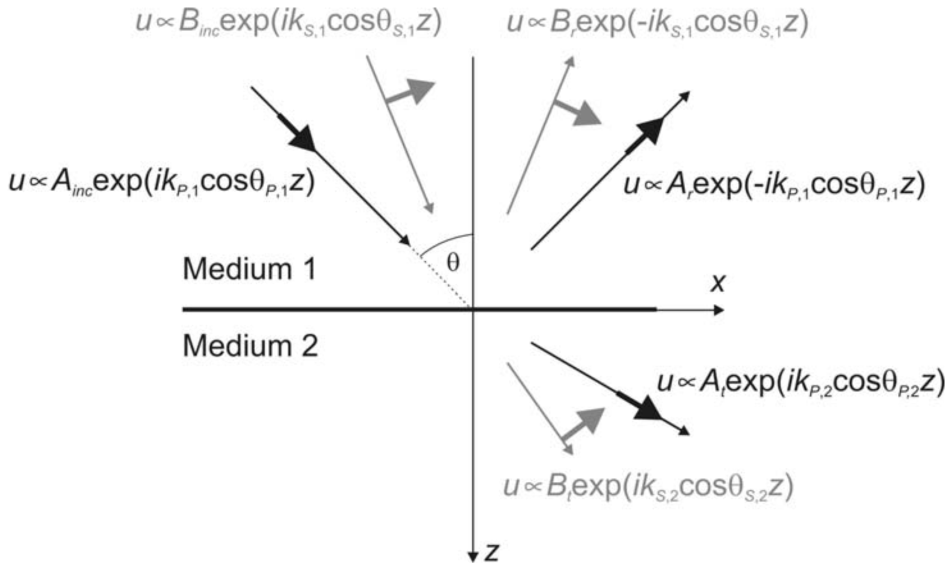

Figure 1. Model geometry and notation. Black arrows indicate *P* waves, gray arrows – *SV* waves. Thick arrows indicate positive displacement directions. Factors dependent on *z* in the expressions for *P*- and *SV*-wave displacements are shown. Second subscripts in *k* and θ indicate the propagation media.

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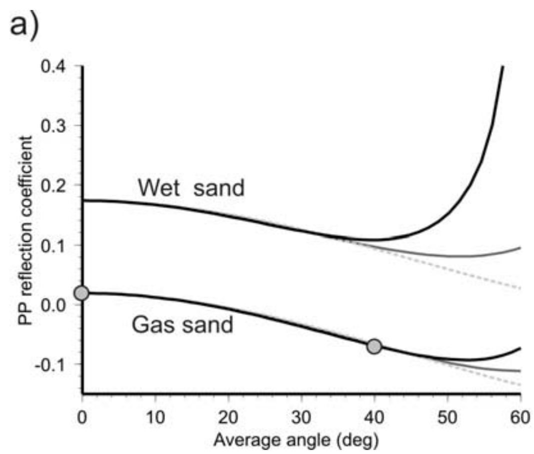

Figure 2. a) *PP* and *PS* reflection coefficients in shale over wet- and gas-sand models from VerWest (2004). Model parameters are: V_P = 2.77 km/s, V_S = 1.52 km/s, and ρ = 2.29 g/cm^3 for shale, V_P = 3.85 km/s, V_S = 2.24 km/s, and ρ = 2.34 g/ cm^3 for wet sand, and V_P = 3.08 km/s, V_S = 1.98 km/s, and ρ = 2.14 g/ cm^3 for gas sand. The average angle is defined as θ_a = asin[$p(V_{P1}+V_{P2})$] (VerWest, 2004). Black lines show the exact solutions; solid gray lines give solutions with *P*- and *S*-wave refracted angles θ_2 (θ in equation 15) in layer 2 (sand) replaced with their average values, and dashed gray lines correspond to θ_2 taken the same as in the shale layer. Gray dots in plot a) indicate the points used in inversion (Figure 3).

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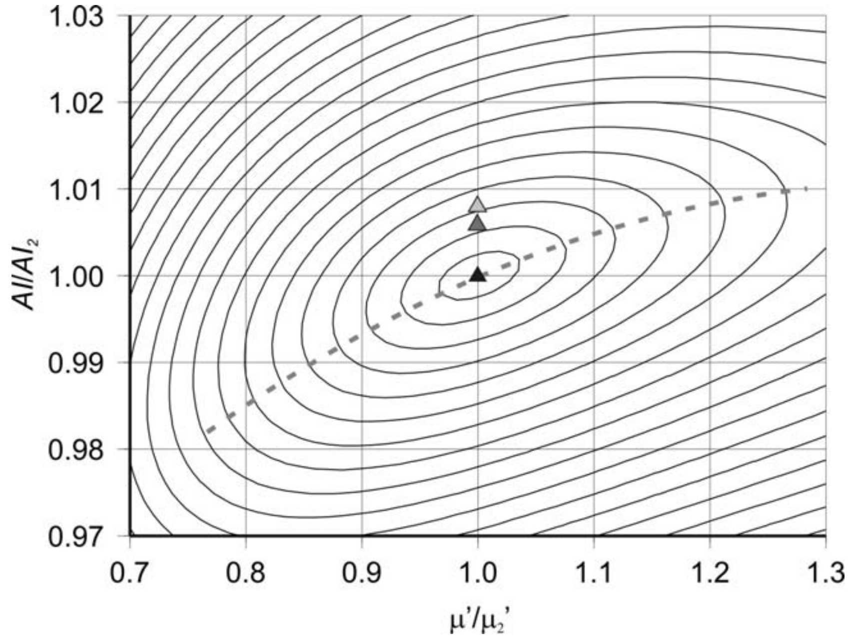

Figure 3. Illustration of inversion for **Z** at a single boundary in shale over gas-sand model. Axes correspond to relative deviations of *AI* and μ' from the true values in the gas-sand layer. Contours correspond to RMS misfits from two points marked with circles in Figure 2a, using the exact **Z** (black lines in Figure 2a) and drawn at intervals of 0.001. Black, dark-, and light-gray triangles correspond to solutions obtained by using the corresponding θ dependencies shown in Figure 2a. Gray dashed line indicates the direction of correlated uncertainties in *AI* and μ'

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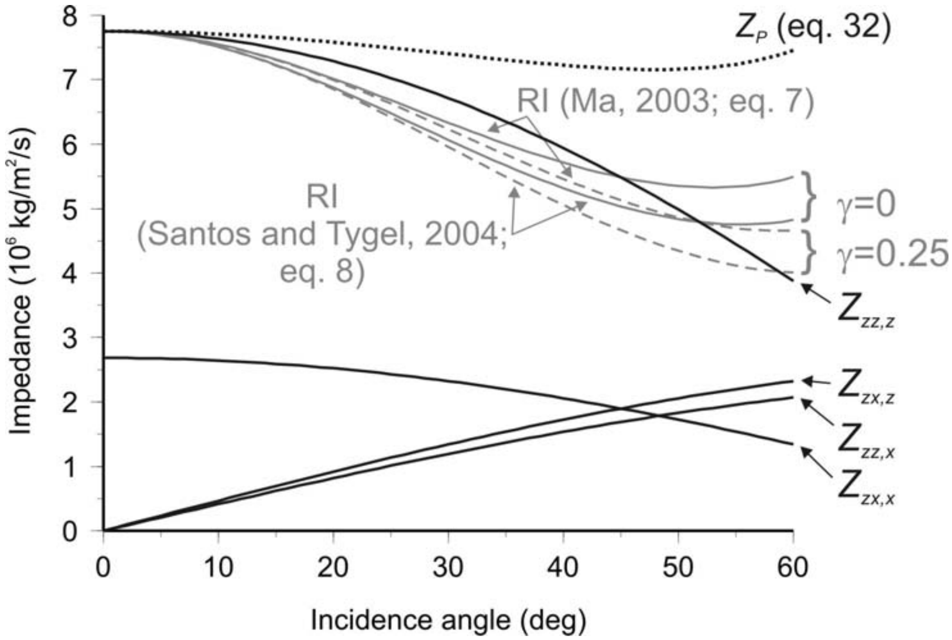

Figure 4. Comparison of the approximate forms of RI for different γ values (gray lines, equations 7-8, labeled) to the elements of P-wave matrix EI (black lines, equation 30, labeled) for an elastic medium with V_P = 3400 m/s, V_S = 2000 m/s, and ρ = 2.28 g/cm^3. Dotted line shows the Z_P value in equation 32.

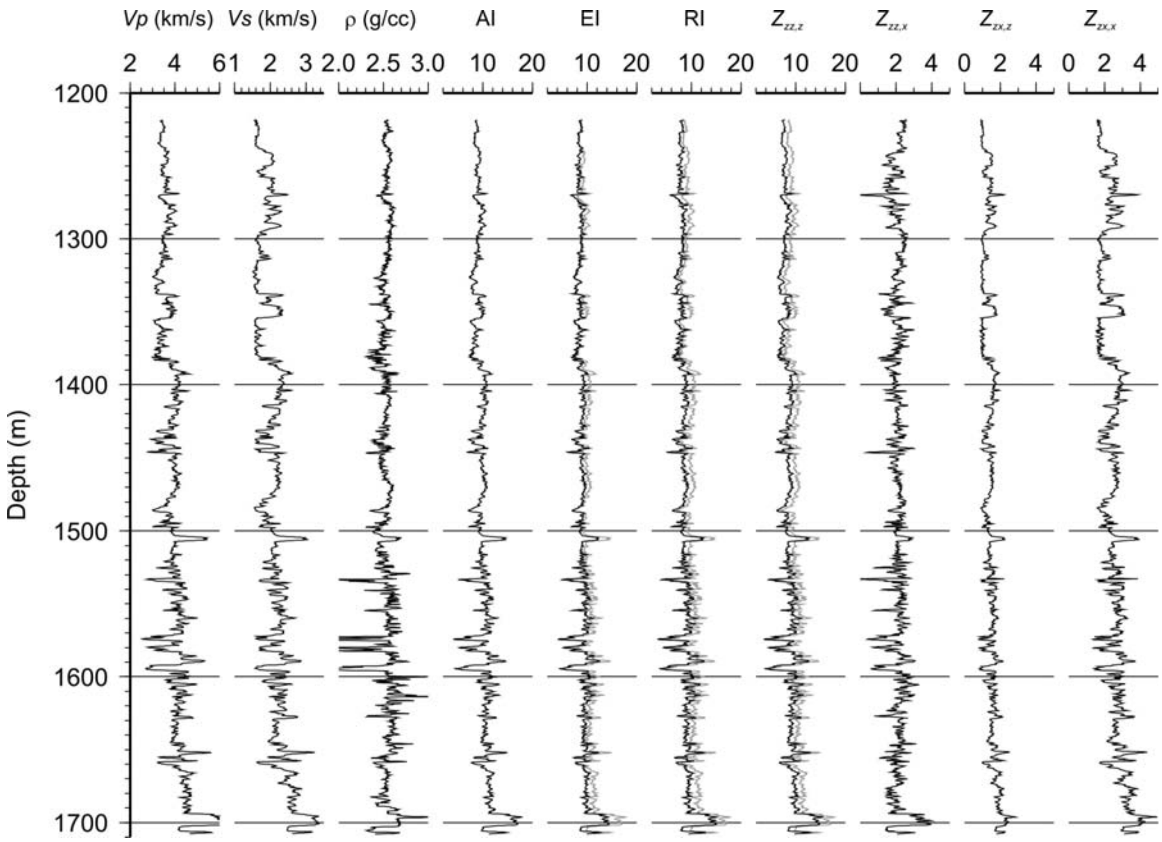

Figure 5. Velocity, density, AI, normalized EI (equation 6), RI (equation 8), and components of matrix **Z** impedance (equation 30) at 30° incidence angle in a well from Blackfoot project (CREWES, 2004). For EI log, normalization constants V_{P0}, V_{S0}, and ρ_0 are selected at the top of the log. Impedance units are km/s·g/cm^3. In EI, RI, and $Z_{zz,z}$ plots, AI log is also shown in gray for comparison. Density log was not edited, resulting in spurious impedance lows between 1500 – 1600 m depths.