Earth’s Attenuation and $Q$

On the use and causes of frequency-dependent $Q$ in seismology

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Seismological attenuation coefficient and \( Q(f) \)
This document represents a work in progress intended to be reworked in a future book on seismological \( Q \). Currently, it represents a collage of several manuscript on seismological \( Q \) that I wrote recently for various journals but that were hitherto not published. As one reads it through, it may become apparent why some of this material may be difficult to publish. I critically reconsider several key points on which the models of seismological \( Q \) established over the past 50 years have been based, and discover that:

- In many observations, the reported frequency dependence of \( Q \) may be related to inaccurate corrections for geometric spreading. In fact, I have not yet found a case in which a frequency-dependent in situ \( Q \) would be required to explain the observations! This applies to numerous observations of body, surface, \( Lg \), and coda waves in several frequency bands from \(~1000 \text{ s} \) to \(~100 \text{ Hz} \). The concept of "scattering \( Q \)" also appears quite difficult for differentiating from the geometrical spreading or intrinsic \( Q \) in observations.

- Looking into the theory of wave attenuation, the use of “quality factor” \( Q \) for describing the Earth’s medium appears to be not well justified and mostly driven by imprecise analogies. In particular, I question the accepted interpretation of attenuation as related to elastic relaxation mechanisms and complex values of elastic moduli in the frequency domain.

- In modeling wave propagation in anelastic media, I also question the so-called “visco-elastic” theories. Although perfectly self-consistent in themselves, they lead to incorrect representation of boundary conditions, resulting, for example, in wrong expressions for the Acoustic Impedance in the presence of anelasticity. This also leads to inaccurate formulas for surface-wave \( Q \), and incorrect assumptions about the equivalence of the velocity and \( Q^{-1} \) sensitivity kernels that are typically used in attenuation tomography.
This draft is posted in a hope of providing a relatively coherent and up-to-date story for those interested in the concept of frequency-dependent seismic attenuation.

The general topic of these papers is the effect of theoretical assumptions on the measured $Q(f)$. In **Chapter 1**, I discuss the general problem of $Q(f)$ observations, starting with introduction of inaccurate assumptions (such as about known geometrical spreading) and then justifying the assumptions in retrospect. I also propose a simple, assumption free and data-driven attenuation analysis method. When reading this Chapter, the following paper is also recommended:


In **Section 1.1**, I discuss the relation of variable geometrical spreading to the “scattering $Q$” and present several examples of assumption-free re-interpretations from several key publications on $Q(f)$. As it shows, the new view sometimes changes the interpretations dramatically and updates published values of $Q$ by factors of 20-30. This Section continues the discussion of the following paper and contributes to it examples in different frequency bands:


In **Section 1.2**, the same argument is extended to the case of long-period body waves. I show that the well-known change in P-wave $t^*$ values from ~0.2 s at short periods to ~1s at long periods may be caused by an unaccounted for ~6% error in the geometrical spreading used in estimating the long-period $t^*$.

**Chapter 2** is devoted to the attenuation theory. This is work in progress and is
still missing its introductory part. Unlike the traditional approach starting from wave-equation solutions derived in simple cases (typically, in a uniform medium) and postulating the relation of attenuation to retarded elastic responses, I consider the more fundamental Lagrange formulation of the wave mechanics, In Section 2.1, I show that the visco-elastic approach fails in describing the Acoustic Impedance (i.e., reflectivity) in the presence of attenuation. In Section 2.2, I present numerical modeling of geometrical spreading in realistic crustal structures in order to illustrate its variability. Although the fact of variability may appear trivial (yet nevertheless ignored in the measurements of $Q(f)$!), the specific point of this Chapter is to explain the typically positive values of uncompensated geometrical spreading observed in crustal body-wave observations. Such positive geometrical spreading leads to positive apparent frequency dependence of $Q(f)$, which is commonly observed.

In Section 3.1, I offer a simple yet spectacular re-interpretation of the well-known mantle absorption band (i.e., increased absorption within the mantle surface-wave frequency band). From the attenuation-coefficient point of view, this band is apparent, and may be caused by combinations of lower-$Q$ layers within the outer core and the upper mantle, and high-$Q$ crust. The flanks of the apparent absorption band are formed by changes in geometrical spreading of the different seismic waves. Note that this model does not require any frequency-dependent rheological properties.

Section 3.2 presents a tomographic study of short-period coda $Q$ attenuation in Russian Eurasia by using the Soviet Peaceful Nuclear Explosion data. The inversion for frequency-dependent apparent $Q(f)$ is performed in the new assumption-free form and provides important correlations with geological structures in this and other areas of the world. The following paper is also recommended when reading this Chapter:


Finally, Appendix A contains a comment on a recent paper on frequency-
dependent coda $Q$ published in *the Bulletin of the Seismological Society of America*. Unfortunately, papers of this kind are published in nearly every issue of BSSA and other seismological journals, and so many similar comments can be written. The authors perform their analysis strictly (albeit maybe somewhat uncritically) within the broadly accepted guidelines and produce “evidence” of $Q(f) \alpha f$ where neither attenuation nor scattering is likely involved.

In a final remark, note that in the papers below, I use the $\alpha(f)$ notation for the attenuation coefficient: $\alpha(f) = \pi f/Q(f)$. However, in more recent work, I prefer changing this notation to $\chi(f)$, because “$\alpha$” often denotes the spatial attenuation coefficient.
1. Problem: Earth’s sphericity and structure as causes of the frequency-dependent apparent $Q$

**Summary**

In attenuation measurements, the interplay of “practical” assumptions about geometrical spreading and permissive frequency-dependent attenuation law $Q(f) = Q_0(f/f_0)^n$ represents a logical trap caused by an implied preference for a frequency-dependent $Q$. A new description using the attenuation coefficient $\alpha(f) = \gamma + \pi f/Q_e$ emphasizes the data constraints and resolves this problem. In this description, $\gamma$ represents the uncompensated geometrical spreading, and $Q_e$ is the attenuation, which turned out to be frequency-independent in all data examples considered so far. The model is illustrated on an example of coda attenuation from a Peaceful Nuclear Explosion in Siberia and on modeled 50 – 300-s surface waves. These and other studies show that short-period crustal body waves are characterized by positive $\gamma_{SP}$ values of $(0.6 - 2.0) \cdot 10^{-2}$ s$^{-1}$ interpreted as related to the upper-crustal reflectivity. Long-period surface waves show negative $\gamma_{LP} \approx -1.9 \cdot 10^{-5}$ s$^{-1}$, which could be caused by the Earth’s sphericity. This provides a simple geometrical explanation of the apparent absorption band within the Earth. The band is formed by a high $Q_e \approx 1100$ within the crust and low $Q_e \approx 120$ within the uppermost mantle, and its cut-off frequencies correspond to $\gamma_{LP}$ and $\gamma_{SP}$. Relaxation-spectra or scattering models may not be needed for explaining the observed apparent $Q(f)$, or at least they should be applied after the geometrical effects are accounted for. Overall, based on its physical sense, simplicity, stability, convenience, and geological consequences, the $(\gamma, Q_e)$ model appears to be far superior to the conventional $(Q_0, \eta)$ view.

**Introduction**

Nearly every issue of seismological journals contains articles presenting observations of frequency-dependent attenuation. Reported $Q$ values often increase nearly proportionally to the frequency $f$ or even faster (for *Bulletin* examples from this...
year only, see Castro et al., 2008; Kinoshita, 2008; Morasca et al., 2008; Mukhopadhyay et al., 2008, Oth et al., 2008, and with somewhat slower $Q(f)$ dependencies, Drouet et al., 2008; Ford et al., 2008). The concept of frequency-dependent scattering or rheological $Q(f)$ (Aki and Chouet, 1975; Liu et al., 1976) has been imbued in minds of more than one generation of seismologists and is now accepted as a matter of fact. However, a fundamental problem with steep $Q(f)$ is the fact that values $Q \propto f$ lie entirely within the uncertainty of attenuation-free geometrical spreading (GS), and therefore such dependencies are likely to be measurement artifacts. Even with such a long history and broad acceptance, frequency-dependent $Q(f)$ still requires scrutiny, and it is important to keep in mind what evidence we actually have for it, and decide whether other interpretations could be better.

The principal argument on which many of the $Q(f)$ studies focus is that frequency-dependent attenuation is certainly theoretically possible. Without attempting a comprehensive overview, I note that modeling broadly shows that depending on the statistical properties of the scattering medium, frequency dependence of the elastic (scattering) $Q_s$ can range from nearly frequency-independent $Q_s$ (Frankel and Clayton, 1986) to $Q_s \propto f$ or steeper (Chernov, 1960; Dainty, 1981; Sato and Fehler, 1998).

Theoretical intrinsic $Q_i(f)$ dependencies also range from near-constant to nearly proportional to the frequency, as predicted by ‘creep’ or ‘relaxation’ rheological models (Jackson and Anderson 1970; Liu et al., 1976). Such similarity of observable properties leads to a virtual impossibility of separating these quantities in the data without making various stringent assumptions, of which the key one is about the GS. It is broadly recognized that the common GS assumptions are in most cases inadequate; however, from a theoretical standpoint, they nevertheless do not break the validity of the ensuing models.

However, in observational seismology, one still needs to establish how well such assumptions and models relate to the Earth and measurement procedures, and whether the fact of frequency dependence of $Q$ is invariant in their respect. Such testing tends to be subdued in the use of elaborate inversion methods, curve-fitting, and in the pursuit of detailed solutions from usually limited and noisy datasets. Assumptions have become a
legitimate part not only of theoretical treatments but also of data interpretations. “Practical” assumptions may be acceptable and necessary when used to simplify understanding of the problem; however, it appears that in typical $Q(f)$ inversion, GS assumptions often complexify it by encouraging overly detailed parameterizations.

When using model parameterizations that contain more degrees of freedom than constrained by the data, the results are mostly controlled by the underlying “a priori constraints,” and a danger of biased or “preferred” solutions arises. Note that in most cases currently found in the literature, the quality of the $Q(f)$ solutions is only judged by their ability to fit the seismic amplitude data. This criterion is insufficient and misleading, because in under-constrained inversion, multiple models fit the data equally well. For example, frequency-independent $Q$ re-interpretations given in Morozov (2008) fit the data from Kinoshita (2008) similarly to the author’s $Q(f)$ model or better, and two additional examples of this kind are given below. Because $Q(f)$ inversions exploit the freedom of theoretical assumptions, neither the assumed GS nor the resulting parameters may relate to the Earth’s or wavefield’s properties.

Thus, there is an apparent disconnect between the theoretical arguments for a possible in situ frequency-dependent $Q$ within the Earth and observations of $Q(f)$ from seismological data. Observational evidence for the frequency-dependent $Q$, such as advanced in support for the absorption band concept (Doornbos, 1983), are limited to the apparent $Q(f)$ derived by using the theoretical GS compensation. As shown in this note, the band-like shape of the apparent $Q(f)$ within $\sim 3 \cdot 10^{-3} – 30$ Hz can be explained by the lithospheric layering and Earth’s sphericity, which affect the GS in opposite ways. Generally, through the use of the uniform-space GS or sometimes models for source spectra, attenuation interpretations appear to be biased by the assumptions of frequency-dependent $Q(f)$ and fixed GS, creating an endless logical loop of argument.

**GS - $Q(f)$ trade-off: a loop of uncertainty**

Trying to decide whether the origin of the $Q(f)$ uncertainty is in inaccurate but necessary “practical” assumptions about the GS or in the willingness to infer a frequency-dependent $Q$ resembles the classic chicken-and-egg problem. The $Q(f)$ is
implied in the model from the very beginning, and it absorbs any deviations of the
assumed GS from its true values. Both approximate GS and $Q(f)$ cannot exist without and
cross-feed each other. However, my solution to this dilemma is that the implied $Q(f)$ still
comes first. Theorists tend to like elegant models (such as scattering- or relaxation-
spectra based), and experimentators prefer using GS obtained from simple theoretical
arguments – both parties thereby encouraging complex and permissive $Q(f)$
dependencies. The experiments may simply show what theories are designed to predict.

A frequency-dependent $Q \propto f$ is equivalent to GS, as can be seen from the
expression from which $Q(f)$ is usually inverted (e.g., Fan and Lay, 2003):

$$P(t, f) = G(t)e^{-\eta f/f_0},$$

(1)

where $G(t)$ denotes the GS (assumed to be frequency-independent for simplicity), $P(t, f)$
is the path factor from which the source and site effects were removed, and $Q(f)$ is
usually taken in the form $Q(f) = Q_0(f_0/f)^\eta$. Values of $\eta \approx 1$ are often expected from
scattering $Q_s$, which was used to explain the $S$-wave attenuation at $1 – 30$ Hz (Dainty,
1981). If one allows a steep $Q_s(f)$ dependence close to $\propto f$ and at the same time
recognizes the inaccuracy and spatial variability of GS, any discussions of their
differences become observationally pointless. $Q_s \propto f$ cannot be distinguished from GS by
any technique, either by using spectral ratios or GS compensation. If frequency-
independent intrinsic attenuation $Q_i$ is added to $Q_s$: $Q^{-1}(f) = Q_s^{-1} + Q_i^{-1}$, practically any
observed $Q(f)$ dependence in eq. (1) can be reproduced by setting $0 \leq \eta \leq 1$. If $Q_i$ is
considered to be frequency-dependent, apparently any apparent $\eta$ at all can be expected.
Furthermore, contributions from the elastic and frequency-dependent intrinsic attenuation
in $Q^{-1}(f)$ in eq. (1) are also impossible to differentiate without additional theoretical
assumptions (Wennerberg and Frankel, 1989).

Without making a clear physical distinction between the GS and $Q(f)$,
interpretation may become tricky and erroneous. For example, Figure 1 shows codas of
the Peaceful Nuclear Explosion (PNE) Kimberlite-3 recorded within the Siberian Craton.
Within the model proposed earlier (Morozov and Smithson, 2000), the scattered-wave
GS should be compensated by the spreading scattering volume, leading to $G(t) \approx 1$ in eq. (1). Therefore, the coda amplitude decay was originally interpreted as caused entirely by attenuation (Morozov and Smithson, 2000). At the same time, the time-domain log-amplitude slopes are practically independent of the frequency (Figure 1), leading to $Q(f) \approx 330f^{0.85}$ in eq. (1) for this PNE (with $f$ measured in Hz; Figure 2b). However, this is an erroneous result, which in fact indicates a failure of the model of coda-GS compensation. Such low $Q_0$ and high $\eta$ are not expected for the stable Siberian Craton. The fact that crustal $Q$ at 0.5 – 1 Hz cannot be as low as ~200 - 300 in this area is apparent from previous observations of $P_g$ traveling to over 1600 km along the Kimberlite profile at these frequencies (Morozov et al., 2002).

By contrast, recognizing the importance of GS, we see that the frequency-independent coda slopes simply indicate that their origin is geometrical, and attenuation is low. Similar observations of steep $Q(f)$ dependencies which should actually correspond to much higher $Q$ and stronger GS can be found in many other crustal studies (e.g., Castro et al., 2008; Kinoshita, 2008; Morasca et al., 2008; Mukhopadhyay et al., 2008; Oth et al., 2008; see also references in Morozov, 2008).

**Separation of geometrical spreading and attenuation**

Despite the difficulty in their formal separation from a single $P(t, f)$ dependence (1), an important difference in the physics of the GS and $Q$-related processes still exists and can be used for their differentiation. A practical approach to such separation was described by Morozov (2008). I will not repeat the arguments of that paper here but only summarize the two key principles on which the procedure was based:

1) The attenuation described by $Q$ is a secondary effect which can only be established when GS is known. In other words, GS can exist without attenuation, but $Q(f)$ in a structure with unknown spreading is meaningless. Note that only $\sim$10% variations of the GS exponent $\nu$ are sufficient to explain most of the observed $Q(f)$ dependencies (Morozov, unpublished).

2) The effect of attenuation typically increases with frequency. This is
indicated by the factor \( f \) in the nominator of the exponent in eq. (1), corresponding to the general definition of \( Q^{-1} \) as fractional energy dissipation per oscillation cycle in mechanics. Of course, by assuming scattering on “larger” structures with \( Q_s \propto f \), attenuation can completely replace GS (Dainty, 1981), but such large-scale structures should likely be related to the deterministic lithospheric layering, and therefore – to GS again. Thus, it appears reasonable to postulate the effect of \( Q \) to equal zero for \( f=0 \) and to increase with the frequency.

Therefore, before \( Q \) can be estimated, GS must be carefully measured as the \( f=0 \) case. To accomplish this, starting from some initial approximation \( G_0(t) \) [for example, \( G_0(t) = ct^{-\mu} \) commonly used], the logarithm of the path factor (1) can be written as:

\[
\ln P(t, f) = \ln G_0(t) - \alpha(f)t .
\]        (2)

This is the standard scattering-theory approximation meaning that logarithmic amplitudes dissipate approximately proportionally to the travel times or distances (e.g., Chernov, 1960). In principle, the attenuation coefficient \( \alpha(f) \) could also be time-dependent; however, such detail may be incommensurate with the limited time ranges of typical measurements (e.g., coda time lags or S-wave propagation times across a local network) and with only a limited knowledge of the background \( G_0(t) \).

Further, it is convenient to explicitly separate the frequency-independent part \( \gamma = \alpha(0) \) of the attenuation coefficient \( \alpha(f) \):

\[
\alpha(f) = \gamma + \kappa f^\nu .
\]        (3)

In this expression, the meaning of \( \gamma \) is the improved estimate of the true GS:

\[
G(t) = G_0(t)e^{-\gamma} .
\]        (4)

Parameter \( \kappa \) should be related to the effective attenuation caused by intrinsic dissipation and small-scale scattering, and therefore it can be written by using a quality factor \( Q_e \): \( \kappa = \pi Q_e \). Under “small-scale,” I understand the range of scales below those of the dominant seismic structure, as determined by the crustal and basin thickness and velocity
gradients, which should all be included in GS. Although parameter \( \gamma \) can be linked to turbidity \( g \) (e.g., Dainty, 1981; Padhy, 2005), such association is only limited to scattering-theory descriptions of \( P(t,f) \), which do not include the effects of the lithospheric structure not accounted for by \( G_0(t) \). Theoretically, both \( \kappa \) and \( Q_e \) can be frequency-dependent if \( \alpha(f) \) shows a non-linear behavior on \( f \); however, such frequency dependence still remains to be found in seismological data (Morozov, 2008).

From eq. (1), the resulting procedure for separating the GS and \( Q \) effects and for analyzing \( Q \) for real frequency dependence is remarkably simple: A) extract \( \alpha(f) \) values by fitting \( \ln[P(t,f)/G_0(t)] \) at different frequencies in the time domain, and B) analyze the dependence of \( \alpha(f) \) on the frequency according to eq. (3), and in particular examine the limit of \( \gamma = \alpha(0) \). Step A) is part to virtually any \( Q(f) \) inversion, and the attenuation coefficients can be crudely estimated from many published \( Q(f) \) results as \( \alpha(f) = \frac{\pi}{Q(f)} \) (Morozov, 2008). However, surprisingly, \( \alpha(f) \) values are rarely plotted, and step B) is practically never performed, with only turbidity-related studies (Dainty, 1981; Padhy, 2005) coming close to doing this. Note that the “stacked spectral ratios” (e.g., Xie, 2007, and many other papers by Xie and Mitchell) actually represent the \( \alpha(f) \) quantities but are unfortunately plotted in logarithmic frequency scales, which do not allow measuring the intercept values (\( \gamma \)) and recognizing the linearity of \( \alpha(f) \) dependences.

Figure 2 illustrates the sharp difference in the interpretation resulting from the model-free approach [eq. (3)]. By measuring the log-amplitude coda slopes within the \( Lg \) coda time window (Figure 1), the values of \( \alpha(f) \) are directly obtained for the corresponding frequencies (Figure 2a). The \( \alpha(f) \) dependence shows a linear trend [eq. (3)] showing an under-compensated GS with \( \gamma \approx 0.75 \times 10^{-2} \text{ s}^{-1} \) and a frequency-independent \( Q_e \approx 2500 \) (Figure 2a). For comparison, by conventionally attributing the entire residual coda amplitude decay to \( Q \), one would obtain \( Q(f) \approx 330f^{0.85} \) (Figure 2b). As mentioned before, such steep apparent frequency dependence is caused by the nearly frequency-independent amplitude decays in Figure 1.

Due to their independence on inaccurate assumptions, the resulting (\( \gamma, Q_e \)) values facilitate consistent interpretations and comparisons with other studies. In this PNE
example, the value of $Q_e$ is 8-10 times higher than $Q(f)$ at 0.5-1 Hz (Figure 1), which explains the crustal-guided waves ($P_g$ and $L_g$) propagating to over 1600-km distances across the Siberian Craton. The value of $\gamma$ is also remarkably close to that measured from recordings from another PNE Quartz-4 in the East European Platform, and also consistent with predictions from numerical coda modeling (Morozov et al., 2008). Therefore, at least in this area, the simple ($\gamma, Q_e$) model is clearly advantageous and shows that the strong apparent $Q(f)$ (Figure 2b) is a geometrical artifact.

Another important type of geometrical effects causing apparent frequency dependence of $Q(f)$ was observed in surface-wave (Anderson et al., 1965), $L_g$ (Mitchell, 1991), and PNE coda studies (Morozov et al., 2008). These authors showed that layered structures with frequency-independent but depth-dependent intrinsic shear-wave $Q$ result in an apparent frequency dependence of the observed $Q$. Spherical mode summations show that surface-wave $Q$ values quickly decrease with frequency, in agreement with the observations (Anderson et al., 1965; Figure 3b). At a first glance, such frequency dependence may be caused by longer waves sampling progressively deeper layers with higher $Q$ (Figure 3a). However, this may not be the main reason! When transformed to the attenuation coefficient form, values of $\alpha(f)$ again fall on almost straight lines within the entire modeled frequency band (Figure 3c). The resulting frequency-independent values of $Q_e$ are 121 for Rayleigh waves and 112 for Love waves (Figure 3b). Interestingly, these values are close to each other and also to the lowest $Q$’s near 0.02 Hz in Figure 3b. Such frequency-independent $Q_e$ suggests that at all periods, the attenuation of both Rayleigh and Love waves may be principally accumulated at the sub-crustal depths (~38 – 60 km), because such $Q$ values are present only within this depth range in the model (Figure 3a).

The resulting attenuation-coefficient intercept values $\gamma$ are negative, showing that the GS is over-compensated by the spherical-wave $G_0(t) = t^{1/2}$ correction. For Rayleigh waves, $\gamma \approx -4 \cdot 10^{-5}$ s$^{-1}$, and for Love waves, $\gamma \approx -1.9 \cdot 10^{-5}$ s$^{-1}$ (Figure 3c). Such over-compensated GS may be caused by the Earth’s sphericity causing the surface waves to spread slightly slower than expected from a cylindrically-symmetric model. As shown below, such low $\gamma$ values are still significant for long-period waves.
The combined examples in Figures 2 and 3 explain the observed absorption band differently from Anderson and Given (1982). The increase in apparent long-period $Q$ toward lower frequencies is almost entirely produced by the negative $\gamma$ values, denoted by $\gamma_{LP}$ here:

$$Q_{LP}(f) = \frac{\pi f}{\gamma_{LP} + \frac{\pi f}{Q_e}} = \frac{Q_e f}{f - f_{c,LP}},$$  \hspace{1cm} (5)$$

where the long-period “cross-over” frequency $f_{c,LP}$ equals $-\gamma_{LP}Q_e/\pi$. For Rayleigh waves, $f_{c,LP} \approx 1.5 \cdot 10^{-3}$ Hz, which is significantly lower than the typical surface-wave frequencies. Therefore, $Q(f)$ steeply increases when frequency drops to $f_{c,LP}$ (gray lines in Figure 3b).

For short-period body waves, $\gamma$ is usually positive, and the corresponding $f_c$ can be defined as: $f_{c,SP} = \gamma Q_e/\pi$ (Morozov, 2008). Consequently, the short-period $Q(f)$ is:

$$Q_{SP}(f) = \frac{\pi f}{\gamma_{SP} + \frac{\pi f}{Q_e}} = \frac{Q_e f}{f + f_{c,SP}}.$$ \hspace{1cm} (6)$$

With typical crustal values of $Q_e \approx 1100$ and $\gamma_{SP} \approx 0.6 \cdot 10^{-2}$ s$^{-1}$ for tectonically stable areas and $\gamma_{SP} \approx 2 \cdot 10^{-2}$ s$^{-1}$ for tectonically active ones (Morozov, 2008), we obtain: $f_{c,SP} \approx 2.8$ Hz and 7.8 Hz for these respective cases. Expression (6) shows that $Q_{SP}(f)$ increases with the frequency at short periods, and consequently, the two characteristic frequencies $f_c$ delineate a band of reduced apparent $Q(f)$ (Figure 4). Note that the origin of this band is purely geometrical, and its attenuation levels are controlled by only two frequency-independent values of $Q_e = 1100$ (within the crust) and $Q_e = 121$ (within the uppermost mantle) shown by the gray bars in Figure 4.

It is also important to compare the mantle absorption band model (ABM; Figure 4; Anderson and Given, 1982) to the predictions from expressions (5) and (6). In ABM model, the attenuation level is controlled by a fixed minimum level $Q_m = 80$, which corresponds to $Q_e$ in our interpretation. However, in ABM, $Q_m$ is assumed to be constant within the entire mantle, whereas Figure 2 shows that $Q_e$ may mostly be formed by the increased attenuation within the relatively thin, low-$Q$ subcrustal lithosphere. For $f >>$
$f_{c,LP}$, eq. (5) yields $Q(f)$ values inversely proportional to $f$: $Q_{LP}(f) \approx Q_{e,LP}/f$, which corresponds to the low-frequency flank of ABM with parameter $\tau_2 = Q_c/f_{c,LP} = -\pi/\gamma_{LP}$. For body waves at $f << f_{c,SP}$, $Q(f)$ is nearly proportional to the frequency [eq. (6)]: $Q_{SP}(f) \approx Q_e f/f_{c,SP}$. Such $Q \propto f$ dependence defines the upper flank of the ABM band, in which parameter $\tau_1$ becomes: $\tau_1 = Q_e/f_{c,SP} = \pi/\gamma_{SP}$. Therefore, quantifies $\tau_1$ and $\tau_2$ defined by Anderson and Given (1982) as the cut-off levels in the Debye distributions of relaxation times may simply be reciprocals of the uncompensated GS parameters for body and surface waves, respectively. Note that or model shows that the upper flank of this band becomes higher ($Q$ increasing) and flatter ($\eta$ decreasing) with tectonic age because of reducing values of $f_{c,SP}$ and $\gamma$ [Figure 4; for more on this, cf. Morozov (2008)], whereas the ABM model accounts for the crust by shifting the $\tau_1$ value to zero (Anderson and Given, 1982). Further comparison of these models is complicated by the complex (i.e., over-parameterized) nature of ABM using multiple wave modes (with likely variable GS) and depth-dependent ($\tau_1$, $\tau_2$) combined with a constant $Q_m$ and various other assumptions.

**Discussion and conclusions**

In the presented examples, an analysis of geometrical spreading allowed us drawing several insightful conclusions directly from attenuation data within a broad frequency band. However, if $Q(f)$ is allowed to trade-off with GS, both real and modeled spectral amplitude-decay data can still be fit in either ($\gamma$, $Q_e$) (Figures 2a and 3c), $Q(f)$ (Figures 2b and 3b), or presumably many other forms. This ambiguity is well known, yet the ultimate goal of seismic interpretation is in finding an unambiguous and useful description of the Earth. Therefore we still need to look for a reliable attenuation model. The reasons for preferring any of such forms should be in the underlying nature of the measured quantities and consistency of the results, and not in the data fit alone.

Independence of any inaccurate assumptions and stability of the results is just one major argument in favor of the ($\gamma$, $Q_e$) approach. Another argument is that $Q_e$ turns out to be frequency-independent in many cases (Figures 2, 3, and also Morozov, 2008), which is convenient for interpretation and avoids complexity where it is not dictated by the data. This interpretation reveals that frequency-dependent $Q$ is not as pervasive as it
is thought today (see Figure 4). Further, because a reference $G_0(t)$ (i.e., a fixed $\gamma$) is always implied in $Q(f)$ interpretations, there is no reason to believe that this reference should stay constant across significant areas. Strictly speaking, this makes comparisons of regional variations in $Q(f)$ impossible or unrelated to the properties of the lithosphere. The resulting mapping (e.g., like done by Morasca et al., 2008) is useful but remains mostly related to the waveform (e.g., coda) properties. By contrast, the $\alpha(f)$ approach simply addresses such GS variability by measuring the regional variations of $\gamma$.

For a more quantitative yet general comparison of the $(\gamma, Q_e)$ vs. $(Q_0, \eta, \text{and assumed } \text{GS})$ approaches, note that widespread observations of high and positive values of $\eta$ in the $Q(f) = Q_0(f/f_0)^\eta$ law suggest that in many areas and for most crustal seismic waves, scattering and attenuation should drop quickly at smaller scale-lengths and higher frequencies. This is contrary to the common observations of geological heterogeneity and difficult to accept as a common property of the Earth material. By contrast, the $(\gamma, Q_e)$ model gives a simple explanation to such observations: in areas where the apparent $\eta$ is positive, $Q_e$ is in fact significantly higher than $Q_0$ (Figure 2), and the GS is under-compensated by the traditional $G_0(t)$ corrections, which corresponds to $\gamma > 0$. Such faster-than-$G_0(t)$ GS from earthquake sources indeed appears to be a common crustal property, which can be related to the upper-crustal reflectivity [see Appendix in Frankel et al. (1990); more on this will be given elsewhere]. Values of $\gamma$ also correlate with tectonic age, which was explained by age-related changes in the velocity structure (Morozov, 2008). Thus, from the viewpoints of physical sense, simplicity, stability, convenience, and geological plausibility, the $(\gamma, Q_e)$ model [eqs. (6)-(3)] appears to be far superior to the conventional $Q(f)$ view. In many, if not most cases, the reported $Q(f)$ may represent artifacts of insufficiently accurate solutions for GS and should be interpreted with caution.

References


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Earth’s structure as the cause of apparent $Q(f)$


Mukhopadhyay, S., J. Sharma, R. Massey, and J. R., Kayal (2008). Lapse-time dependence of coda \( Q \) in the source region of the 1999 Chamoli Earth’s structure as the cause of apparent \( Q(f) \)
Earth’s structure as the cause of apparent $Q(f)$


Figures

Figure 1. Stacked amplitude envelopes from six recordings within 660 – 760-km distances from the Peaceful Nuclear Explosion Kimberlite-3 in Siberian Craton. Records are filtered within several frequency bands (labeled). Regional phases and the coda attenuation measurement interval are indicated. Times are aligned at the Lg arrival, and log-amplitude envelopes are shifted in order to separate the curves. Note that the coda amplitude decay rates are almost independent of the frequency.
Figure 2. Interpretation of PNE Kimberlite-3 coda amplitude decays (Figure 1):

a) by using the uncompensated GS ($\gamma$) and frequency-independent effective $Q_e$ [eq. (3)]; b) by assuming a theoretical GS compensation, resulting in $Q(f)$ steeply increasing with frequency. Note that $Q_e \gg Q(f)$ within the measured frequency band.
Earth’s structure as the cause of apparent $Q(f)$

Figure 3. Modeled surface-wave attenuation data from Anderson et al. (1965):

a) The layered attenuation model MM8.

b) $Q$ for Rayleigh and Love waves modeled by spherical mode summation (black curves) and predicted from the corresponding $\gamma$ and $Q_e$ (determined from plot c) as: $Q(f) = \pi f (\gamma + \pi f Q_e)$ (gray curves). Note that both types of modeled curves are practically identical.

c) The same results transformed into attenuation coefficients $\alpha(f)$ [eqs. (6) and (3)]. Note that both $\alpha(f)$ dependencies are practically linear within the entire frequency band, with small negative $\gamma = \alpha(0)$ seen when extrapolated to $f = 0$ Hz (dotted lines).
Earth’s structure as the cause of apparent $Q(f)$
1.1. Causes of frequency-dependent apparent seismological Q

Abstract

Variability of the Earth’s structure makes a first-order impact on attenuation measurements which often does not receive adequate attention. Geometrical spreading (GS) is spatially variable, and with traditional simplified GS compensation, it can appear as biases in both $Q_0$ and $\eta$ parameters in the conventional frequency-dependent attenuation law $Q(f) = Q_0 f^\eta$. A new model using the attenuation coefficient $\chi(f) = \gamma + \pi f Q_e$ resolves this problem by directly measuring the residual GS, denoted $\gamma$, and effective attenuation, $Q_e$. The model is illustrated by re-interpreting several published real datasets, including nuclear-explosion and local-earthquake cadas, $Pn$, and synthetic 50-300-s surface waves. Some of these examples were key to establishing the $Q(f)$ concept. In all examples considered, $\gamma$ significantly deviates from zero, and $Q_e$ can be considered frequency-independent. Short-period crustal body waves are characterized by positive $\gamma_{SP}$ values of $(0.6 - 2.0) \times 10^{-2}$ s$^{-1}$ interpreted as related to the upper-crustal reflectivity. Long-period surface waves show negative $\gamma_{LP} \approx -1.9 \times 10^{-5}$ s$^{-1}$, which could be caused by insufficient modeling accuracy at long periods. The above $\gamma$ values also provide a simple explanation for the apparent absorption band within the Earth. The band is interpreted as formed by $Q_e \approx 1100$ within the crust decreasing to $Q_e \approx 120$ within the uppermost mantle, with frequencies of its flanks corresponding to $\gamma_{LP}$ and $\gamma_{SP}$. Therefore, the observed absorption band could be of purely geometrical nature, and relaxation or scattering models may not be necessary for explaining the observed apparent $Q(f)$.

Key words:

Attenuation; Body waves; Coda; Crust; Geometrical spreading; Mantle; Structure; Surface waves

Introduction

Attenuation of seismic energy within the crust and mantle is among the most...
intriguing physical effects studied in seismology. Frequency dependence of the seismic quality factor within the mantle, $Q(f)$, was first pointed out by Gutenberg (1958), and since then, the concept of frequency-dependent scattering or rheological $Q(f)$ (Jackson and Anderson, 1970; Aki and Chouet, 1975; Liu et al., 1976) became imbued in minds of more than one generation of seismologists. Most $Q$ models today represent the Earth as a combination of absorption bands with $Q(f) = Q_0f^\eta$ transitions across their flanks (Anderson et al., 1977). The bands vary with depth (Anderson and Given, 1982) and correlate with tectonic structures (Der and McElfresh, 1976, 1980; Der et al., 1986). Across a broad range of surface- and body-wave frequencies from ~0.01 to ~100 Hz (e.g., Der et al., 1982, 1986; Lees et al., 1986; Abercrombie, 1998), $Q$ values consistently increase ($\eta > 0$) in a vast majority of observations.

However, attenuation is measured in the background of other amplitude variations and the difficulty of differentiating its apparent attributes from the true in situ properties is known (Der and Lees, 1985; White, 1992). Predominant observations of positive $Q(f)$ dependencies often reaching and exceeding the linear $Q \propto f$ rate still suggest serious concerns about the measurement methodology. The cases of $\eta = 1$ are impossible to separate from the attenuation-free geometrical spreading (GS), and therefore such observations could actually represent inaccurate GS estimates. Similarly, slower positive and negative $Q(f)$ dependencies are also likely to contain GS contributions. Even with such a long history and broad acceptance, frequency-dependent $Q(f)$ still requires scrutiny, and it is important to clearly understand what evidence for it we actually have. Generally, two groups of such evidence are usually advanced: theoretical and observational.

Theoretically, frequency-dependent attenuation is certainly possible. Without attempting a comprehensive overview, I note that modeling broadly shows that depending on the statistical properties of the scattering medium, frequency dependence of

1 A comprehensive list of references would include hundreds of publications and grow every month. I only refer to several key papers that shaped the concept of frequency-dependent attenuation.
elastic (scattering) $Q_s$ can range from nearly frequency-independent $Q_s$ (Frankel and Clayton, 1986) to $Q_s \propto f$ or steeper (Chernov, 1960; Dainty, 1981; Sato and Fehler, 1998). Theoretical intrinsic $Q(f)$ dependencies range from near-constant to nearly proportional to the frequency, as predicted by ‘creep’ or ‘relaxation’ rheological models (Liu et al., 1976). Note that similarity of observable properties leads to difficulties in separating these quantities in the data and requires making stringent assumptions, of which the key one is again about the GS (Wennerberg and Frankel, 1989).

It is well known that causality constraints in the form of the Kramers-Krönig (K-K) identities also require $Q$ to be frequency-dependent (e.g., Futterman, 1962). However, this result may often be somewhat over-emphasized. K-K constraints are very general and only correlate the variations of the phase velocity and $Q$ over the entire infinite frequency band and in the form of integral transforms that converge quite slowly. For example, Futterman (1962) showed that for $Q$ values of ~30 at ordinary seismic frequencies, $Q^{-1}$ should drop at frequencies below ~$10^{-99}$ Hz. This frequency is far below any measurable level, and therefore the question whether $Q$ is frequency-dependent within the seismological band is still not answered by K-K relations.

Observationally, further evidence for frequency-dependent $Q$ in Earth materials comes from laboratory studies. For example, Faul et al. (2004) and other authors presented frequency-dependent intrinsic $Q$ estimates in crustal and mantle rock samples at seismic frequencies. However, in correlating the $Q$ values arising from different types of observations, it is important to keep in mind the type of quantity that is being measured. Because no single parameter describing the ability of the medium to dissipate the elastic energy exists, $Q$ is usually introduced as a phenomenological proxy depending on the type of observation (Bourbié et al., 1987). Bourbié et al. (1987, Chapter 3) summarized a number of such measurements and noted that although most of them can be described by the corresponding visco-elastic models (and consequently $Q(f)$ dependencies), there is little agreement between the resulting values of $Q$. Therefore, while generally setting the target for looking for frequency-dependent attenuation within the mantle, laboratory or evidence or theoretical conjectures still should not influence rigorous and robust analysis of seismological data.
Considering the measurements of in situ attenuation in observational seismology, we need to establish what aspects of $Q$ are invariant in respect to the structural variability which is disregarded by the conventional measurement procedure. As a first-order parameter describing such variability, we can use the exponent $\nu$ in GS power law $t^\nu$, where $t$ is the travel time, or $\gamma$ in an alternate GS approximation $e^{\gamma t}$ used in Morozov (2008) and below. The sensitivity of both $Q_0$ and $\eta$ parameters in the commonly used $Q(f) = Q_0 f^\eta$ power law to the assumed GS is well known (e.g., Kinoshita, 1994), but its quantitative implications for interpretation in terms of in situ Earth’s properties still seem not well appreciated. GS models are usually considered as “reasonably” accurate, yet this is insufficient, because smaller than $\sim$10% variations in $\nu$ can eliminate the reported $Q(f)$ dependencies in many cases. Such level of GS variability should be common in the Earth, as caused by variations in crustal thickness, velocity gradients, layering, reflectivity, and other attributes of the lithospheric structure (Morozov, 2008). No single GS model can be constructed as a common reference for $Q(f)$ measurements across a significant area. As a consequence of using simplified models for GS (e.g., $\nu = \frac{1}{2}$, 1, or any other constant or different functional form), structural variations become imprinted in $Q(f)$. Because of their mutual trade-off with the residual GS, $Q_0$ and $\eta$ should also not be interpreted separately, and their stable combinations need to be sought. One of such useful combinations is $\gamma$ (or $\nu$), as shown by Morozov (2008) and will be illustrated below.

Unfortunately, analysis of GS effects tends to be subdued in the use of elaborate $Q(f)$ inversion methods, curve-fitting, and in pursuit of detailed solutions for $Q$ from typically limited and noisy datasets. Interpreters may often be inspired by elegant models, and experimentators prefer using GS given by simple analytical formulas – both parties thereby encouraging complex and permissive $Q(f)$ dependencies. Assumptions have become a part of not only theoretical treatments but also of data measurements, in which cases they are termed “practical.” However, assumptions may be acceptable only when used to simplify the understanding of the problem. As I show below, GS assumptions made early in $Q(f)$ inversion may in fact make it more complex by encouraging under-constrained parameterizations.

When using model parameterizations with more degrees of freedom than
constrained by the data, the results may become mostly controlled by “a priori constraints,” dependency on “starting models” develops, and a danger of biased or “preferred” solutions arises. For example, in standard absolute-amplitude attenuation measurements (e.g., Aki, 1980), there were three unknowns: GS, $Q_0$, and $\eta$; however, the data supported only a two-parameter inversion, and therefore GS was forced to match some preferred model, for which a uniform space was selected. Morozov (2008) took a different approach, noting that GS can be measured directly before frequency dependence of $Q$ can be tested.

Thus, it appears that perspectives of exciting theoretical insights may have caused a drift from conservative, data-driven and model-independent observations towards complex $Q(f)$ models. A vast body of observational evidence has been advanced in favour of the frequency-dependent $Q$, such as supporting the mantle absorption band (e.g., Doornbos, 1983) and scattering models (e.g., Aki and Chouet 1975; Aki, 1980). However, note that such observations still present the apparent $Q(f)$ constrained by simplified GS compensations. Similarly, observations of $t^*$ values of $\sim 1$s for long-period body $P$ waves (compared to $\sim 0.2$ s at short periods) are usually viewed as another strong evidence for the mantle $Q(f)$ increasing with frequency (e.g., Der et al., 1986). Yet again, long-period $t^*$ measurements were based on GS compensation of absolute-amplitude long-period data, and an only $\sim 6\%$ correction in GS could bring these values to the short-period level (Morozov, submitted I²). Note that GS-independent (i.e., spectral-ratio) measurements can only be carried out within the higher-frequency band, in which there is no compelling evidence for a frequency dependent $t^*$ (Der et al., 1986).

Note that in many cases found in the literature, the quality of $(Q_0, \eta)$ solutions is only judged by their ability to fit the seismic amplitude data. This criterion is insufficient


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and misleading, because in under-constrained inversion, multiple models can fit the data equally well. For example, GS measurements (Morozov, 2008) fitted the local body wave data from Kinoshita (2008), but with the ~20-30 times larger, frequency-independent $Q$ values. Similar examples are given below. The true criteria of model fidelity lie not only in the data fit but also in its correspondence to the physics of the process, mutual independence of parameters, and validity of its assumptions.

In this paper after introducing the GS - $Q(f)$ trade-off problem on a data example and discussing the role of GS models in attenuation measurements, I offer a simple solution to the trade-off by explicit measurement of the residual GS remaining after GS compensation. The approach is illustrated by re-interpreting several contrasting examples (nuclear-explosion and earthquake coda, $Pn$, and surface waves), some of which have been at the foundation of the $Q(f)$ concept. Removal of the ad hoc GS assumptions leads to substantial differences from the conventional solutions. In particular, frequency-dependent $Q$ is no longer found, and the resulting $Q$ values are significantly increased. Finally, the mantle absorption band between frequencies of ~3\cdot10^{-3} – 30 Hz is also explained by slight uncertainties in the Earth’s GS, and particularly related to the variations in the crust/mantle structure. These effects appear to influence the GS in opposite ways at low and high frequencies, leading to the apparent absorption band. Instead of the frequency-dependent rheological $Q(f)$, this band can be explained by attenuation stratified in depth, particularly with high $Q$ values of ~1100 within the crust and low ~120 within the uppermost mantle.

**GS – Q(f) trade-off**

The equivalence of the frequency-dependent $Q \propto f$ to GS can be seen from the expression from which $Q(f)$ is usually inverted (e.g., Fan and Lay, 2003)

$$P(t, f) = G(t)e^{-\eta f / Q(f)},$$  \hspace{1cm} (1)

where $G(t)$ denotes the GS (here assumed to be frequency-independent for simplicity), $P(t, f)$ is the path factor from which the source and site effects were removed. Values of $\eta \approx 1$ in $Q(f) = Q_0 f^\eta$ are often found for scattering $Q_s$, such as the one used to explain the...
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*S*-wave attenuation at 1 – 30 Hz (Dainty, 1981). $Q_s \propto f$ cannot be distinguished from GS by any technique, either using spectral ratios or GS compensation, and for this reason, the use of $Q_s$ should be careful. If frequency-independent intrinsic attenuation $Q_i$ is added to $Q_s$: $Q^{-1}(f) = Q_s^{-1} + Q_i^{-1}$, then almost any observed positive $Q(f)$ dependence with $\eta \in [0, 1]$ can be reproduced. Finally, if $Q_i$ is also considered to be frequency-dependent, almost any apparent $\eta$ at all can be expected.

Without making a clear physical distinction between the GS and $Q(f)$, interpretation may become tricky and erroneous. For example, Figure 1 shows codas of the Peaceful Nuclear Explosion (PNE) Kimberlite-3 recorded within the Siberian Craton. According to the model proposed earlier (Morozov and Smithson, 2000), the scattered-wave GS was considered as compensated by the spreading scattering volume, leading to $G(t) \approx 1$ for the coda. Therefore, coda amplitude decay was initially interpreted as caused entirely by crustal attenuation. At the same time, time-domain log-amplitude slopes are practically independent of the frequency (Figure 1), leading to $Q_0 \approx 330$ and $\eta \approx 0.85$ for this PNE (Figure 2b). However, this was an erroneous result, which in fact indicated a failure of the initial model of coda-GS compensation. Such low $Q_0$ and high $\eta$ are unlikely in the stable Siberian Craton. The fact that crustal $Q$ at 0.5 – 1 Hz cannot be as low as ~200 - 300 in this area is apparent from observations of $Pg$ traveling to over 1600 km (likely the longest distance on Earth) along the PNE profiles in this area (Morozov et al., 2002). Numerical modeling of PNE coda (Morozov et al., 2008) also showed that steeply frequency-dependent coda $Q$ can be observed without any frequency-dependence in the rheological $P$- and $S$-wave $Q$ values.

Below, by utilizing the variability of GS, we will see that frequency-independent coda slopes in Figure 1 simply indicate their geometrical origin and low crustal attenuation. Similar observations of steep $Q(f)$ dependencies which should actually be explained by increased GS can be found in many crustal studies (e.g., Kinoshita, 2008; Mukhopadhyay et al., 2008; see also references in Morozov, 2008).

The role of geometrical spreading models

GS models play the key role in most conventional interpretations of attenuation.
Similarly to many other physical phenomena, it is important to differentiate between two categories of such models: 1) “mathematical” GS typically defined on the basis of some theoretical considerations, for example, by considering a “wave front” spreading with propagation time, and 2) phenomenological models trying to characterize the effects of GS without attempting its detailed mathematical characterization.

Unfortunately, the existing paradigm of not only modeling but also measurement of the frequency-dependent attenuation leans heavily toward the “mathematical” model types. The reported $Q(f)$ values and $(Q_0, \eta)$ parameters are either explicitly [for example, see the label in Figure 3a from Aki (1980)] or implicitly dependent on the assumed mathematical GS forms. For the very limited number of tractable cases, such theoretical models typically use the $G_0(t) \propto t^{-\beta}$ dependencies mentioned above.

However, the mathematical $G_0(t) \propto t^{-\beta}$ dependence does not occur in nature for practically any cases of interest. The concept of wave fronts themselves breaks down in a crustal model with velocity gradients and contrasts, in which triplications, reflections, and mode conversions are common. The concept of a “ray” which could be followed in order to track the $G(t)$ dependence is also absent in a realistic medium (this effect is sometimes referred to as “multi-pathing”). Note that even in the purely theoretical cases of stronger structural contrasts, such as $Pn, Sn$ (e.g., Yang et al., 2007), or $PL$ (Aki and Richards, 2002), the $t^{-\beta}$ dependence is violated and also becomes frequency-dependent. Therefore, theoretical models are not helpful in defining what realistic GS is. By using full waveform modeling in 3D, one could in principle accurately solve for the wave amplitudes in the absence of attenuation; however, this may still be hampered by insurmountable difficulties of insufficient knowledge of the velocity/density structure, and uncertainties in the source radiation patterns and receiver site effects. Thus, neither the $G_0(t) \propto t^{-\beta}$ nor any of its improved modifications can be considered sufficiently accurate and accepted without verification. Note that high accuracy expected from such models, because the total effect of $Q(f)$ typically consists in only a few per cent of the GS.

By contrast, phenomenological models do not require detailed descriptions of the mechanism of the process under consideration but may be based on general principles, such as the conservation of energy and time/spatial continuity. Such a model for GS was
proposed in Morozov (2008) and is further employed here. In this model, the GS factor is allowed to weakly deviate from the best-known background is the “theoretical” $G_0(t)$ discussed above

$$G(t) = G_0(t)e^{-\gamma t},$$  \hspace{2cm} (2)

where $\gamma$ is the “geometrical attenuation” parameter (Morozov, 2008). Note that one should not attribute excessive significance to the functional form of the amplitude correction factor $e^{-\gamma t}$ in eq. (2). This model does not attempt simulating any particular wave-spreading process, which would again be an intractable problem. It is empirical, and as shown in the following section, the exponential form selected in eq. (2) originates from the scattering theory and highlights the important similarity between measuring the attenuation (i.e., $Q^{-1}$) and GS variations ($\gamma$).

Mathematically, suitability of the correction parameter $\gamma$ is obvious for cases where $\gamma t \ll 1$, in which other approximations [e.g., $G_0(t)(1-\gamma t)$ or $G_0(t)t^{\gamma t}$] would work as well (Morozov, 2008). It is just a first-order deviation from $G_0(t)$. For local-earthquake studies, the meaning and characteristic values of $\gamma$ are illustrated in Figure 4 showing numerical modeling of GS in a two crustal models: a) simple positive-gradient model with IASP91 crustal thickness, and b) an more realistic crustal structure inverted from detailed wide-angle seismic in the East European Platform (Russia; Morozova et al., 1999). Modeling was performed by using the 1D reflectivity method by Fuchs and Müller (1971), for a point source located at 7-km depth. The peak (mostly body-wave; black symbols) and whole-trace (mostly surface-waves; grey symbols) energies were compensated with the theoretical $r^2$ factor and plotted ($r$ is the source-receiver distance in this case). As one can see, with the use of appropriate coordinate scales (ln$r$ or $r$, respectively) the compensated amplitudes appear as near-linear within $\sim$ 0-70 km distance ranges in both the $t^{\gamma t}$ (middle row in Figure 4) and $e^{-\gamma t}$ forms (bottom row). Note the positive value of $\gamma = 0.008 \text{ s}^{-1}$ in the realistic model (Figure 4b, bottom). Such values appear typical for stable continental crust (Morozov, 2008) and could be explained by the presence of downward-reflecting sedimentary cover (Figure 4b, top). Only in featureless models like the one shown in Figure 4a, the simple $t^\gamma$ GS compensation is adequate, and
therefore $\gamma = 0$ (i.e., $\nu = 1$).

Thus, variations of GS (i.e., of the attenuation-free amplitude decay within the Earth structure) from the background model $G_0(t)$ should be common, and the key idea of the present approach consists in measuring them before estimating $Q(f)$. The background model can be elaborate (for example, such as shown in Figure 4b), spatially-variable in 3D, and also frequency-dependent. Yet even with the currently best-known model, the true GS should still be allowed to differ from it, which can be reflected in parameter $\gamma \neq 0$. As shown below, the definition of $G(t)$ in eq. (2) becomes based on the measured frequency-independent part of the attenuation coefficient instead of any assumptions about the mechanisms of wave propagation. This phenomenological $G(t)$ can therefore be viewed as model-free, in the sense of its being entirely controlled by the data and independence of the choice of the (reasonably close) theoretical or numerical model for $G_0(t)$. The background model only serves as a common reference, similarly, for example, to using a coordinate origin in geometry.

Separation of geometrical spreading and attenuation

Attempting to give a phenomenological definition for GS that would reflect its intuitive meaning for interpretation and not depend on simplistic wave-front models, I suggest: “GS is the effect of the large-scale, attenuation-free structure on seismic amplitudes.” The meaning of “large-scale” here is of course relative and defined by our viewing certain structures (such as crustal gradients, boundaries, blocks, and topography) as “deterministic,” as opposed to small-scale structures that are considered “random,” treated statistically, and described by attenuation. Fortunately, although the inversion for the full GS may be very complex and uncertain, its small variations can be readily measured together with the variations of attenuation.

Considering that the $P(t, f)$ dependence (1) represents essentially all the data available for GS and attenuation measurement, it is clear that their separation could only be done based on differentiating between its frequency-independent and dependent parts. Fortunately, such differentiation corresponds to the important difference in the physics of the GS- and attenuation-related processes, as described by Morozov (2008). I will not
3) The attenuation described by $Q$ is a secondary effect which can only be established when GS is known. In other words, GS can exist without attenuation, but $Q(f)$ in a structure with unknown GS (i.e., the large-scale structural effect) is meaningless.

4) The effect of attenuation typically increases with frequency. This is indicated by the factor $f$ in the numerator of the exponent in eq. (1), corresponding to the definition of $Q^{-1}$ as fractional energy dissipation per oscillation cycle in mechanics. Thus, it appears reasonable to postulate the effect of the in situ $Q$ to equal zero at $f = 0$ and to increase with the frequency.

Further, let us assume that $P(t, f)$ is close to some background approximation $G_0(t)$ with the difference being due to a variation in GS and attenuation. By selecting the appropriate scaling for $G_0(t)$, we can make these quantities equal at $t = 0$, and consider the first-order deviation of $\ln[P(t, f)/G_0(t)]$ in $t$:

$$\ln P(t, f) = \ln G_0(t) - \chi(f) t.$$  

(3)

This is the standard scattering- (perturbation-) theory approximation (e.g., Chernov, 1960). Once again, its validity is limited to time intervals for which $\chi(f)t \ll 1$, which is typically satisfied with correct choices for $G_0(t)$. For the same reason, higher-order in $t$ terms are also ignored in eq. (6). Note that $\chi(f)$ was denoted $\alpha(f)$ in Morozov (2008) and renamed here to avoid collision with the temporal attenuation coefficient (Aki and Richards, 2002).

Further, it is convenient to explicitly isolate the frequency-independent part $\gamma = \chi(0)$ of the attenuation coefficient $\chi(f)$:

$$\chi(f) = \gamma + \kappa f.$$  

(4)

In this expression, $\gamma$ is interpreted as the improved estimate of the true GS in eq. (2). Parameter $\kappa$ should be related to the effective attenuation caused by intrinsic dissipation
and small-scale scattering, and therefore it can be written by using an effective quality factor $Q_e$: $\kappa = \pi Q_e$. The meaning of this factor can also be understood as follows: if, in the attenuation-free crust of Figure 4b, one “turns on” an attenuation of $Q_e^{-1}$, the amplitudes recorded at any point would reduce by the observed factor $\exp(-\kappa f)$. This quantity thus resembles the “apparent resistivity” in electrical surveying.

Generally, both $\kappa$ and $Q_e$ can be frequency-dependent if $\chi(f)$ shows a non-linear behaviour with varying $f$. However, in all datasets I reviewed so far (Morozov, 2008 and below), such frequency dependencies were not required, and so the question of frequency dependence of $\kappa$ remains open.

Scattering is particularly difficult to differentiate from GS and intrinsic attenuation effects. $Q_s$ can be defined only in relation to some “theoretical” background model, and with the use of realistic (e.g., phenomenological) models, the corresponding attenuation coefficient $\chi_s(f) = \pi f Q_s^{-1}$ becomes distributed between $\gamma$ (frequency-independent part) and $\kappa$ (frequency-dependent part). In many cases, $Q_s$ quickly increases with frequency, which means that it is dominated by the variations of $\gamma$ (Morozov, 2008). For example, numerical simulations in realistic lithospheric structures with intrinsic $Q^{-1} = 0$ (e.g., Figure 4b; Morozov et al., 2008) show myriads of reflections and mode conversions resulting from crustal discontinuities. In particular, reflections from major, spatially separated contrasts, such as the Moho, are nearly frequency-independent, and consequently the associated $Q_s$ would be nearly proportional to the frequency. Such frequency dependence was indeed predicted for scattering on “larger” heterogeneities (Dainty, 1981). However, treating the Moho, as well as other major crustal features as random scatterers in a uniform space and describing them by $Q_s$ appears inadequate and misses most points of seismic interpretation. A description of the Earth by a combination of variable GS, intrinsic $Q_s$ and $Q_e$ would be over-parameterized, and its regularization by making grossly inaccurate assumptions about GS would lead to spurious values of $Q_s$.

For these reasons, I suggested (Morozov 2008, 2009a,b) that in real-data observations, $Q_s$ is indistinguishable from GS and $Q$ and should be abandoned in their favour.

Note that separation of the frequency-independent and frequency-dependent terms
in eq. (3) represents practically the only way for extracting two parameters from the single observed $\chi(f)$ dependence. For a consistent terminology, it would be best to simply characterize the Earth’s attenuation structure in terms of $\gamma$ and $\kappa$, without separating them into GS, intrinsic attenuation, and scattering. If such separation is nevertheless desired, simplifications have to be made. Because we cannot rely on model-based arguments, I interpreted $\gamma$ as the residual GS and combined the intrinsic attenuation and small-scale scattering in $\kappa$ (Morozov, 2008). Alternatively, if we had some additional evidence for a frequency-dependent GS: $\gamma = \gamma + \gamma f$ (for example, such as modelled in Yang et al., 2007 or Morozov et al., 2008), then the intrinsic attenuation and scattering would account for the residual $\kappa' = \kappa - \gamma$. In any case, separation of the effects of scattering from either $\kappa$, $\kappa'$, or $\gamma$ appears unfeasible. The existing approaches to such separation (e.g., Wu, 1985; Jin et al., 1994) rely on comparing the $\chi(f)$ at different distances from the source, which again implies the artificial constraint of $\gamma = 0$.

From eq. (1), the resulting procedure for separating the GS and $Q$ effects and for analyzing $Q$ for frequency dependence is as follows: A) extract $\chi(f)$ values by calculating linear regressions of $\ln[P(t,f)/G_0(t)]$ in $t$ at different frequencies, and B) analyze the resulting dependence of $\chi(f)$ on the frequency according to eq. (3). In particular, we need to examine (extrapolate) the limit of $\gamma = \chi(0)$ and linearity of the $\chi(f)$ dependence. Note that because $Q_e$ typically turns out to be frequency-independent, measurement of $\gamma$ does not require amplitude measurements at $f = 0$ and can also be accomplished by linear regression (eq. 3).

Although procedure A)-B) above should normally be applied to the source- and receiver-corrected observed amplitude $[P(t,f)]$ data (e.g., to spectral ratios), the attenuation coefficient can also be estimated from published $Q(f)$ results (Morozov, 2008) by writing

$$\chi(f) = \pi f Q^{-1}(f) = \pi f^{1-\eta} Q_0^{-1}. \tag{5}$$

Measurement of this quantity is part to virtually any $Q(f)$ inversion. Unfortunately, $\chi(f)$ values are rarely plotted prior to their conversion to $Q^{-1}(f)$, and step B) is practically
never performed. Only in turbidity-type studies (Dainty, 1981; Padhy, 2005), linear approximations for \( \chi(f) \) were considered. Note that the “stacked spectral ratios” (SSRs; e.g., Xie, 2007, and many other papers by Xie and Mitchell) actually represented \( \chi(f) \) but plotted in logarithmic frequency scales, which did not allow measuring the intercept values (\( \gamma \)) and recognizing the linearity of \( \chi(f) \) dependences.

In terms of fitting the attenuation data, parameterizations of the attenuation coefficient in terms of (\( \gamma \), \( Q_e \)) (eq. 3) and (\( Q_0 \), \( \eta \)) (eq. 5) may appear equivalent. Indeed, as discussed in some detail in Morozov (2008), within finite observation frequency bands, both of these parameterizations can be used to fit the data, and they can be transformed into each other. However, the two parameterizations are strongly different conceptually. Formula (5) does not allow a non-zero value of \( \chi(f) \) at \( f = 0 \) whereas formula (3) does. As a result, when the observation frequencies drop below the “cross-over” frequency \( f_c = \gamma Q_e / \pi \), approximation (5) becomes unstable, i.e., leads to high values of \( \eta \). Comparative analysis of the \( \chi(f) \) data and their \( Q(f) \) interpretations (Morozov, 2008, and examples below) indeed shows that higher \( \eta \) values are commonly inferred in observations conducted at lower frequencies \( f < f_c \).

As stated above, with the available attenuation data quality, differentiation between parameterizations (3) and (5) cannot be done based on the data fit alone. With both approximations, nearly straight data trends are typically revealed in the (\( f, \ln \chi \)) and (\( \ln f, \ln \chi \)) coordinates, respectively. Note that both \( Q_0 \) and \( \eta \) are also typically dependent on the frequency band, which makes dependence (5) totally general (if not overly complex). Nevertheless, there also is a substantial theoretical evidence in favour of the more basic formula (3), which can be summarized as follows:

1) Close relation to the observable attenuation coefficient \( \chi(f) \), as opposed to \( Q \), which depends on the incident wavelength and therefore is not a true medium property (for more detail on this, see Bourbié et al., 1987);

2) Direct and simple relation to the scattering theory (eq. 6);
3) Simple mathematical relation to $\chi(f)$ (basically, a Taylor series approximation), as opposed to a complex power-law, which may be justified for scattering on self-affine structures but not in the general case;

4) Explicit recognition of the residual GS variation in terms of $\gamma$;

5) Correctness of terminology and interpretation: for example, a perturbation of the amplitudes caused by varying crustal velocity gradient is attributed to changing GS ($\gamma$) and not to "attenuation" (i.e., $Q_0$ and $\eta$) and particularly to "scattering $Q$" (Morozov, 2009a,b);

6) Tolerance to changes in the background $G_0(t)$ model: only $\gamma$ varies in a predictable manner, as opposed to both $Q_0$ and $\eta$ trading off;

7) Linearity of the theoretical $\chi(f)$ dependencies expected for: a) surface waves in layered structures with zones of pronounced attenuation highs (such as the upper mantle in the last example below), and b) body waves with perturbed GS (such as the crust in Figure 4b);

8) Observations of linear $\chi(f)$ trends in many datasets and within broad frequency bands (examples below and in Morozov, 2008, 2009a), apparently not requiring anything else but the simple linear law (3);

9) Predictability of $\gamma$ by numerical waveform modeling (Morozov et al., 2008) and its correlation with tectonic structures even in cases in which the classification based on ($Q_0$, $\eta$) fails (Figure 2; Morozov et al., 2008).
Examples

Figure 2 illustrates the sharp difference in the interpretation resulting from the phenomenological GS approach to our PNE example. By measuring the log-amplitude coda slopes within the $L_g$ coda time window (Figure 1), values of $\chi(f)$ are directly obtained for the corresponding frequencies (Figure 2a). The $\chi(f)$ dependence shows a linear trend [eq. (3)] indicating an under-compensated GS with $\gamma \approx 0.0075$ s$^{-1}$ and a frequency-independent $Q_e \approx 2500$ (Figure 2a). This $Q_e$ is much higher than $Q_0 \approx 330$ interpreted in the conventional way, and the steep apparent $Q(f)$ dependence with $\eta \approx 0.85$ (Figure 2b) becomes explained by the dominance of the geometrical contribution in coda amplitude decays (Figure 1).

Due to their independence of theoretical model assumptions, the resulting ($\gamma, Q_e$) values allow consistent interpretations and comparisons with other studies. The value of $\gamma$ is remarkably close to that measured from recordings from another PNE Quartz-4 in the East European Platform, and also almost exactly corresponds to the predictions from waveform coda modeling (Morozov et al., 2008). Therefore, at least in this area, ($\gamma, Q_e$) model is clearly advantageous and shows that the strong apparent $Q(f)$ (Figure 2b) is a geometrical artefact.

To see that the above argument applies not only to PNE codas, consider an example of local earthquake codas in central California, Hawaii, and central and western Japan (Aki, 1980) (Figure 3a). These observations made the key contributions in establishing the scattering-$Q(f)$ concept. In our notation, Aki (1980) used a theoretical GS of $t^{-1}$ (labelled in Figure 3a) in eq. (6) to compensate the time-domain log-amplitude coda slopes and converted $\chi(f)$ into $Q^{-1}(f)$ by using the inverse of eq. (5). Two observations were made from the resulting curves (Figure 3a): 1) three of the four $Q^{-1}(f)$ dependencies appeared to converge at the higher frequencies, and 2) station TSK in central Japan showed a distinctly different attenuation mechanism from the other areas.

However, re-plotting the same data in terms of $[-\chi(f)]$ decay rates reveals linear attenuation patterns (dashed lines in Figure 3b) and suggests different conclusions. The convergence of the inferred $Q^{-1}(f)$ near $\sim 25$ Hz (Figure 3a) could be principally related to
dividing the values of attenuation coefficients $\chi(f)$ by the frequency. This division also somewhat exaggerated the difference between the results from stations OIS and OTL below 5 Hz (Figure 3a). By contrast, a comparison of the intercepts and slopes of the attenuation coefficient trends (dashed lines in Figure 3b), shows that: 1) all GS terms $\gamma$ are positive, 2) in Hawaii and for both areas in Japan, $\gamma$ values are lower, from (0.02 – 0.03 s\(^{-1}\)) than in central California (0.06 s\(^{-1}\)); 3) attenuation values in Hawaii and western Japan are quite similar, with $Q_e \approx 600$; 4) in central Japan, attenuation is low, with $Q_e \approx 2300$; and 5) in terms of both parameters, central California is distinctly different from the other three areas, with its $\gamma \approx 0.06$ s\(^{-1}\) and $Q_e \approx 1250$. On top of the linear $[-\chi(f)]$ trends, some “spectral scalloping” also becomes clear, particularly the reduced amplitudes near 6 Hz and increased at ~1-2 Hz and 12 Hz in (Figure 3b). These amplitude variations are consistent in all four cases and also present in the $Q^{-1}(f)$ form, although less clearly because of the logarithmic frequency scale (Figure 3a).

To further illustrate the attenuation model (3) for regional body waves, consider the apparent $Pn$ $Q$ results from a recent INDEPTH study by Xie (2007) (Figure 5). The quantity plotted along the vertical axis in Figure 5a is the stacked spectral ratio, $SSR = \frac{\eta}{Q_0} = \frac{f}{Q(f)}$ (Figure 5a), and therefore only plotting of $[-\chi(f)] = -\pi SSR$ against a linear frequency scale is needed in order to examine the attenuation coefficient (Figure 5b). As Figure 5 shows, a linear $\chi(f)$ dependence fits the spectral ratios similarly to the $Q_0$$^\eta$ function, maybe even somewhat better at lower frequencies.

The best-fit geometrical $Pn$ spreading parameter is $\gamma \approx 0.002$ s\(^{-1}\), and $Q_e$ is approximately 340 (dashed black line in Figure 5b). From the same data, Xie (2007) gave values of $\eta = 0.14$ and $Q_0 = 278$, which is ~25% lower than $Q_e$. For such moderate $\eta$, values of $(Q_0, \eta)$ can be approximately transformed to $(\gamma, Q_e)$ (Morozov, 2008), resulting in $Q_e = 400$ and $\gamma = 0.005$ s\(^{-1}\) from our values (grey line in Figure 5b). This transformation seems to slightly over-estimate the $Q_e$ directly measured from the $[-\chi(f)]$ plot (Figure 5b), although still acceptably. Note that this is the only real data example I have found in which a solution with $\gamma = 0$ appears acceptable (i.e., the GS compensation error lies within data uncertainties). By setting $\gamma = 0$, we obtain: $Q_e = 310$ (dotted line in

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This value of $Q_e$ is the closest to $Q_0 = 278$ by Xie (2007). Because of $\eta > 0$, all of the resulting values of $Q_e$ are above the corresponding $Q_0$.

From eq. (3), the slope of the SSR’s in log-log plots (Figure 5a) is:

$$\frac{d \ln \chi}{d \ln f} = \frac{d \chi}{df} \frac{f}{\chi} = \frac{1}{1 + \frac{f_e}{f}},$$

where the “cross-over” frequency $f_c = \gamma Q_e/\pi$. The meaning of $f_c$ is such that for $f = f_c$, attenuation effects equal those of the residual GS (Morozov, 2008). For this dataset, $f_c \approx 0.2$ Hz, and therefore the slope is approximately equal 1 for the observation frequencies, which are all $f >> f_c$. Note that the slope of $\ln\text{SSR}(\ln f)$ is exactly 1 for the best-fit line $Q(f) = Q_0 f^n$ used by Xie (2007).

In Figure 5a, Xie (2007) also illustrated the trade-off of both $(Q_0, \eta)$ parameters with the assumed background geometrical spreading of the form $r^\nu$, one of which was frequency-dependent. Attenuation-coefficient parameters ($\gamma, Q_e$) are model-independent (Figure 5b), and therefore it is sufficient to use a single (any one) background model for plotting. With the use of the frequency-dependent geometrical spreading of $Pn$ waves (e.g., Yang et al., 2007), a part of the resulting $[-\chi(f)]$ slope would be absorbed by the $\nu(f)$ dependence, leading to modified values of $\gamma$ and $Q_e$. Apart from statistical data-fitting errors, this is the only uncertainty in these parameters.

Another important geometrical effect causing apparent $Q(f)$ was illustrated by numerical modeling in surface-wave (Anderson et al., 1965), $Lg$ (Mitchell, 1991), and PNE coda amplitudes (Morozov et al., 2008). These studies demonstrated that layered structures with depth-dependent intrinsic shear-wave $Q$ lead to frequency-dependent attenuation observed on the surface. Spherical-mode summations show that long-period surface-wave $Q$’s quickly decrease with frequency, in agreement with the observations (Anderson et al., 1965; Figure 6b). At the first glance, such dependence may be caused by longer-period waves sampling progressively deeper layers with higher $Q$ (Figure 6a). However, this may not be the main reason. Most surface-wave modes have their largest amplitudes in the uppermost mantle, where the attenuation is the highest, and this depth
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range should in fact have the dominant effect on the attenuation at all frequencies. In the attenuation-coefficient form, values of \( \chi(f) \) fall on almost straight lines within the entire modeled frequency band (Figure 6c). Interestingly, the effect of surface-wave depth sampling changing with frequency results in effectively “geometrical” (independent of \( f \)) shift in \( \chi(f) \). The resulting frequency-independent values of \( Q_e \) are 121 for Rayleigh waves and 112 for Love waves (Figure 6b). Note that these values are close to each other and also to the lowest \( Q \)'s near 0.02 Hz (Figure 6b). Such frequency-independent \( Q_e \) suggests that at all periods, the attenuation of both Rayleigh and Love waves should be principally accumulated at the sub-crustal depths (~38 – 100 km), because such \( Q \) values are present only within this depth range in the model (Figure 6a).

Note that the attenuation-coefficient intercept values \( \gamma \) are negative for the surface waves, showing that their GS is over-compensated in the model. For Rayleigh waves, \( \gamma \approx -4 \cdot 10^{-5} \text{ s}^{-1} \), and for Love waves, \( \gamma \approx -2 \cdot 10^{-5} \text{ s}^{-1} \) (Figure 3c). Reasons for such over-compensated modeled GS still need to be examined; one of them could be the use an analogy of \( 1/2 Q \) with a phase shift in complex-valued phase velocity (Anderson et al., 1965). Such analogy appears inaccurate and may lead to underestimated \( Q \) values (Morozov, submitted II). Nevertheless, at this point, it is important to note that negative values of \( \gamma \) are present in \( Q(f) \) data at long periods, and they can be measured by the \( \chi(f) \) method.

**Apparent mantle absorption band**

The combined examples in Figures 2-6 suggest a different explanation of the observed mantle absorption band (Anderson et al., 1977; Anderson and Given, 1982). The increase in apparent long-period \( Q \) toward lower frequencies results from negative long-period \( \gamma \) values, denoted by \( \gamma_{LP} \) here:

\[ \]
$Q_{LP}(f) = \frac{\pi f}{\gamma_{LP} + \pi f/Q_e} = \frac{Q_e}{1 - f_{c,LP}/f},$ (7)

where the “cross-over” frequency $f_{c,LP}$ equals $(-\gamma_{LP}Q_e/\pi)$. For Rayleigh waves, $f_{c,LP} \approx 1.5 \cdot 10^{-3}$ Hz, which is significantly lower than the typical surface-wave frequencies. Therefore, $Q(f)$ steeply increases when frequency drops to $f_{c,LP}$ (grey lines in Figure 3b).

For short-period body waves, $\gamma$ is positive, and the corresponding $f_{c,SP} = \gamma Q_e/\pi$ (Morozov, 2008). Consequently, the short-period $Q(f)$ is:

$Q_{SP}(f) = \frac{\pi f}{\gamma_{SP} + \pi f/Q_e} = \frac{Q_e}{1 + f_{c,SP}/f}. \quad (8)$

With the typical crustal values of $Q_e \approx 1100$ and $\gamma_{SP} \approx 0.006$ s$^{-1}$ for tectonically-stable areas and $\gamma_{SP} \approx 0.02$ s$^{-1}$ for tectonically-active ones (Morozov, 2008), we obtain $f_{c,SP} \approx 2.8$ Hz and 7.8 Hz, respectively. Expression (6) shows that $Q_{SP}(f)$ increases with frequency at short periods, and consequently, the two characteristic frequencies $f_{c,LP}$ and $f_{c,SP}$ delineate a band of reduced apparent $Q(f)$ (Figure 4). Note that the character of this band is purely geometrical, and its attenuation levels are controlled by only two frequency-independent values of $Q_e = 1100$ (within the crust) and $Q_e = 121$ (within the uppermost mantle), shown by grey bars in Figure 4.

Let us also compare the predictions from expressions (5) and (6) to the mantle absorption band model (ABM; Anderson and Given, 1982). In ABM, the high-attenuation $Q$ level is controlled by a fixed minimum level $Q_m = 80$, which is analogous to $Q_e$ in our interpretation. However, in ABM, $Q_m$ is assumed to be constant within the entire mantle, whereas Figure 7 shows that $Q_e$ may mostly be formed by the increased attenuation within the relatively thin, low-$Q$ subcrustal mantle. For $f >> f_{c,LP}$, eq. (5) yields $Q(f)$ values inversely proportional to $f$: $Q_{LP}(f) \approx Q_e f_{c,LP}/f$, which corresponds to the low-frequency flank of ABM with cut-off parameter $\tau_2 = Q_e f_{c,LP} = -\pi/\gamma_{LP}$. For body waves at $f << f_{c,SP}$, $Q(f)$ is nearly proportional to the frequency (eq. 6): $Q_{SP}(f) \approx Q_e f_{c,SP}$. Such $Q \propto f$ dependence forms the upper flank of the ABM band, in which parameter $\tau_1$ becomes $\tau_1 = Q_e f_{c,SP} = \pi/\gamma_{SP}$. Therefore, quantifies $\tau_1$ and $\tau_2$ defined by Anderson and
Given (1982) as the cut-off levels in the Debye distributions of relaxation times may simply be reciprocals of the residual GS parameters for body and surface waves, respectively. Note that our model shows that the short-period flank of this band becomes higher (with \( Q_0 \) increasing) and flatter (\( \eta \) decreasing) with tectonic age because of the decreasing \( f_{c,SP} \) and \( \gamma \) (Figure 7; for more on this, cf. Morozov, 2008). For comparison, the ABM model accounts for the crust by shifting the \( \tau_1 \) value to zero. The present model also readily incorporates the differences between the oceanic and continental lithospheres by including the corresponding \( \gamma \) and \( Q_e \) values in its short-period flank (Figure 7). Further comparison of these models is complicated by the complex (i.e., over-parameterized) nature of ABM using multiple wave modes (with likely variable GS) and depth-dependent parameters (\( \tau_1, \tau_2 \)) in combination with a constant \( Q_m \), and various other assumptions.

**Discussion**

In the examples presented here, an analysis of geometrical spreading allowed us drawing several important conclusions directly from the data within a broad frequency band. However, if \( Q(f) \) is allowed to trade-off with GS, both real and modeled spectral amplitude-decay data can still be fit in either (\( \gamma, Q_e \)), (\( Q_0, \eta \)), or apparently many other forms. This ambiguity is well known, yet the ultimate goal of seismic interpretation is in finding an unambiguous and useful description of the Earth. The reasons for preferring any of such forms should be in the underlying nature of the measured quantities and consistency of the results, and not in the spectral-amplitude data fit alone.

Independence of inaccurate assumptions and stability of the results is just one major argument in favour of the (\( \gamma, Q_e \)) approach. Another argument is that \( Q_e \) turns out to be frequency-independent in many cases (examples above, and also in Morozov, 2008), which is convenient for interpretation and avoids complexity where it is not dictated by the data. This interpretation reveals that frequency-dependent \( Q \) may not be as pervasive as it is often thought. Further, because some reference \( G_0(t) \) (i.e., a fixed \( \gamma \)) is always implied in (\( Q_0, \eta \)) interpretations, there is no reason to believe that this reference should stay constant across significant areas. This makes comparisons of
regional variations in \((Q_0, \eta)\) complicated, because certain correlated combinations of these parameters (specifically, \(\gamma\)) mostly describe the structure and not attenuation. Mapping of \((Q_0, \eta)\) is useful to summarize the wave amplitude properties, but the trade-off between these quantities may make correlations of these parameters to geology ambiguous. By contrast, the \(\chi(f)\) approach addresses the GS variability by measuring the regional variations of \(\gamma\) and \(Q_e\) separately, and these quantities are therefore directly transportable and comparable.

As a general comparison of the \((\gamma, Q_e)\) vs. \((Q_0, \eta, \text{and assumed GS})\) approaches, note that the broadly observed high and positive values of \(\eta\) for most crustal seismic waves suggest that scattering and/or attenuation should drop quickly at smaller scale-lengths and higher frequencies. This is contrary to common observations of geological heterogeneity increasing at shorter scale-lengths and difficult to accept as a common property of the Earth material. By contrast, the \((\gamma, Q_e)\) model explains such observations quite simply: in areas where the apparent \(\eta\) is positive, \(\gamma > 0\), and \(Q_e\) is in fact significantly higher than \(Q_0\) (Figure 2). In other words, GS is under-compensated by the traditional \(G_0(t)\) corrections. Such faster-than-\(G_0(t)\) GS from earthquake sources appears to be a common crustal property, which can be related to the upper-crustal reflectivity [see Appendix in Frankel et al. (1990); more on this will be given elsewhere]. Values of \(\gamma\) also correlate with tectonic ages, which should be related to changes in the velocity structure with age (Morozov, 2008). Thus, from its simplicity, stability, convenience, and links to geology, the \((\gamma, Q_e)\) model (eqs. 6-3) appears to be superior to the conventional \(Q(f)\) view. In many cases, the reported \((Q_0, \eta)\) values may contain effects of insufficiently accurate solutions for GS and should be interpreted with caution. This warning particularly applies to the cases of larger \(\eta\) and separation of elastic and anelastic attenuation, which heavily relies on simplistic GS models combined with elaborate inversions (Morozov, 2009).

Fortunately, transformation (5) allows approximate cancellation of this uncertainty and estimation of GS effects from existing \(Q\) data, as done in this paper. However, for estimation of fitting errors and evaluation of other frequency-dependent
amplitude effects (such as shown in Figure 3b), $\chi(f)$ measurements from raw data are still required.

**Conclusions**

The accepted methodology of measuring the frequency-dependent seismic attenuation using the quality factor $Q(f)$ is prone of uncertainties related to the variations of geometrical spreading (GS) in geologic structures. In many cases, frequency-dependent attenuation $Q(f) = Q_0f^\eta$ derived in time-domain, absolute-amplitude $Q$ measurements could be an artefact of the residual (uncompensated) GS. An alternate characterization by means of the attenuation coefficient $\chi(f)$ offers an unambiguous interpretation free from such uncertainties. This approach describes the attenuation together with the GS variation by using the geometrical ($\nu$ or $\gamma$) and effective ($Q_e$) attenuation parameters, corresponding to the frequency-independent and dependent parts of $\chi(f)$, respectively. These parameters are estimated directly from the data and without any theoretical assumptions.

The approach applies to many types of seismic waves (e.g., surface-, body-, $L_g$, and coda) in different frequency bands, and was illustrated on re-interpretations of several key published results. In all cases considered, $\chi(f)$ showed linear dependencies on the frequency within the available data uncertainties. The resulting $Q_e$ values were therefore frequency-independent and typically significantly higher than the corresponding reported $Q_0$. Notably, $\gamma$ values were positive for crustal body waves and negative for upper-mantle surface waves. Such values suggested a GS-based explanation of the Earth’s absorption band, whose magnitude is controlled by the contrast between crustal ($Q_e \approx 1100$) and uppermost-mantle ($Q_e \approx 120$) attenuation, and the low and high cut-off frequencies – by the geometrical “cross-over” frequencies $f_c \approx 1.5 \cdot 10^{-3}$ Hz and $\sim 3 – 8$ Hz, respectively.

In several cases, the $\chi(f)$ view led to significant changes in the interpretations. In particular, the values of ($\gamma$, $Q_e$) suggested a different grouping of the results in local-coda studies by Aki (1980). The frequency-independent $Q_e$ indicates no need for relaxation mechanisms or scale-length selective scattering within the crust or mantle, at least from

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the data considered in this paper.

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References


Figures

Figure 1. Stacked amplitude envelopes from six recordings within 660 – 760-km distances from the Peaceful Nuclear Explosion Kimberlite-3 in Siberia. Records are filtered within several frequency bands (labelled). Regional phases and the coda attenuation measurement interval are indicated. Times are aligned at the $Lg$ arrival, and $\ln(\text{Amplitude})$ envelopes are shifted in order to separate the curves. Note that the coda amplitude decay rates are almost independent of the frequency.
Figure 2. Interpretation of PNE Kimberlite-3 coda amplitude decays (Figure 1): a) by using the residual GS ($\gamma$) and frequency-independent effective $Q_e$ [eq. (3)]; b) by assuming a theoretical GS compensation, the resulting in $Q(f)$ steeply increases with frequency. Note that $Q_e >> Q(f)$ within the measured frequency band.
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Figure 3. a) Local-earthquake coda $Q^{-1}(f)$ from Aki (1980). Labels indicate seismic stations: PAC – central California, OIS – western Japan, TSK – central Japan, and OTL – Hawaii. b) The same data transformed to attenuation coefficients $-\chi(f)$ [eq. (6)] and plotted in a linear frequency scale. Only one line is shown for station OTL. Note the interpreted linear $\chi(f)$ trends (dotted lines and labels).
Figure 4. Models of GS results for: a) a hypothetical gradient model of the crust; b) realistic crustal structure in Russia (Morozova et al., 1999). Top row: $V_P$ and $V_S$ velocity models; Middle row: geometrical spreading within near-offset ranges, in logarithmic distance scale; Bottom row: the complete distance range in linear scale. Grey diamonds show the total recorded energy and black crosses – peak energy in two (radial and vertical) components combined. Both squared amplitudes are geometrically-compensated by using the theoretical factor $(\text{range})^2$. Dashed lines labelled with $\nu$ values indicate the approximations of geometrical spreading using the $r^\nu$ law at near offsets, and lines with labels $\gamma$ show the same ranges approximated by $e^{-\gamma r}$ dependencies.
Figure 5. a) $Pn$ stacked spectral ratios (SSR) from Figure 9 in Xie (2007) for two $(Q_0, \eta)$ models inverted by using different GS power-law $t^{-\nu}$ exponents (black and grey). Label $m$ corresponds to $\nu$ used in the text. Error residuals shown in labels. b) The same data in $[-\chi(f)]$ form plotted in a linear frequency scale. Lines represent different options for linear fitting (see text).
Figure 6. Modeled surface-wave attenuation data from Anderson et al. (1965):

a) Layered attenuation model MM8.

b) $Q$ for Rayleigh and Love waves modeled by spherical mode summation (black curves) and predicted from the corresponding $\gamma$ and $Q_e$ determined from plot c) as: $Q(f) = \frac{\pi f}{(\gamma + \frac{\pi f}{Q_e})}$ (grey curves). Note that both types of modeled curves are practically identical.

c) The same results transformed into attenuation coefficients $\chi(f)$ [eqs. (6) and (3)]. Note that both $\alpha(f)$ dependencies are practically linear within the entire frequency band, with small negative $\gamma = \chi(0)$ seen when extrapolated to $f = 0$ Hz (dotted lines).
Figure 7. Apparent absorption band formed by geometrically over-compensated long-period surface waves [cross-over frequency $f_{c,LP}$, eq. (5)] and under-compensated short-period body waves [$f_{c,SP}$ in eq. (6)]. Two versions of $f_{c,SP}$ are given, corresponding to the areas of stable (black lines) and active tectonics (grey). Two horizontal gray bars indicate the levels of $Q_e$ (crustal and upper-most mantle) that determine the band. Dotted line labelled ABM shows the mantle absorption band model (Anderson and Given, 1982), with relaxation cut-off times $\tau_1$ and $\tau_2$ indicated. See text for discussion.

1.2. Frequency dependence of $t^*$ for long-period body waves

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Abstract

Frequency-dependent attenuation observations are sensitive to inaccurate knowledge of the Earth’s structure, which to the first-order is described by geometrical spreading (GS). Nevertheless, unlike commonly thought, GS can be rigorously accounted for during attenuation measurements, and the corresponding uncertainties removed. A generalized assumption-free approach applicable to all types of attenuation measurements is proposed, and leads to two types of techniques: 1) direct measurement of the residual GS and attenuation, and 2) making attenuation measures invariant in respect to the unknown GS. The approach is compared to the traditional $t^*$ and $Q$-based methods by using two well-known real-data examples. First, the observed increase of body $P$-wave attenuation from $t_P^* \approx 0.2$ s at short periods to $\sim 1-2$ s at long periods is interpreted as apparent and related to a $\sim 6\%$ bias in the underlying assumed GS. A correction for GS variation is applied, resulting in effective $t_P^* \approx 0.2$ s, which is independent of the frequency. In the second example, local-earthquake coda $Q$ in three areas of active tectonics is re-examined, and its frequency dependence is also explained by the variations of GS. Therefore, at least in these as well as several other cases, frequency-dependent in situ $Q$ is not needed for explanation of the available observations.

Introduction

Attenuation of seismic energy within the crust and mantle is among the most intriguing physical effects studied in seismology, and a profusion of attenuation models were developed in the past fifty years. Frequency dependence of the observed mantle attenuation was first pointed out by Gutenberg (1958), and since then, most models represented the Earth as a single or multiple absorption bands (e.g., Anderson et al., 1977) which may also be complexly distributed in depth (e.g., Anderson and Given, 1982) and correlate with tectonic structures (Der and McElfresh, 1976, 1980; Der et al., 1982, 1986). With all the uncertainties, inconsistencies, and differences between the models using different data and analysis techniques, one point appears to be well established, namely the frequency-dependence of the Earth’s quality factor, $Q$. Across a
broad range of frequencies from \( \sim 0.01 \) to 100 Hz (for summaries of some of the relevant data, see Jackson and Anderson, 1970; Anderson and Given, 1982; Der et al., 1982, 1985, 1986; Lees et al., 1986; Abercrombie, 1998), the attenuation of surface and body waves consistently decreases, suggesting an increase in \( Q \).

However, attenuation measurements are also known for their difficulty of differentiating the apparent measured attributes from the true in situ properties (i.e., \( Q \)) and may be prone of pitfalls (Der and Lees, 1985; White, 1992). The situation is complicated by the fact that the low-frequency end of the spectrum is dominated by time-domain (i.e., various types of absolute-amplitude based) measurements which heavily rely on theoretical assumptions and models, and in particular on the compensation of geometrical spreading (GS). Sensitivity of the frequency-dependent \( Q \) to such compensation is well-known (e.g., Kinoshita, 1994). Broad acceptance of standard reference models of GS, such as \( G(t) = t^{-1} \) for local-earthquake body waves (here and below, \( t \) is the travel time, and \( f \) is the frequency) allows comparisons between the \( Q(f) \) models derived for different areas; however, this abstract reference reduces these \( Q(f) \) curves back to descriptions of the seismic amplitudes and not of the Earth’s properties. For an unbiased interpretation of the in-situ \( Q \), the results should be referenced to the local GS.

The key premise of the present study is that \( G(t) \) is still imperfectly known, likely spatially variable, and also may be frequency-dependent (Yang et al., 2007). Therefore, its universal compensation in the standard models is unlikely accurate, and the residual GS should be measured instead of being assumed to equal zero. Even small errors in GS manifest themselves as significant apparent frequency-dependent attenuation. For example, estimates below show that only a \( \sim 6\% \) error in GS could account for most of the observed \( Q(f) \) differences between the long-and short-period body waves. Similarly, in recent re-interpretations of several crustal-scale datasets, even very strong \( Q(f) \) dependencies disappeared when GS variations were taken into account (Morozov, 2008, and unpublished).

Below, I describe a general attenuation measurements framework taking into account the variability of GS. As special cases, the method comprises two groups of
approaches commonly used in frequency-dependent attenuation measurements: 1) the
time-domain approach based on the concept of $Q$, used in many surface-wave and coda
studies (e.g., Aki and Chouet, 1975), and 2) the $t^*$ -based approach, which is extensively
used in body-wave studies (e.g., Der and Lees, 1985). These approaches are discussed
below, with emphasis on removing the effects of GS variability from the resulting
attenuation parameters.

To outline the multivariate $Q(f) - t^*(f) - GS$ inter-relationship we are facing
before focusing on its specific aspects, note the following relations between the concepts,
quantities, and unknowns entering all descriptions of the Earth’s attenuation structure:

1) The rock quality factor, $Q$ is ultimately a mechanical rock property averaged
within the seismic wavelength and period. As a material property, $Q$ is
instrumental in relating the attenuation to impedance and dispersion (e.g., via the
Kramers-Kröning relations; Aki and Richards, 2002). This type of attenuation
model is preferable at the in-situ level and at local scales and in modeling, where
the medium can be considered approximately uniform. Numerous insightful
rheological creep, relaxation spectra, and scattering models have been proposed
(e.g., Jackson and Anderson 1970; Liu et al., 1976; Sato and Fehler, 1998) in
order to explain possible $Q(f)$ dependencies. Nevertheless, GS errors may still be
present at all distance scales, causing biases in the measured $Q(f)$. These biases
should be removed before significant inferences about the in-situ $Q$ are made.

2) By contrast to $Q$, $t^*$ is a cumulative property of the wave path and arises from
observations. Although this property is dependent on both $Q$ and velocity
structure, its averaged character makes it relatively insensitive to local variations
of $Q$. Values of body-wave $t^*$ are almost constant within the ~25 – 90° distance
ranges (Der et al., 1986), making this quantity convenient in body-wave studies
Variability of geometrical spreading and nuclear-test monitoring. Its trade-off with errors in GS was insufficiently studied; however, as I show below, it is significant and equivalent to that of $Q(f)$.

3) The concept of frequency “absorption bands” is broadly used in respect to both $Q(f)$ and $t^\ast(f)$ descriptions. In the $Q(f)$ form, flanks of the band are typically assumed to behave as $Q = Q_0 f^\eta$, where $Q_0$ is the attenuation quality at 1 Hz (e.g., Aki and Chouet, 1975; Anderson and Given, 1982). Note that $t^\ast$-type descriptions use no presumed functional forms of frequency dependence, and therefore they may be less influenced by theoretical prejudices. However, a general problem of all absorption-band models is in their introducing additional degrees of freedom in the attenuation model, on top of the spatial variability of $Q$ and GS. This can make the models over-parameterized and uncertain while fitting the data well. Below, I formulate an alternate, non-parametric approach from which the presence of an absorption band is established from the data.

4) Understanding the Earth’s structure is among the key goals of seismology, yet the effects of structural variability still have not received sufficient attention in attenuation studies. They are commonly replaced with “reasonable” or “practical” assumptions which, however, are also known to be inaccurate. In this paper, I focus on the variations of GS as only one aspect of such variability. As it can be seen, GS trades off with all other model parameters 1)-3) above and may render the frequency-dependencies of both $t^\ast$ and $Q$ unreliable.

The goal of the following analysis is to describe an interpretation method that is unprejudiced in favor of any particular GS, $t^\ast(f)$, or $Q(f)$ models. As I show below, the $t^\ast$ and $Q$ methods can be viewed as special (constrained) cases of the generalized
attenuation-coefficient approach (Morozov, 2008), from which the effects of GS variations become clear and can be removed.

**Attenuation coefficient, $t^*$, and $Q$**

In a “reasonably,” but still imperfectly known Earth structure, the GS-compensated, path contribution to the seismic amplitude $A(f, t)$ can be written as:

$$A(f, t) = \exp \left[ - \chi^*(f, t) \right], \quad (1)$$

with only the normalization condition $\chi^*(f; t=0) = 0$ imposed on function $\chi^*$. Further, for attenuation measurements involving relatively short time lags, $\chi^*(f, t)$ can be considered approximately proportional to $t$:

$$\chi^*(f, t) = \chi(f) t. \quad (2)$$

This is the standard scattering-theory approximation, which is the first-order term in the Taylor series for $\chi^*(f, t)$ in respect to $t$. I will refer to both quantities $\chi^*$ and $\chi$ in eqs. (1-2) as the generalized attenuation coefficient, which contain the effects of variable or inaccurately compensated GS and the elastic and anelastic attenuation losses. Note that $\chi$ was denoted ‘$\alpha$’ in Morozov (2009), but here I prefer changing the notation in order to avoid collision with the spatial attenuation coefficient.

Denoting the frequency-independent parts of both types of $\chi$ by $\gamma^* = \chi^*(0)$ and $\gamma = \chi(0)$, we have:

$$\chi^*(f, t) = \gamma^* + \frac{\pi}{Q_e} f, \quad \text{and} \quad \chi(f) = \gamma + \frac{\pi}{Q_e} f, \quad (3)$$

where $Q_e$ is the “effective attenuation” quality, and $\gamma$ describes the frequency-independent uncompensated GS, and quantities with asterisks refer to the corresponding path-average properties. Both $Q_e$ and $Q_e^*$ may be frequency-dependent; however, frequency dependence has actually not been required to explain all observations that I considered so far (Morozov, 2008; also see below and unpublished). Note that $\gamma$ and $Q_e^*$ are measured in frequency units.
In standard approaches, the attenuation coefficients (1-2) are replaced with parameters $t^*$ and $Q_e$ (Der and Lees, 1985; Aki and Chouet, 1975, respectively):

$$t^*(f,t) = \frac{\chi^*(f,t)}{\pi f}, \quad \text{and:} \quad Q(f) = \frac{\pi f}{\chi(f)}. \quad (4)$$

From the general form in eq. (3), we therefore have:

$$t^*(f,t) = (Q_e^*)^{-1} + \frac{\gamma^*}{\pi f}, \quad \text{and:} \quad Q^{-1}(f) = Q_e^{-1} + \frac{\gamma}{\pi f}. \quad (5)$$

With $Q_e = \text{const}$, the second of this formulas was used by Dainty (1981) to describe the $S$-wave $Q^{-1}(f)$ at $1 – 30$ Hz. Note that representing the r.h.s. or both equations (4) as the $t^*$ and $Q$ parameters are motivated the expectation of $\chi(f) \to 0$ with $f \to 0$. This is true in a boundless and uniform space, in which the GS is accurately known and can be compensated. In such a case, $t^*$ and $Q$ are frequency-independent and simply related to the in situ $Q$. However, if such a medium also has a non-trivial velocity structure causing variations in $\gamma$, then both $t^*$ and $Q$ in eqs. (4) would still exhibit spurious variations with frequency. For example, when $\gamma > 0$, $t^*$ would decrease, and $Q –$ increase with frequency. Such behavior is often observed and appears typical for crustal coda waves, for which $\gamma \approx 0.01$ s$^{-1}$ (Morozov, 2008). Similar cases are discussed below.

By contrast to $t^*$, which is measured from GS-compensated amplitudes, Der and Lees (1985) also defined the apparent $t^*$, here denoted by $\tilde{t}^*$:

$$\tilde{t}^* = -\frac{1}{\pi} \frac{\partial \ln A}{\partial f} = \frac{1}{Q_e^*} \left(1 - \frac{\partial \ln Q_e^*}{\partial \ln f}\right). \quad (6)$$

This quantity is determined from spectral-ratio measurements and is independent of GS (i.e., $\gamma$). From eqs. (5) and (6), note that their relative difference is also caused by the GS factor ($\gamma$) and is inherently frequency-dependent:

$$\frac{t^* - \tilde{t}^*}{\tilde{t}^*} = \frac{\gamma^* Q_e^* + \frac{\partial \ln Q_e^*}{\partial \ln f}}{1 - \frac{\partial \ln Q_e^*}{\partial \ln f}} \approx \gamma^* Q_e^*. \quad (7)$$
Also compare to this ratio to the $t^*$ bias function by Der and Lees (1985).

Another useful way to understanding the effects of uncompensated GS on $t^*$ measurements is by directly relating $\bar{t}^*$ to $t^*$ (Der and Lees, 1985):

$$\bar{t}^* = t^* + \int \frac{dt^*}{df}.$$  \hspace{1cm} (8)

For a measured $\bar{t}^*$, this differential equation can be integrated to obtain the “true” $t^*(f)$ (Der and Lees, 1985). However, this integration is non-unique, and its uncertainty is given by the solution to the homogeneous counterpart of equation (8), which is: $t^* = af^1$, with $a$ being an arbitrary constant. By taking $a = \gamma^* / \pi$, we see from eq. (4) that this homogenous solution is again nothing else but the uncompensated GS.

**Applications: body-wave $t^*(f)$ and coda $Q(f)$**

As shown above, both the $t^*(f)$ and $Q(f)$ types of attenuation descriptions can be viewed as special cases of the general attenuation-coefficient (2-3). By definition of their parameters (4), these special cases are designed to work with perfectly compensated GS. In the presence of uncompensated or variable GS (which should be the common case), the general $\chi(f)$ form appears to be most suitable and less prone to uncertainties. Below, I present two examples showing how the $\chi(f)$ approach modifies the existing interpretations. The examples are selected from two important publications representative of the $t^*(f)$ and $Q(f)$ approaches.

The attenuation coefficients $\chi^*(f)$ or $\chi(f)$ should ideally be measured directly from the data. In fact, they are inherent in nearly every attenuation measurement, but unfortunately are practically never presented. However, $\chi^*$ or $\chi$ can also be estimated from reported $t^*$ and $Q$ values by inverting eqs. (1) and (4):

$$\chi^* = \pi ft^*, \text{ and: } \chi = \frac{\pi f}{Q}.$$  \hspace{1cm} (9)

Note that the second of these expressions is close to the “stacked spectral ratios” (SSRs) used in many studies by Mitchell and co-authors (e.g., Mitchell, 1991). However, SSRs have only been plotted in logarithmic frequency scales, which missed the point of our
analysis. According to eqs. (3), the key approach to interpreting the $\chi^*(f)$ or $\chi(f)$ dependencies should consist in establishing their linearity (or non-linearity) with frequency. In the following examples, we will plot the attenuation-coefficient (9) dependencies in linear frequency scales and try answering two questions from them: 1) whether they can be considered linear in $f$, and 2) whether non-zero intercept values $\chi(f = 0)$ are indicated by the data.

**Example 1: Body-wave $t^*_b(f)$**

In several extensive studies, Der and McElfrish (1977), Der and McElfrish, 1980; Der et al. (1982a,b, 1986), Der and Lees (1985), Lees et al. (1986), and Sharrock et al. (1995) examined the dependencies of $t_P^*$ and $t_S^*$ of various teleseismic waves on frequency and tectonic types in several areas of the world. Broadly, their observations suggested that: 1) $t^*$ for body waves (and consequently $\chi^*$), are nearly independent of travel time beyond $\sim 25^\circ$ ranges (Der and Lees, 1986), 2) $t^*$ values increase under tectonically active areas, where the zone of increased $Q$ in the mantle may also become shallower (e.g., Sharrock et al., 1995), and 3) $t_P^*$ decreases with frequency from $\sim 1$ s within the long-period band to $\sim 0.2$ s for short-period phases (Der and McElfrish, 1980; Der et al., 1982). To date, such variations have invariably been associated with the corresponding variations in the in situ $Q$ decreasing within tectonic areas and with frequency, respectively.

However, variation of the in situ $Q$ is not the only possible explanation, and variations of GS, i.e., systematic differences in the crustal and lithospheric structure related to high heat flow and orogenic processes could also cause apparent $Q$ variations. Recently, we demonstrated this by numerical coda modeling in realistic lithospheric structures (Morozov et al., 2008), and showed that the GS parameter $\gamma$ (eq. (3)) consistently correlates with tectonic age, whereas $Q_\infty$ may remain broadly variable (Morozov, 2008). The dependence of the in situ $Q$ on frequency is particularly questionable, because it strongly trades off with GS (see eq. (5)).

In Figure 1a, I reproduced the $r^*(f)$ and $t^*(f)$ data summary from Der et al. (1986b), derived from their analysis of multiple $P$- and $S$- body-wave phases at $25^\circ$ - $90^\circ$
distances, with paths lying within the shield areas of Eurasia. The interpreted $i^*$ and $t^*$ dependencies modeled by ray tracing in a layered, frequency-dependent $Q$ model EURS are shown by dashed lines (Figure 1a). An earlier shield-path model (Der et al., 1982) is also overlain on this plot (gray dotted line in Figure 1a). This model also closely corresponds to the absorption-band mantle model by Minster (1978a,b).

However, the same $i^*$ and $t^*$ data can also be explained by a linear $\chi^*(f)$ dependence (eq. (3)) with $\gamma^* \approx 0.06$ and $Q_e^* \approx 5.5 \text{ s}^{-1}$ (thick solid line in Figure 1a). Note that this line fits the GS-independent data ($P$-wave spectral measurement in 1 – 10 Hz range) even better than the existing model, and it may be better following the trend of rising $t^*$ at the lower frequencies. The marginal fit of the multi-phase $S$ and rise-time data could be due to poorer reliability of these measurement techniques. Difficulties in fitting multiple data types and impracticality of formal inversion were also emphasized in the original $t^*$ research (e.g., Der et al., 1986a,b; Sharrock et al., 1995).

Figure 1b illustrates the simple interpretation technique that was actually used to analyze the $t^*$ data above. By using eq. (9), $\chi^*/\pi$ was first derived, and then trial “reductions” (subtractions of terms linear in $f$) were applied until the resulting data distribution appeared near-horizontal (Figure 1b). This gave the optimal intercept ($\gamma^*$) and slope ($\pi/Q_e^*$) of the $\chi^*(f)$ line. Note that a curve with somewhat frequency-dependent $Q_e^*$ (similar to the dashed black curve in Figure 1b) might fit the data a little better; however, this conclusion still does not seem to be warranted by the data quality (Der et al., 1986b).

With some concerns about the data fit and possible mild frequency dependence of $Q_e$, the principal result remains clear: a slight positive shift of $\gamma^* \approx 0.06$ in GS could explain the observed increase of $t^*$ from ~0.2 s at 1-10 Hz to ~1-2 s at 0.01-0.02 Hz. The effective attenuation is practically frequency-independent and equals $Q_e^* \approx 5.5 \text{ s}^{-1}$, which corresponds to $t_e^* = 1/Q_e^* \approx 0.18 \text{ s}$. Note that the level of $\gamma^* \approx 0.06$ corresponds to only a ~6% error in GS compensation used in inverting the long-period $SS$ and $ScS$ amplitudes (Figure 1a). Such level of forward-model uncertainty can certainly be expected; nevertheless, it should not be transformed into a frequency dependence of mantle $Q$.

The above analysis shows that the traditional $t^*$ value also represents an
“apparent” quantity in the sense of its dependence on GS compensation, which is an integral part of the measurement procedure. Values of $t^*$ change whenever the real GS differs from the inferred theoretical level. By contrast, the apparent $\bar{\tau}$ measured from spectral ratios turns out to be close to the true $Q_e^*$ property of the Earth (eq. (6)). For this reason, spectral measurements should be significantly more reliable in constraining the attenuation.

A method of GS-assumption-independent interpretation of attenuation utilizing the above $\bar{\tau}^*$ property is shown in Figure 2. Instead of deriving $\bar{\tau}^*$ from $\bar{\tau}$ by integrating eq. (8) (with the ensuing uncertainties), one can determine $\bar{\tau}^*$ from published $\bar{\tau}$ data by using the same equation. The empirical $dt^*/df^*$ trend for use in eq. (8) can be estimated from the same data (Figure 2a). The resulting $\bar{\tau}^*$ values simulate spectral measurements within the long-period band, and they are independent on the uncertainty of GS. Therefore, $\bar{\tau}^*$ values can be reliably interpreted and modeled, and in the case of the present data, they can be satisfied with the frequency-independent $\bar{\tau}^* \approx 0.18 \text{ s}$ mentioned above (Figure 2b).

Our derivation of $\chi^*$ by “reverse-engineering” the published $\bar{\tau}^*$ values still inherits some of their limitations and approximations, such as: 1) the $P$- and $S$-wave data are tied together by simple scaling $ts^* = 4tp^*$, 2) $t^*$ are considered as independent of $t$, 3) the same GS rate $\gamma^*$ is assumed for $P$- and $S$-waves, their reflections and multiples, and 4) the free-surface reflection coefficient for $ScS_n$ multiples is assumed to equal 1 (Der et al., 1986a,b; Lees et al., 1986). None of these limitations are necessary in the $\chi^*$ approach, and the corresponding parameters can be included in the model. Inverting the raw data spectra, amplitudes, and pulse shapes directly within the $\chi^*$ model would certainly improve the data fit in Figure 1. However, unfortunately, this would involve revisiting a very large and complex dataset, which is not available to the present analysis.

**Example 2: Local earthquake coda $Q(f)$**

To briefly illustrate the relation of a $Q(f)$-type models to $\chi(f)$, let us review the data from local earthquakes in central California, Hawaii, and central and western Japan by Aki (1980) (Figure 3a). Aki (1980) converted the time-domain log-amplitude coda slopes into $Q^{-1}(f)$ by assuming a theoretical GS of $1/t$ form, and made two observations
from the resulting curves (Figure 3a): 1) three of the four \( Q^{-1}(f) \) dependencies appeared to converge at the higher frequencies, and 2) station TSK in central Japan suggested a distinctly different attenuation mechanism.

However, re-plotting the same data in terms of \(-\chi(f)\) decay rates reveals linear attenuation patterns suggesting different relationships between the datasets (dashed lines in Figure 3b). It appears that the convergence of the inferred \( Q^{-1}(f) \) near \( \sim 25 \) Hz (Figure 3a) could be incidental and related to dividing the values of attenuation coefficients \( \chi(f) \) by the frequency. This division also somewhat exaggerated the difference between the results from stations OIS and OTL below 5 Hz (Figure 3a). By contrast, comparison of the intercepts and slopes of the attenuation coefficient trends (dashed lines in Figure 3b), shows that: 1) GS factors are close to \( \gamma \approx 0.02 - 0.03 \) s\(^{-1}\) for Hawaii and for both areas in Japan; 2) attenuation values in Hawaii and western Japan are quite similar, with \( Q_e \approx 600 \); 3) in central Japan, attenuation is low, with \( Q_e \approx 2300 \); and 4) in terms of both parameters, central California is distinctly different from the other three areas, with its \( \gamma \approx 0.06 \) s\(^{-1}\) and \( Q_e \approx 1250 \).

On top of the linear \(-\chi(f)\) trends, some “spectral scalloping” is also visible in Figure 3b, particularly the reduced amplitudes near 6 Hz and increased at \( \sim 1-2 \) Hz and 12 Hz. These amplitude variations are consistent in all four cases, which suggests that they could be related to systematic variations of the scattered-wavefield amplitude with frequency. Such variations are likely related to frequency-dependent geometrical spreading (e.g., reflection tuning) and not to attenuation. This issue requires further investigation by analyzing the raw data and modeling. However, note that the division by the frequency in order to produce the apparent \( Q(f) \) early in the interpretation process makes these systematic variations less clear (Figure 3a).

Once the intercept (\( \gamma \)) values are established even relatively crudely, the potential frequency-dependence of \( Q_e \) can be examined by the method similar to that in Figure 2. A GS-invariant \( \overline{Q} \) can be defined similarly to eq. (8):

\[
\overline{Q}^{-1} = \frac{d\chi}{\pi df} = Q^{-1} + f \frac{d(Q^{-1})}{df},
\]  

where \( \chi(f) \) is the apparent attenuation coefficient and \( Q^{-1} \) is the inverse quality factor. This equation allows for the estimation of a GS-invariant quality factor that is independent of frequency and can be used to better understand the attenuation mechanisms in different regions.
and for the correction terms $\tilde{f} d(\tilde{Q}^{-1})/df$, the data trend lines (dashed lines in Figure 3b) can be used. Because of taking a derivative in frequency, such $\tilde{Q}$ measures the change in spectral slope during wave propagation, i.e., simulates $Q$ values that would result from spectral-ratio measurements. The corrected $\tilde{Q}^{-1}$ for the data in Figure 3a are shown in Figure 4. As we see, $\tilde{Q}^{-1}$ values appear to be constant from $\sim$3 to $\sim$25 Hz, although the amplitude variations discussed above cause significant distortions near $\sim$1 Hz.

The resulting values of $Q_e$ are significantly higher than the corresponding reported $Q_0$ (Aki, 1980), with the differences increasing with larger $\eta$. It can be shown that $\eta$ is roughly proportional to $\gamma Q_e$ (Morozov, 2008), and the $Q_e-Q_0$ difference also increases for larger $\gamma$ or $Q_e$. For example, station TSK shows an update from $Q_0 \approx 300$ to $Q_e \approx 2300$, and station PAC – an over 20-fold increase from $Q_0 \approx 60$ (Figure 3a) to $Q_e \approx 1250$. Such significant changes in $Q$ could upset the association of these areas with zones of active tectonics, which are expected to have lower $Q$ values (e.g., Aki, 1980).

However, as illustrated in a recent worldwide compilation (Morozov, 2008), $Q_e$ and $Q_0$ may be less reliable indicators of the tectonic age, whereas $\gamma$ appears to systematically correlate with it.

The measured values of $\gamma \approx 0.02 – 0.06$ s$^{-1}$ are consistently high and located well above the $\gamma_D = 0.008$ s$^{-1}$ threshold separating the stable and active tectonic settings (Morozov, 2008). Such levels of $\gamma$ correspond to significant but realistic deviations of GS amplitudes from the $1/t$ law assumed by Aki (1980). For example, with $\gamma = 0.05$ s$^{-1}$, seismic amplitudes would deviate from spherical spreading by $\sim$40% after 10 seconds of propagation (i.e., at $\sim$35 km of distance for $S$ waves, which is the order of crustal thickness). Such deviations are quite likely in an area of strong lithospheric velocity gradients and complex structure. For a stable area with a more uniform, higher-velocity crust, this deviation would be only $\sim$8%. Such level of GS variability should be widespread around the world.

**Discussion**

Let us return to our first example (Figure 1) of teleseismic body waves. Der and
McElfresh (1980) were among the first to note that values of $t_P^* \sim 1$ s and $t_S^* \sim 4$ s could not be correct for all frequencies and body-wave paths, and also pointed out the regional variability of short-period $t^*$. They concluded that $t_P^*$ and $t_S^*$ should be frequency-dependent within the short-period band, because otherwise the 4-5 Hz energy would never be seen in $P$ waves. Such frequency dependence of $t^*$ is by now a clearly established fact; however, its commonly implied relation to the Earth’s $Q$:

$$t^*(f) = \int_{\text{Ray path}} \frac{dt}{Q(f)}$$ (11)

still remains questionable. This expression automatically projects the frequency dependence of $t^*$ into that of the in-situ $Q$, whereas, as shown above, the $t^*(f)$ dependence contains an uncertainty related to the selection of background GS and other approximations made in forward amplitude modeling of long- and intermediate-period waves.

A robust and assumption-independent replacement for the path integral (11) arises directly from expressions (3):

$$\gamma^* = \int_{\text{Ray path}} \delta G dt$$ , (12)

and:

$$(Q_e^*)^{-1} = \int_{\text{Ray path}} Q^{-1} dt$$ , (13)

where $\delta G$ is some path integration kernel (yet to be determined) whose effect should be the logarithm of the uncompensated GS (i.e., $-\ln G(t)$ in Introduction) in the actual velocity/density structure. The measured $(Q_e^*)^{-1}$ in eq. (13) therefore becomes a path-average of the in situ $Q^{-1}$.

As shown above, $Q_e$ is practically frequency-independent from the present data (Figure 1), and after a $\sim6\%$ GS correction, the long-period $t^*$ values become close to those observed at short periods (i.e. $\sim0.2$ s for $P$ waves). From eq. (13), this means that all the data in Figure 1 can be explained by a frequency-independent Earth $Q$ model!
Variability of geometrical spreading

Inversion for the depth and regional $Q$ variations is a complex problem certainly impregnated with further uncertainties, which will be addressed elsewhere. However, from the data reduction to the $(\gamma^*, Q_e^*)$ form and from eqs. (12, 13), it is clear that the in-situ $Q$ model can be frequency-independent unless additional data provide evidence to the contrary.

Along with contrasting $t^*$ values at long- and short-period measurements, another evidence is sometimes advanced in favor of the mantle $Q$ increasing with frequency, namely the observation of the high-frequency “teleseismic $P_n$” from Peaceful Nuclear Explosions (PNEs) in Russia (Ryberg et al., 1995). PNE waves at $>5$Hz frequencies were found to travel at $P_n$ velocities to over 3000-km distances within the uppermost mantle of the East European Platform. A multiply-scattering waveguide mechanism was proposed favoring preferential propagation of high-frequency waves (Ryberg and Wenzel, 1999). It is still unclear whether such scattering would actually increase or decrease mantle $Q$ at these frequencies.

However, the scattering-waveguide interpretation of the high-frequency teleseismic $P_n$ from PNEs represents another notable example of mistaking structural effects for a frequency dependence of mantle attenuation. Careful analysis shows that the teleseismic $P_n$ in the East European Platform consists of a series of $P$-wave multiples (or “whispering gallery” modes) propagating above the depth level of ~100 km, below which there is a strong increase in attenuation (Morozov et al., 1998a,b). Consequently, its high-frequency amplitude appears anomalously strong only when compared to the teleseismic $P$ waves that penetrate the attenuative layers (Morozov, 2001). High-frequency (up to ~10-15 Hz) $P$ waves and their multiples travel efficiently in this area merely because of the low attenuation in the uppermost mantle ($Q \approx 1400 - 2000$; Morozov et al., 1998b). The scattered character of the teleseismic $P_n$ arrivals is more naturally explained by crustal (e.g., Morozov et al., 1998a; Nielsen et al., 2001) than by mantle scattering and is not related to the efficiency of $P_n$ propagation.

Conclusions

The question of frequency dependence of crustal and mantle attenuation still does
not appear unequivocally resolved. Reviewing the theory of $t^*$ and $Q$ measurements and inversions, and also revisiting two key data examples from the literature shows that inaccurate assumptions about the geometrical spreading (GS) may be responsible for the observed $t^*(f)$ and $Q(f)$ dependencies. An only $\sim 6\%$ correction of the assumed GS removes the frequency dependence of body $P$-wave $t^*$ and puts them in $\sim 0.2s$ range for all frequencies. This leads to an in situ Earth $Q$ model that is also frequency-independent within the available data constraints.

Instead of a frequency-dependent rheological or scattering $Q$ and absorption bands, the proposed model incorporates a spatially-variable GS. Similarly to $Q$, GS can be directly measured from the data, predicted by forward modeling, and correlated with lithospheric structures. As a conjecture (although supported by recent data compilations in Morozov (2008)), it appears that variations in GS could be principally responsible for the observed $Q(f)$ variability in many cases, and particularly for correlations of $Q$ with tectonic ages.

**References**


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Figure 1. Alternate interpretation of $t^*$ data in Eurasia (Der et al., 1982a,b, 1986a,b; Lees et al., 1986) by using the attenuation coefficient $\chi^*$:

a) Data summary from Der et al. (1986) with $t^*$ and $\tau^*$ curves modeled by ray tracing at $60^\circ$ ranges in their EURS $Q$ model (gray dashed lines). Different data sources, measurement methods, and frequency-$t^*$ value ranges are indicated. The $t^*(f)$ curve for shield areas from Der et al. (1982) is shown by gray dotted line. Thick black line corresponds to the attenuation coefficient linear in frequency, with $\gamma^* \approx 0.06$ and $Q_e^* \approx 5.5$ (eq. (3)).

b) The same data in the form of “reduced” attenuation coefficient. Dashed black line
shows an alternate interpretation with slightly frequency-dependent $Q_v^*$ or $\gamma^*$. 
Figure 2. GS-independent \( t^* \) interpretation technique: a) the same data as in Figure 1a with \( t_P^*(f) \) trend estimated from the data (dashed line); b) the trend removed by using eq. (8) and producing a GS-independent \( \bar{t}^* \). Note that after this correction, the attenuation can be considered frequency-independent, with \( \bar{t}^* \approx 0.18 \) s (dotted line in plot b)).
Figure 3. a) Local-earthquake coda $Q^{-1}(f)$ from Aki (1980). Labels indicate seismic stations: PAC – central California, OIS – western Japan, TSK – central Japan, and OTL – Hawaii. b) The same data transformed to attenuation coefficients ($-\chi(f)$) [eq. (3)] and plotted in a linear frequency scale. Only one line is shown for station OTL. Note the linear $\chi(f)$ trends (dotted lines and labels).
Figure 4. GS-independent $\bar{Q}^{-1}$ derived from the data in Figure 3a. Symbols, lines, and labels as in Figure 3.
2. Theory of attenuation

2.1. Anelastic Acoustic Impedance and critique of the correspondence principle

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Abstract

A general definition of seismic wave impedance is proposed as a differential operator transforming the displacement boundary conditions into the traction ones. This definition corresponds to the standard acoustic impedance at all incidence angles and allows extensions to attenuative media and to the full elastic case. In all cases, reflection amplitudes at the contact of two media are uniquely described by the ratios of their impedances. Here, the anelastic Acoustic Impedance is studied in detail, and attenuation contrasts are shown to produce phase-shifted reflections. Notably, the correspondence principle (i.e., the complex-valued elastic module approach) leads to incorrect phase shifts of the impedance due to attenuation, and consequently to wrong waveforms reflected from attenuation contrasts. Boundary conditions and the Lagrange formulation of elastodynamics suggest that elastic constants should remain real in the presence of attenuation. Therefore, the correspondence principle and complex-valued elastic moduli should be used with caution when applied to heterogeneous media.
Introduction

Acoustic impedance (AI) is the key quantity used for characterizing the reflection and transmission of seismic waves. Since the 70’s, AI has become the primary concept used in seismic reflectivity inversion and interpretation (Lindseth, 1979). However, despite its canonical character and widespread use, the theoretical background of this concept is still not entirely clear and requires some attention. In particular, modifications of AI due to anelasticity are still not well understood. AI is often interpreted heuristically as, for example, the product of density and velocity or generator of a given reflectivity time series. However, as shown below from the basic principles of wave mechanics, AI can be rigorously defined as a property of field boundary conditions. This definition allows extending this concept to full visco-elastic cases.

Wave propagation in anelastic media has been extensively studied (e.g., Lockett, 1962; Anderson and Archaemmeau, 1964; Cooper and Reiss, 1966; Borcherdt, 1977; Borcherdt and Wennerberg, 1985; Richards, 1984), but almost exclusively by using some form of the correspondence principle. The essence of this approach is in describing the attenuation as a modification of elasticity, with real-valued elastic moduli replaced with their complex-valued counterparts in frequency domain (Bland, 1960; Carcione, 2007). In particular, in a medium with attenuation quality factor $Q$, the shear modulus $\mu$ is assigned a negative complex argument of $[-\tan^{-1}(Q^{-1})]$, which leads to the well-known negative imaginary shift in wave velocities, and consequently in AI. However, this last effect on AI is incorrect because the imaginary part of AI should be positive, as was recently demonstrated by explicit calculations of reflection amplitudes from attenuation contrasts (Lines et al., 2008).

An example of such phase-shifted reflectivity resulting from complex-moduli elastodynamics is shown in Fig. 1. Here, a reflection from the bottom of a layer at 1-km depth with a 10% increase in ReZ across the boundary is modeled by using the “reflectivity” approach (Fuchs and Müller, 1971), in which the attenuation effects are incorporated by using complex-valued medium velocities. The overburden is uniform and lossless ($V_P = 2$ km/s, $V_S = 1$ km/s and $Q_{P}^{-1} = Q_{S}^{-1} = 0$). In the presence of attenuation increase to $Q_{P}^{-1} = Q_{S}^{-1} = 0.2$ below the reflector, the reflection becomes time-advanced

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(Fig. 1a) relative to the attenuation-free case (see the reference time pick indicated by a dashed gray line in Fig. 1c). Its phase shift equals 45°, and after its compensation (Fig. 1b), the reflection becomes similar to the attenuation-free one (Fig. 1c). In the absence of a \( \rho V \) contrast, the reflection from a jump in \( Q^{-1} \) becomes phase-advanced by 90° (Fig. 1d) compared to the attenuation-free case (Fig. 1c).

Phase-advanced reflections from a positive attenuation contrast (Fig. 1a,d) result from interpreting viscous friction as time-retarded elasticity (and consequently the Lamé moduli are generalized to “relaxation-rate” functions). Such descriptions lead to equivalent equations of harmonic-wave propagation in uniform media; however in the presence of a discontinuity, it is not obvious that such retarded forces are actually applied to the boundary. As shown below, by using the standard (instantaneous) Hooke’s law and assuming attenuation to be caused by external (non-elastic) causes, reflection phase shifts become opposite to those shown in Fig. 1a,d.

The literature on complex-moduli visco-elasticity is vast and covers almost every aspect of wave mechanics (for a recent overview see Carcione, 2007). In the time domain, complex-valued elastic moduli correspond to several types of time-delayed stress-strain relations that are often used to explain the dependence of seismic \( Q \) on frequency. Examples of such models include Maxwell, Kelvin-Voigt, and Zener (the standard linear solid) models, and superpositions of such models were used to explain the Earth’s absorption band (Liu et al., 1976). The impact of such models is so strong that they may have even biased the observations of seismological \( Q \) toward pervasive frequency-dependence within the Earth’s crust and mantle (Morozov, 2009a). However, equivalent mechanical models phenomenologically mix together such physically disparate phenomena as the elastic response and viscous friction, which may be possible only in relatively simple cases and not as a general rule. Although providing convenient mathematical solutions to wave equations in uniform media, they complicate understanding of such fundamental quantity as the elastic force, which becomes combined with the force of friction. However, the elastic force is directly measurable and is critical in considerations of the field boundary conditions, and therefore its behavior should be carefully understood. One approach to such understanding is through the
The anelastic AI problem provides a simple example to study the combined effects of wave propagation and boundary conditions. Below, I review several of the existing AI definitions, offer its new generalized form, denoted \( \tilde{Z} \), and derive its dependence on the quality parameter \( Q \) of the medium. In addition, I show that unlike the traditional AI, \( \tilde{Z} \) turns out to be “reactive” (imaginary) for lossless medium, which also makes it a more straightforward analog to the electromagnetic impedance. Although \( \tilde{Z} \) is applicable to the general cases of oblique incidence and \( P/S \) wave mode conversions, most of the discussion below uses normal-incidence acoustic examples. Further, I describe the anelastic AI phase problem and discuss the general applicability of the correspondence principle to solving wave problems in heterogeneous media. By contrast to the broadly accepted view, I argue that the Lamé moduli should be considered as real-valued in the presence of attenuation. I also briefly illustrate an alternate (and also well-known) description of anelasticity based on Lagrange’s formalism. Finally, these statements are further illustrated by considering two simple mechanical systems.

**Acoustic Impedance**

To begin, let us consider the following several definitions of AI. For \( P \) waves at normal incidence, AI is broadly known to equal the product of density (\( \rho \)) and wave velocity (\( V \)) of the medium

\[
Z = \rho V .
\] (1)

However, this is still not the true definition but only the final formula for an isotropic attenuation-free medium. More fundamentally, AI is defined as the ratio of pressure to particle velocity component across the boundary (Brekhovskikh, 1980)

\[
Z = \frac{p}{u_z} ,
\] (2)

which allows extending formula 1 to oblique incidence at angle \( \theta \):

\[
Z_A = \frac{\rho V}{\cos \theta} .
\] (3)
In more complex cases, such as elastic wavefield with attenuation, a yet more general definition for \( Z \) from first principles is required. Such a definition can be derived from the role AI plays in elasto- and electro-dynamics, where \( Z \) is constructed so that the reflection coefficient \( r_{12} \) from a boundary of two media is solely determined by the ratio of their impedances \( Z_1 \) and \( Z_2 \):

\[
r_{12} = \frac{Z_2 - 1}{Z_2 + 1}.
\]

(4)

Note that the well-known identity for the transmission coefficient \( t_{12} = 1 - r_{12} \) also follows from the displacement-continuity boundary condition and does not depend on parameters of the media. Equation (4) also indicates the fundamental scale ambiguity of \( Z \), which can be multiplied by an arbitrary complex factor \( c \):

\[
Z \rightarrow Z' = cZ
\]

(5)

without altering the observed reflectivity.

Based purely on expression (4), several extensions of the normal-incidence AI (eq. 1) to oblique incidence were proposed and called Elastic Impedances (EI) (e.g., Connolly, 1999; Ma, 2003; Whitcombe et al., 2002; Santos and Tygel, 2004; VerWest, 2004; Martins, 2006). However, all of these definitions were based on the scaling ambiguity of impedance (eq. 5) and heuristic integrations of reflectivity time series and did not rigorously represent properties of the medium (for more on this, see Morozov, in review II). Such definitions will not be considered here.

Seeking a rigorous definition of AI which would incorporate both oblique incidence and attenuation, we need to look into the wave boundary conditions. Note that eq. (4) follows from solving the boundary conditions on the reflecting interface. In the acoustic case, these conditions require the continuity of normal displacement and stress across the interface:

\[
\begin{align*}
|u_z|_1 &= |u_z|_2, \\
|\sigma_z|_1 &= |\sigma_z|_2,
\end{align*}
\]

(6)
where $\sigma_{zz}$ is the normal stress component, and subscripts 1 and 2 indicate the propagating media. From Hooke’s law, stress is always proportional to spatial derivatives of the displacement:

$$\begin{bmatrix} \sigma_x \\ \sigma_{xx} \end{bmatrix} = -\tilde{Z} \begin{bmatrix} u_x \\ u_z \end{bmatrix},$$

where $\tilde{Z}$ is a linear matrix differential operator. For example, for an elastic field, it equals

$$\tilde{Z} = \begin{pmatrix} (\lambda + 2\mu)\partial_z & \lambda\partial_x \\ \mu\partial_x & \mu\partial_z \end{pmatrix}.$$  

where $\lambda$ and $\mu$ are the Lamé constants. Equations (7) and (8) give apparently the most general definition of impedance.

In the acoustic case and for a planar harmonic wave (in which $u_x/u_z = \tan \theta$), a scalar impedance of each medium can be defined as

$$\tilde{Z}_A = -\frac{\sigma_{zz}}{u_z} = \left[-\tilde{Z} \begin{bmatrix} 1 \\ \tan \theta \end{bmatrix}\right]_z.$$  

Since $\sigma_{zz} = -\tilde{Z}_A u_z$ in each of the two media, the reflection amplitude becomes determined entirely by the values of $\tilde{Z}_A|_1$ and $\tilde{Z}_A|_2$, and it can be easily seen that equation 4 is satisfied with

$$r_{12} = \frac{u_{z\text{ reflected}}}{u_{z\text{ incident}}}|_1.$$  

Thus, the impedance can be uniquely defined as a factor relating the stress and displacement boundary conditions in the incident, reflected, or transmitted waves (eqs. 7-9). Note that in an acoustic case, definition (2) has practically the same meaning, except that the velocity is used instead of the displacement, and consequently factor $c = -i\omega$ is added, where $\omega$ is the angular frequency. Therefore, as it can be verified from eq. (9), the following quantity is identical to the standard acoustic AI (eq. 3) at all angles:

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\[
Z_A = \frac{\tilde{Z}_A}{-i\omega}.
\]  

(11)

Factor \(-i\omega\) in eq. (11) is insignificant for describing the reflectivity (see expression 5), but it may become important when considering non-stationary waves. However, in the following we will consider harmonic fields and mostly use the conventional \(Z_A\) defined in eqs.(3) and (11).

**AI in the presence of attenuation**

By solving the wave boundary conditions on a welded contact of two elastic media, Lines et al. (2008) noted that attenuation contrasts cause phase-rotated reflections. Morozov (2009b) pointed out that such reflections can be expressed by incorporating the \(Q\) factor in AI

\[
Z = \rho V\left(1 + \frac{i}{2Q}\right),
\]  

(12)

and using formula (4).

However, a problem arises when deriving this expression from the correspondence principle, which describes the effect of attenuation by a negative complex argument of the medium velocity (Aki and Richards, 2002)

\[
V(Q) = V\left(1 - \frac{i}{2Q}\right).
\]  

(13)

When used to extend the AI in eq. (1), this formula gives an opposite sign of \(\text{Im}Z\) compared to the exact expression (12). Thus, the correspondence principle should be used with caution in this type of problems. A detailed discussion of the reason for such discrepancy and its general solution are given later in this paper.

To derive the correct expression (12) in a somewhat abbreviated form of Lines et al. (2008), consider a plane harmonic \(P\) wave normally-incident on a welded contact of two media (Figure 2a). In the presence of attenuation, its potential can be written as

\[
\phi(x,t) = A \exp(-i\omega t + ikr - ar) = A \exp(-i\omega t + ik'r),
\]  

(14)
where $\omega$ is the circular frequency, $\mathbf{k}$ is the wavenumber vector, and $\mathbf{\alpha}$ is the attenuation vector. Complex-valued vector $\mathbf{k}' = \mathbf{k} + i\mathbf{\alpha}$ is the effective wavenumber including both the propagation and attenuation effects. For simplicity, consider a homogeneous wave with parallel $\mathbf{k}$ and $\mathbf{\alpha}$. The spatial attenuation coefficient $\mathbf{\alpha}$ can then be related to $\mathbf{k}$ through the spatial attenuation quality factor $Q$ as

$$\mathbf{\alpha} = \frac{\mathbf{k}}{2Q}. \quad (15)$$

From the for $P$-wave potential (14) in each of the two media, we obtain the displacement

$$u_i(r,t) = \partial_i \phi(r,t) = i k'_i \phi(r,t), \quad (16)$$

velocity

$$u_i(r,t) = \omega k'_i \phi(r,t), \quad (17)$$

strain

$$\varepsilon_{ij}(r,t) = \partial_i u_j(r,t) = -k'_i k'_j \phi(r,t), \quad (18)$$

and stress

$$\sigma_{ij}(r,t) = -\left(\lambda \delta_{ij} k'_n k'_n + 2\mu k'_i k'_j\right) \phi(r,t), \quad (19)$$

where $\lambda$ and $\mu$ are the Lamé constants, and $\delta_{ij}$ is the Kronecker symbol (unit matrix element). Now let us only consider the normal-incidence case. From eqs. (17 and (19, we can relate the traction and velocity boundary conditions by a single factor $Z$:

$$\sigma_{zz}(r,t) = -Z u_z(r,t), \quad (20)$$

where $Z$ becomes the conventional impedance (eq. 11)

$$Z = \frac{(\lambda + 2\mu)k'}{\omega} = \rho V \left(1 + i \frac{2Q}{\omega}\right), \quad (21)$$

as in eq. (12).

Equation (21) shows that $Z$ varies with $Q$ as the wavenumber, and not as the
complex-valued phase velocity (eq. 13), and hence $\text{Im}Z$ is positive. In the presence of a $Q$ contrast, $r_{12}$ becomes complex-valued and corresponds to phase-shifted reflectivity (Lines et al., 2008). For example, for small contrasts in $\rho V$ and $Q^{-1}$, eq. (4) gives

$$r_{12} \approx \frac{\delta \ln Z}{2} \approx \frac{\delta \ln(\rho V)}{2} + i \frac{\delta(Q^{-1})}{4},$$

showing that reflections from positive attenuation contrasts are delayed in phase by $\delta \phi = \arg(r_{12}) \approx \arctan\left[\frac{\delta Q^{-1}}{\delta \ln(\rho V)/2}\right]$. Note that although the reflections in some cases appear advanced in time (Fig. 1), their causality is disturbed by neither positive nor negative phase shifts in eqs. (21) and (22). Because these phase shifts are frequency-independent, the corresponding group velocity delay is $d(\delta \phi)/d\omega = 0$. Nevertheless, correct signs of phase delays are important for determining the shapes of wavelets reflected from the boundary. In particular, reflections from pure attenuation contrasts (Fig. 1d) change polarities after switching from the explicit to the correspondence-principle formulation.

**Reactive/resistive properties of AI**

As mentioned above, although generalized to the case of attenuative medium, the AI in expression (12) is still somewhat unsatisfactory in the sense that its $\rho V$ part (describing the reflectivity) is real-valued, but $\rho V/Q$ (corresponding to energy dissipation) – imaginary. In the electrical circuit theory, from which the concept of impedance has originated, the impedance is usually decomposed in an opposite manner:

$$Z = R + iX,$$

where $R \geq 0$ is the resistance (i.e., energy dissipation) and $X$ is the reactive (energy-conserving) part. A closer analogy to electromagnetic impedance can be obtained by using $\tilde{Z}$ (eqs. 8 and 9), which relates stress to the displacement. This AI can be interpreted as the effective “spring constant” of the wave acting on the boundary (compare eq. (7) to the Hooke’s law)
\[ \tilde{Z} = -i(\lambda + 2\mu)k' = -i\omega\rho V\left(1 + \frac{i}{2Q}\right). \]  

This parameter can also be described as the “complex modulus” of the wave (Bland, 1960). Fig. 2b illustrates this analogy, which is, however, somewhat loose because both displacements and forces must be considered as additive in the parallel spring arrangement in medium 1. Also, the principal part of this spring constant is imaginary because the phase of traction is shifted by 90° relative to the displacement in an elastic wave.

Equation (4) holds with impedance \( \tilde{Z} \) similarly to the traditional AI. The resistive part of \( \tilde{Z} \), \( \text{Re} \tilde{Z} = \frac{\omega\rho V}{2Q} \), is positive and corresponds to energy dissipation, similarly to the electric resistance. The elastic energy density (Aki and Richards, 2002) in our example is

\[ E(r,t) = \frac{1}{2} \text{Re} [\sigma_{ij}(r,t)\varepsilon_{ij}(r,t)] = \frac{\text{Im}[k'\tilde{Z}^*]2}{2} |\mu_z(r,t)|^2, \]  

where eqs. (16), (18), and (19) were used. Note that the energy density is principally controlled by the imaginary (reactive) part of \( \tilde{Z} \) and decreases with distance \( z \) according to the dissipation law

\[ \frac{\partial \ln E(r,t)}{\partial z} = 2 \frac{\partial \ln |\mu_z(r,t)|}{\partial z} = -2\alpha = -k/Q. \]  

Thus, the quality factor corresponds to a phase shift of \( \tilde{Z} \) from a purely reactive impedance: \( Q^{-1} = 2\arg(\tilde{Z}) \), and \( Q^{-1} \) measures the relative energy dissipation per one wavelength.

In respect to its inversion from reflectivity data, \( \tilde{Z} \) is close to the traditional AI (eq. 1), which is heavily rooted in seismology and acoustics and embedded in a multitude of practical applications. Because the frequency dependence of \( \tilde{Z} \) does not affect the reflectivity, it should cause no changes in AI inversion approaches. The additional \( \tan^{-1}(Q^{-1}/2) \) phase factor can be incorporated in AI inversion, providing a more accurate
treatment of the effects of attenuation. At the same time, as Morozov and Ma argued recently (in press), strictly speaking, impedance-type attributes are never inverted for but rather constructed from the reflectivities by utilizing non-seismic information and exploiting the scaling ambiguity (eq. 5). Such inversion can be carried out for any attribute satisfying equation 4, including $\tilde{Z}$ or $Z_A$.

**Wave attenuation and the complex-valued elastic moduli**

The reason for the impedance discrepancy produced by the correspondence principle is that in reality, the $V(Q)$ in eq. (13) represents the phase velocity and not the material wave speed $V_m$, which is merely a combination of elastic parameters (such as $V_m = \sqrt{(\lambda + 2\mu)/\rho}$ for $P$ waves). Note that the material and phase velocities equal each other only in a uniform, isotropic, boundless, and attenuation-free medium. Phase velocity is related to the complex wavenumber as $V = \omega/k'$, and consequently its imaginary part is negative when $\text{Im}k' > 0$. A complex-valued material $V_m$ would require that $\lambda$ and/or $\mu$ also attain negative imaginary parts, and indeed, the concept of complex-valued $\mu$ was often used for explaining attenuation (e.g., Anderson and Archambeau 1964; Borcherdt and Wennerberg, 1985; Aki and Richards, 2002). However, in heterogeneous structures, a single phase velocity corresponds to a distribution of $V_m$, and analytical relations between them can hardly be justified in the general case. Assumptions of such analytical relations may produce elegant but inaccurate or erroneous solutions. For example, surface-wave energy dissipation derived by using the correspondence principle (Anderson et al., 1965) exceeds the cumulative dissipation within the medium by 10-20%, thereby violating the conservation of energy (Morozov, in review I).

In wave mechanics, the description of the displacement field consists of two components: 1) the wave equations, or the corresponding Lagrangian, and 2) boundary or radiation conditions. The wave equations are correctly transformed and greatly simplified by the correspondence principle, which was used in most studies (e.g. Anderson and Archambeau, 1964; Cooper and Reiss, 1966; Borcherdt, 1977; Borcherdt and Wennerberg, 1985; Richards, 1984; Aki and Richards, 2002). However, when applied to
the boundary conditions 6, this transformation makes \( \lambda \) and/or \( \mu \) in formula (21) complex-valued, and the stress,

\[
\sigma_{ij} = \lambda \varepsilon_{ik} \delta_{ij} + 2 \mu \varepsilon_{ij},
\]

acquires an additional phase delay \([-2\tan^{-1}(Q^{-1}/2)]\) relative to the elastic case (as can be seen, for example, from relation \( \lambda + 2 \mu = \rho V^2(Q) \) for \( P \) waves). This delay overwhelms the attenuation-related \( \tan^{-1}(Q^{-1}/2) \) phase advance of \( \varepsilon_{ij} \) in respect to \( u_i \), reverses the phase of \( \Delta I \) in eq. (12), and leads to incorrect phases of reflections from attenuation contrasts. The often implied assumption “…it is understood that the boundary conditions for the two problems are identical…” (Bland, 1960, p.96) appears to be unviable with complex-valued elastic moduli.

Thus, boundary conditions show that elastic parameters \( \lambda \) and \( \mu \), as well as density \( \rho \), should remain real-valued in the presence of attenuation. This can also be seen from the Lagrangian of the elastic field (see Bourbié et al., 1987), which remains real-valued when extended to complex-valued \( u_i \):

\[
L(u, \dot{u}) = \int \left( \frac{\rho}{2} \ddot{u}_i \dot{u}_i^* \right) + \left( -\frac{\lambda}{2} \varepsilon_{ij} \varepsilon_{ij}^* + \mu \varepsilon_{ij} \varepsilon_{ij}^* \right) d^3 r
\]

Here, dots indicate the time derivatives, and \( \dot{\cdot} \) denotes complex conjugation.

Rigorously, the attenuation is described not by modifying \( \lambda \) and \( \mu \) but by adding an external dissipation force to the right-hand side of Lagrange’s equations (Bourbié et al., 1987; Razavy, 2005)

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{u}_i} \right) - \frac{\delta L}{\delta u_i} = -\frac{\delta D}{\delta u_i}.
\]

Here, \( D \) is a (usually) second-order functional describing energy dissipation (not to be confused with the “specific dissipation function” \( Q^{-1} \) used by Bland (1960), Anderson and Archambeau (1964) and others). Several realistic examples of dissipation functionals for porous fluid-filled rock were given by Bourbié et al. (1987). For a simple example, consider the following hypothetical form of \( D \) constructed similarly to the elastic energy but using velocities:
\[ D\{\dot{u}\} = -\int \left( \frac{\lambda}{2} \dot{e}_{ij} \dot{e}_{ij}^* + \mu \dddot{e}_{ij} \dddot{e}_{ij}^* \right) d^3 r. \]  

(30)

With such \( D \), the \( P \)- and \( S \) waves are not coupled by attenuation, and the homogeneous equation of motion (29) for a \( P \) wave in a uniform medium becomes

\[ \ddot{u}_i - V_m^2 \dot{u}_i - \xi V_m^2 \dddot{u}_i = 0. \]  

(31)

From its plane-wave solution propagating in the direction of axis \( z \)

\[ u_i(z,t) = A \exp(-i \omega t + ik'z), \]  

(32)

the dispersion relation gives a positive \( \text{Im} k' \), as expected:

\[ k' = \frac{\omega}{V_m \sqrt{1 - i \omega \xi}} \approx \frac{\omega}{V_m} \left( 1 + \frac{i \omega \xi}{2} \right). \]  

(33)

Comparing this equation to expression 6, we see that in this example, the \( Q \) factor is frequency-dependent:

\[ Q = \frac{1}{\xi \omega}, \]  

(34)

and the wave phase velocity can be viewed as complex-valued but different from \( V_m \):

\[ V_{\text{phase}} = \frac{\omega}{k'} = V_m \sqrt{1 - i \frac{1}{Q}}. \]  

(35)

In agreement with the correspondence principle, the last two terms in eq. (31) can indeed be lumped together in an complex-valued squared material \( V_m' \):

\[ V_m^2 \rightarrow (V_m')^2 = V_m^2 \left( 1 - i \xi \omega \right), \]  

(36)

with the corresponding modifications of \( \lambda \) and \( \mu \). However, although this transformation makes \( V_{\text{phase}} \) equal \( V_m' \), it does not extend to the boundary conditions and should not be interpreted literally.

The sensitivity of the boundary conditions to the correspondence-principle transformation does not only affect the reflection/transmission and AI problems considered in this paper. For example, in modeling the surface waves in a layered Earth,
this transformation applied to layers with contrasting $Q$ shifts the phases of the boundary-value equations related to stress (see equations 4 in Anderson and Archambeau, 1964). Consequently, the eigenfunctions of the wavefield modify similarly to the reflection coefficients in the AI problem above. The resulting distributions of the kinetic and potential energies also change, leading to predictions of $Q^{-1}(f)$ different from those given in Anderson and Archambeau (1964) (Morozov, in review I). Thus, it appears that the correspondence principle only reliably applies to wave-equation solutions in boundless uniform media. Unfortunately, such cases are of little practical utility in seismology.

To conclude this section, another note on physical terminology and semantics is appropriate. The dissipation functional $D$ contains time derivatives of $u_i$ and is functionally closer to the kinetic energy than to the elastic-energy part of the Lagrangian (eq. 28). This once again shows that dissipation is related more to the kinetic energy than to modifications of the elastic moduli. For example, viscosity forces are proportional to pore-fluid flow velocities, similarly to the forces applied to dashpots in equivalent mechanical models (Bland, 1960). The elastic energy does not convert to heat directly but only by causing relative particle motions. Dissipation is therefore kinetic by its nature and could be formally associated with an imaginary part of mass density; however, this analogy is commonly interpreted as “imperfect gravity” and discarded (Anderson and Archambeau, 1964). Therefore, the main premise of the currently established attenuation model, namely that the strain energy is dissipated by “imperfect elasticity” (e.g., Anderson and Archambeau, 1964) appears unfounded. Moreover, $D$ is added as an external dissipative force in the equations of motion (29), it is not included in the Lagrangian and should not be viewed as a modification of either the kinetic or potential energy. Complex-valued elastic moduli may be convenient for describing the similarities between solutions to elastic and anelastic wave equations, but their rigorous physical meaning still remains to be established, and they do not always guarantee correct solutions for $Q^{-1}$ (Morozov, in review I).

**Mechanical analogs**

For sceptics who may think that the above critique of the complex-moduli
concept is based on the definition of AI or subtleties of wave processes, let us see how it fails in ordinary mechanics. Consider a linear oscillator with friction proportional to the velocity (Fig. 3a):

\[ m\ddot{x} = -m\omega_0^2 x - \xi m\omega_0 \dot{x}, \]  

(37)

where \( x \) is the displacement, \( m \) is the mass, and \( \omega_0 \) is the natural frequency. The spring constant equals \( k = m\omega_0^2 \). The second term in the right-hand side of this equation is the dissipation force, in which we assume \( \xi \ll 1 \).

The general solution to eq. (14) is \( x(t) = \text{Re}[A\exp(-i\omega'_0 t)] \), where

\[ \omega'_0 \approx \omega_0 \left(1 - \frac{i\xi}{2}\right) \]  

(38)

is the “complex frequency,” and \( A \) is an arbitrary complex-valued amplitude. One can introduce a complex-valued spring constant (or also mass)

\[ k' = m\omega'_0^2 \approx k(1 - i\xi) \]  

(39)

and eq. (14) will take the form of a free-oscillator equation

\[ m\ddot{x} = -m\omega'_0^2 x. \]  

(40)

If an external force \( f(t) \) is added to the r.h.s. of eq. (14), the solution can also be derived from its Fourier transform \( f(\omega) \) by using the complex frequency only:

\[ A(\omega) = \frac{f(\omega)}{m(\omega'_0^2 - \omega^2)}. \]  

(41)

Thus, in the mathematical world of complex-valued solutions to the homogeneous equation (14) (i.e., the “relaxation spectrum”), the correspondence principle describes the solutions very effectively.

However, considering the expressions for the force in the real world, the limitations of this description become apparent. First, eq. (16) attributes a property of the dashpot (\( \xi \)) to the spring. This is possible only in retrospect, because of the simplicity of the particular homogenous solution (15), and only for an equivalent model for which we
have one dashpot per spring. The elastic force of the spring is mixed with the attenuation and cannot be obtained from the equivalent equation (17). Second, the elastic force equals \(-kx\), i.e. it is always at a 180° phase relative to \(x(t)\) and not simply related to the “complex force” \(F = -k'Ae^{i\omega t}\). The complex force has an additional phase advance of about \(\xi\) in respect to \(x(t)\). Nevertheless, the force is an easily measurable physical quantity (e.g., by measuring the extension of the spring) and represents the key element of most boundary conditions. Boundary conditions based on equating the complex forces (in fact, on equating the \(mx\) products in eq. (17) instead of the elastic forces) may lead to phase-shifted solutions that have no counterparts in the real world.

For more complex systems, for example consisting of two dashpots with three springs (Fig. 3b), it is unclear how to distribute the attenuation parameters among the spring constants. As shown in the Appendix, by solving the dynamic equations, one can derive the two normal oscillation modes, and the corresponding natural frequencies can again be modified to incorporate the attenuation. However, these parameters only describe the solutions rather than real mechanical parameters of the system, and do not simply extend the spring parameters \(k_1, k_2\) and \(k_3\) into the complex plane. This situation is analogous to the complex-valued phase velocities differing from real-valued material velocities in heterogeneous media.

Finally, as in wave examples above, a rigorous formulation of attenuation can be obtained without the complex-valued \(k'\), by using the following Rayleigh dissipation function in Lagrange’s equations (Razavy, 2005):

\[
D = \xi \frac{m\omega_0}{2} \dot{x}^2. \tag{42}
\]

Note that the dissipation is associated with the single dashpot and not the whole oscillator (Figure 3a), and it has no relation to \(k\). This expression can be naturally extended to any mechanical systems such as shown in Fig. 3b.

**Conclusions**

Four groups of conclusions can be drawn from the above analysis. First, unlike sometimes assumed, the Acoustic Impedance (AI) depends on the attenuation, although
its relative effect \((i/2Q)\) is small in typical Earth media. Similarly to acoustics and electrical circuit theory, reflection coefficients in all cases can be derived exclusively from impedance contrasts. The positive imaginary part of AI explains the phase-shifted reflections from attenuation contrasts pointed out by Lines at al. (2008).

Second, the correspondence principle is rigorously applicable only to the cases of boundless uniform media. When used to infer the AI in the presence of attenuation, it leads to erroneous signs of \(\text{arg}(Z)\) and phases of reflections. Therefore, the recommended approach is to use the explicit equations of motion and boundary conditions while keeping the density and elastic constants real. In a potentially useful analogy, the new impedance \(\tilde{Z}\) (eq. 24) can be interpreted as the effective “spring constant” of the wave at the boundary. With such interpretation, the process of wave reflection and transmission is viewed as the state of equilibrium of three welded springs.

Third, the alternate definition of \(\tilde{Z}\) leads to a better analogy with electromagnetism and wave mechanics than the conventional AI while retaining its familiar relation to reflectivity. \(\tilde{Z}\) associates the wave impedance with inductance or capacitance rather than with resistance and removes the awkward association of \(Q\) with reactive response.

Finally, the classic AI definition (formula 3) represents a scaled reactive part of the new AI

\[
Z_A = \frac{-\text{Im}\tilde{Z}}{\omega}.
\]

(43)

In most practical applications, \(\tilde{Z}\) or \(Z_A\) can be used interchangeably; however, the advantage of \(\tilde{Z}\) is in its rigorous definition and extendibility to general cases of attenuation and \(P/S\) mode conversions, which are discussed elsewhere (Morozov, in review II).

Acknowledgments

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Appendix A. Normal modes of the mechanical model in Figure 2b

In linear elasticity, viscous forces are attributed to the dashpots in Fig. 3b, and the equation of motion for the two masses are:

\[-m_1\ddot{x}_1 - k_1 x_1 + \xi_1 (\dot{x}_{1,D} - \dot{x}_1) = 0, \tag{A1a}\]
\[-m_2\ddot{x}_2 - k_2 (x_2 - x_{1,D}) - \xi_2 \ddot{x}_2 - k_3 x_2 = 0, \tag{A1b}\]

where \(x_{1,D}\) denotes the coordinate of the cup of dashpot 1. Because the cup is considered massless, its viscous force is compensated by the elastic force of the second spring:

\[\xi_1 (\dot{x}_{1,D} - \dot{x}_1) = k_2 (x_2 - x_{1,D}). \tag{A1c}\]

For harmonic oscillations with frequency \(\omega\) and \(\omega \xi_{1,2} \ll 1\), eq. (A1c) gives

\[x_{1,D} \approx x_2 \left(1 + \frac{i \omega \xi_1}{k_2}\right) - i \frac{\omega \xi_1}{k_2} x_1, \tag{A2}\]

and the system of homogeneous eqs. (A1) becomes:

\[x_i \left(m_1 \omega^2 - k_i + i \omega \xi_i \right) + x_2 \left(- i \omega \xi_1 \right) = 0, \tag{A3a}\]
\[x_i \left(- i \omega \xi_1 \right) + x_2 \left(m_2 \omega^2 - k_3 + 2i \omega \xi_2 \right) = 0. \tag{A3b}\]

If \(\xi_1 = \xi_2 = 0\), the two masses are decoupled (Fig. 3b), and the corresponding normal frequencies from eqs. (A3) are \(\omega_{0,1} = \sqrt{k_1/m_1}\) and \(\omega_{0,2} = \sqrt{k_3/m_2}\), respectively.

In the presence of weak dissipation, by perturbing the frequency \(\omega = \omega + \delta \omega\) in the determinant of the linear system (A3) and keeping only linear terms in \(\xi\), we obtain:

\[(i \omega \xi_1 + 2m_1 \omega \delta \omega \left(m_2 \omega^2 - k_3\right) + \left(m_1 \omega^2 - k_1\right) \left(2i \omega \xi_2 + 2m_2 \omega \delta \omega\right) = 0, \tag{A4}\]

from which the shifts of each of the two normal frequencies \(\omega_{0,n} (n = 1,2)\) are

\[\delta \omega_n = -i \frac{\xi_1 \left(m_2 \omega_{0,n}^2 - k_3\right) + 2 \xi_2 \left(m_1 \omega_{0,n}^2 - k_1\right)}{2 \left[m_1 \left(m_2 \omega_{0,n}^2 - k_3\right) + m_2 \left(m_1 \omega_{0,n}^2 - k_1\right)\right]}, \tag{A5}\]
As expected, these frequency shifts are negative and imaginary, in agreement with the \( \exp[(\text{Im}\omega)t] \) attenuation law. However, they are clearly associated with the normal oscillation modes and can hardly be meaningfully attributed to the spring constants.
Figures

Fig. 1. Numerical models of a P-wave reflection from 1-km depths for a range of source-receiver offsets by using the reflectivity method (Fuchs and Müller, 1971): a) with positive velocity/density and attenuation contrasts; b) same, with an additional $-45^\circ$ phase rotation to compensate the effect of $\delta(Q')$; c) with no attenuation contrast; d) with a pure attenuation contrast. Overburden velocities are $V_P = 2$ km/s and $V_S = 1$ km/s, and density $\rho = 2$ g/cm$^3$. Velocity and attenuation contrasts across the boundary are indicated in the labels. Dashed gray line indicates the position of the near-offset peak in plot c).
Fig. 2. a) Model geometry and notation; b) AI analogy to the equilibrium of elastic springs.
Fig. 3. a) Model of a linear oscillator with viscous friction. b) A more complex model with two dashpots and three springs. Note the uncertainty of attributing the dashpot parameters $\xi_1$ and $\xi_2$ to the springs.

2.2. Numerical modeling of geometrical spreading in realistic crustal structures

Summary

In realistic lithospheric structures, geometrical spreading (GS) is complex and variable, and therefore impossible to model by using simple theoretical dependencies. As I showed earlier (Morozov, 2008), in many cases, frequency dependence of the apparent attenuation quality factor $Q(f)$ corresponds to only ~10% errors in GS and disappears when the GS is measured from the data. The elastic (scattering) quality factor $Q_s$ is also irresolvable from GS uncertainties and $Q_l$ and therefore its use should be avoided.

In this study, GS is modelled in several crustal models by using 1D reflectivity waveform synthetics. Variations of body-wave GS are measured by using parameter $\gamma$, which represents the zero-frequency limit of the generalized scattering coefficient. Modelled values of $\gamma$ are close to those observed, and also predicted by coda modelling earlier. For most crustal structures, values of $\gamma$ are positive within 0 – 100-km distances,
showing that that GS is typically faster compared to the commonly used theoretical predictions. Such positive $\gamma$ explain the positive frequency dependence of the apparent $Q(f)$, which was observed in many studies.

The upper-crustal structure and position of the earthquake hypocenter within it determine the character of GS. Reflectors above the hypocenter tend to increase $\gamma$, and reflectors immediately below the source – to decrease it. With strong reflectivity below the source, $\gamma$ can become negative and lead to GS-compensated amplitude peaks at 30 – 70 km of hypocentral distances, which were sometimes attributed to back-scattering. Strong velocity and attenuation contrasts within the upper crust and above the seismogenic zone increases $\gamma$ in tectonically young structures. Further implications of this model and its relation to the conventional $Q_0$ and $\eta$ parameters are also discussed.

**Introduction**

The difficulty of separating seismic scattering (represented by elastic, or scattering $Q$, $Q_s$) from anelastic attenuation (intrinsic, $Q_i$) is well known, and similarly known is the trade-off of the frequency-dependent attenuation $Q(f)$ with geometrical spreading (GS). However, it is usually less noticed that these two uncertainties are interrelated and to a great degree cross-feed each other. Two different aspects should be distinguished in this problem – namely, the theoretical possibility of the frequency-dependent elastic and anelastic $Q$ and the fact of their observations in the data. Unfortunately, these two aspects are rarely clearly separated in discussions of frequency-dependent $Q$. Empirical attenuation studies often lean towards elegant theoretical models without regard to their unrealistic assumptions and to the ability of the data to support them. In particular, this can be seen in the hugely predominant presentations of the attenuation data in the forms of $Q(f)$ or $Q^{-1}(f)$, in which such theoretical assumptions are embedded.

The possibility of frequency dependence of $Q_i$ and $Q_s$ and also their separability has been well established in numerous theoretical and modeling studies, and I do not question or discuss them in this paper. Without attempting a comprehensive overview, modeling broadly shows that depending on the statistical properties of the scattering
medium, the frequency dependence of $Q_i$ can range from $Q_i \propto f$ or steeper (Chernov, 1960; Dainty, 1981; Fehler et al., 1988) to nearly frequency-independent $Q_i$ in self-similar media (Frankel and Clayton, 1986). Anealastic $Q_i(f)$ dependencies ranging from near-constant to nearly proportional to the frequency $Q_i$ were also predicted by ‘creep’ or ‘relaxation’ models (Liu et al., 1976). Such similarity in observable properties leads to a virtual impossibility of separating these quantities from the data without making stringent assumptions. If the total $Q$ is allowed to be proportional to $f$, it also becomes indistinguishable from GS. As Wennerberg and Frankel (1989) put it, the difficulty is seen as “as a consequence of the fact that certain observable can be interpreted by identical equations resulting from either of the two credible physical theories describing fundamentally different processes.” Note that the basis for “credibility” of these theories lies in the assumption that GS can be perfectly compensated, so that the resulting $Q(f)$ is measured in a uniform isotropic space, and therefore only the $Q_i/Q_s$ separation problem remains. From a purely theoretical standpoint, although inadequate, this assumption does not break the validity of the ensuing models.

By contrast, experimental observations of $Q(f)$ have great consequences for interpretation and thus require careful scrutiny. In all such studies (e.g., Anderson and Given, 1982; Wu, 1985; Toksöz et al., 1988; Mayeda, 1991, 1992; Jin et al., 1994; Del Pezzo et al., 2006, Padhy et al., 2007, and many others), a theoretically-known GS is again the key assumption facilitating the measurements and allowing long-reaching inferences. Unrealistic “background” GS models (uniform space, only acoustic $S$-waves, isotropic propagation, etc.) are always used for simplicity; however, when these models are varied, both the $Q(f)$ and presumably the $Q_i/Q_s$ separation change dramatically (e.g., Kinoshita, 1994). Thus, the frequency-dependent $Q(f)$ and likely the very observation of $Q_i$ in the data could in fact result from their presumed existence in the model underlying the measurements. Ironically, given the persistent uncertainty in GS, it still remains to be established whether the frequency-dependent $Q(f)$ has actually ever been observed in seismological data. To answer this question, a GS-assumption free measurement procedure is needed (Morozov, 2008).

When measured from the averaged wave path effects (e.g., Fan and Lay, 2003):
$P(t, f) = t^{-v} e^{-\pi f t / Q(f)}.$  \hspace{1cm} (1)

the elastic and frequency-dependent anelastic attenuation may be difficult to differentiate from the effects of the structure. In theoretical treatments (e.g., Dainty, 1981), a uniform isotropic background is always considered, and therefore ray bending, reflections, and mode conversions are all broadly described as scattering, which is then included in $Q(f)$ [eq. (1)].

If real GS differs from $t^{-v}$, $Q(f)$ in eq. (1) automatically becomes frequency-dependent, and the elastic $Q_s$ is its most controversial part. For example, $Q_s$ is defined so that a $Q_s$-free case cannot exist in real-Earth structure (such as in an Earth with velocity gradients and the Moho), even without scatterers and attenuation! To see this, note that even with $Q_i^{-1} = 0$ and frequency-independent propagation [$P(t,f)$], an imperfectly known GS leads to $Q_s$ proportional to the frequency:

$$Q^{-1}(f) \equiv Q_s^{-1}(f) \approx \frac{1}{\pi f} \left\langle \ln \left[ P(t, f) t^{-v} \right] \right\rangle,$$  \hspace{1cm} (2)

where the angular brackets $<...>$ denote some kind of averaging over the measurement time range, depending on the inversion algorithm. When $Q_i$ is present, both $Q_i^{-1}(f)$ and $Q_s^{-1}(f)$ have to be inverted from the same functional form (1), making the resulting frequency dependencies intricate and unstable. This is the well-known problem of under-constrained inversion, which requires regularization for its solution – and such regularization is provided by the strictly assumed GS form and by the selected scattering model.

Band-selective effects (such as tuning caused by crustal layering) would lead to more complex frequency dependencies of the path factor (1), which could be accordingly mapped into $Q_s(f)$ and the total apparent $Q(f)$. Deviations of GS from the $r^{-v}$ law (for example, an amplitude increase at ~20 - 70 km from the source caused by crustal reflections) could also be interpreted as caused by “backscattering” and used for separating $Q_s$ from $Q_i$ (Wu, 1985). This may be acceptable in the scattering theory, although still reducing the lithospheric structure to simplistic cross-correlation functions.
of the medium. However, labelling the reflections and GS variations as $Q_s$ and frequency-dependent $Q_i$ in attenuation measurements is counter-productive, as it obscures the key causal relationship of GS to the structure and introduces additional complexity, spurious frequency dependencies, and sensitivity to the choice of the “background” GS.

It appears that the root of the problem is in the widespread use of the apparent quality factor $Q$ for describing attenuation. The GS-corrected path factors in eq. (1) are usually transformed into the $Q(f)$ form before an interpretation begins. and especially in its power-law form (Aki and Chouet, 1975; Aki, 1980):

$$Q(f) = Q_0(f/f_0)^\eta.$$  \hspace{1cm} (3)

Here, $f_0$ is some reference frequency often taken equal 1 Hz, and $Q_0$ is the quality factor at that frequency. Both parameters $Q_0$ and $\eta$ are known to trade-off with GS, and within typical measurement errors, this expression can accommodate the apparent frequency dependencies in eq. (2). As detailed analysis of the $Q(f)$ measurement procedure shows, $\eta$ in eq. (3) is approximately proportional to the residual uncompensated GS (cf. Morozov, 2008). Thus, the dependence (Error! Reference source not found.) is simultaneously based on a fixed theoretical GS and absorbs its inaccuracy, making the definition controversial, tied to model assumptions, and insufficiently constrained by the data.

This confusing complexity of $Q(f)$ measurement can be simply removed by abandoning the use of $Q_s$. The concept of elastic $Q_s$ was introduced in order to provide a uniform, “quality-factor” picture of all processes of wave amplitude decay outside of the theoretical GS (Aki, 1980). This provided interesting theoretical parallels with other fields of physics in which scattering phenomena are pronounced, such as optics, plasma physics, and atmospheric and ocean acoustics. Inspiring concepts of seismic albedo [$B_0 = Q_s^{-1}/(Q_s^{-1} + Q_i^{-1})$; e.g., Mayeda et al., 1991], Boltzmann’s after-effect and Debye relaxation spectra (e.g., Liu et al., 1976) were evolved to explain the resulting frequency-dependent $Q$. Similarly to the atmospheric light absorption band, a seismic absorption band was proposed for the Earth (Aki, 1980; Anderson and Given, 1982), and possible physical mechanisms of elastic relaxation in mantle rocks were identified (Jackson and Anderson,
However, there the similarity ends, because the two physical processes of scattering and intrinsic absorption are not alike. Notably, although brought in to support the scattering-based interpretation of lithospheric attenuation (e.g., Aki and Chouet, 1975), the concept of $Q_s$ is alien to the scattering theory, which describes energy dissipation by using path-length related parameters, such as the attenuation coefficient, mean free path, or differential cross-section (e.g., Chernov, 1960; Sato, 1978). By contrast, the quality factor, $Q$, describes the relative energy dissipation per one cycle of wave oscillation and not per unit of propagation distance. Therefore, $Q_s$ remains an artificial analogy to $Q$, so that, for example, $Q_s$ can be constant only in a specially constructed medium (Frankel and Clayton, 1986).

It is quite clear a priori and was noted some time ago (e.g., Banda et al., 1982) that GS depends on the lithospheric velocity structure. Frankel et al. (1990) pointed out that the effective GS of $S$ waves in NE United States was significantly steeper than $R^{-1}$ for a model of the crustal velocity structure developed for this area, where $R$ was the hypocentral distance. From numerical modeling of that paper, one can also see that the GS does not follow the simple $R^{-\nu}$ dependence with any $\nu$, but such dependence could be used within the distance ranges before the near-critical $SmS$ reflection. For $S$ waves (epicentral distance $\Delta < 100$ km), the values of $\nu$ were consistently higher (~1.5 from modeling, ~1.3 observed in New York State, and ~1.9 in South Africa) than the expected theoretical $\nu = 1$. For $Lg$ waves ($\Delta = 100 - 400$ km), $\nu$ equalled ~0.7 where 0.5 was postulated in preceding frequency-domain $Q(f)$ studies (Frankel et al., 1990).

Only ~10% uncertainties in GS are sufficient to explain many of the observed frequency dependencies of the apparent $Q(f)$. This can be shown by considering the typical under-compensated GS level of $\gamma \approx 0.01$ s$^{-1}$ observed in lithospheric body, $Lg$, and coda measurements (Morozov, 2008; the definition of $\gamma$ is given below). By equating the effect of such $\gamma$ to that of an equivalent of $\delta\nu_0 e^{-\gamma t} \approx t^{-\delta\nu_0}$, we find that $\delta\nu_0 \approx \gamma t / \ln t$. For a typical local earthquake coda observation time of $t \approx 30$ s, this gives $\delta\nu_0 \approx 0.09$. With the background value of $\nu = 1$ (Aki, 1980), such GS variations should clearly be expected in
a heterogeneous lithosphere.

Once we take an observational point of view and recognise the variability of GS within the Earth, the picture changes dramatically (Morozov, 2008). Variations in GS can be measured instead of assuming their theoretical levels, and the resulting $Q$ examined for frequency-dependence. The ambiguities in parameters $Q_0$ and $\eta$ disappear, and parameter $\eta$ is replaced by the parameter $\gamma$, which measures the residual uncompensated GS. At least in the cases in which $Q_s$ was nearly proportional to the frequency (e.g., Dainty, 1981; Mayeda et al., 1992), $Q_s$ disappears and is replaced by $\gamma$ (Morozov, 2008). Separation of the “true” (i.e., small-scale Rayleigh- and Mie-type scattering) scattering from anelastic attenuation still appears unfeasible (Wenneberg and Frankel, 1989), and therefore these two contributions are lumped in a single “effective” attenuation factor $Q_e$ (Morozov, 2008). In many real-data cases I considered so far (whole-Earth oscillation, body, surface, coda, and $L_g$ waves; Morozov, 2008 and unpublished), frequency-independent $Q_e$ was found sufficient to explain the observations. Note that the observed frequency independence makes this quantity stable, transportable, and convenient in interpretation.

In a worldwide compilation of short-period $S$-wave studies (Morozov, 2008), the GS was found typically under-compensated by the standard $t^\nu$ corrections, with positive $\gamma$ values ranging from $\sim 0.002$ to $\sim 0.07 \text{ s}^{-1}$ in different types of observations. Values of $\gamma$ were also found to consistently decrease with tectonic age (Morozov, 2008) and to be predictable by numerical modeling in realistic lithospheric structures (Morozov et al., 2008).

In the present paper, I use numerical modeling of GS in several detailed velocity structures to provide explanation for such positive levels of $\gamma$ and to draw some conclusions about their possible origins. Because the apparent frequency-dependence parameter $\eta$ is approximately proportional to $\gamma$ (Morozov, 2008), this should therefore explain the observed predominance of positive $\eta$ values in lithospheric-scale observations. Along with the generally faster spreading, in cases when upper-crustal reflectors are located below the earthquake sources, slower than $R^{-1}$ amplitude decays.
were also observed in these models at 30 – 70 km distance ranges. In scattering-theory based interpretations, such slower decays were attributed to backscattering (e.g., Toksöz et al., 1988; Mayeda et al., 1991). In addition, thin low-$Q$ layers, such as sedimentary basins, modify the mode content of the wavefield, somewhat accelerate the GS at near offsets, and produce generally “geometrical” effects (Morozov et al., 2008). Thus, once again, backscattering and $Q_s$ could be reduced or eliminated once the effects of the structure are accounted for more fully.

A secondary goal of this paper is to suggest a simple form for the empirical GS law that is less “theoretical” and restrictive than $G(t) = t^{-\nu}$ and could be convenient for measuring the attenuation coefficient and inverted from the data. Our form is thus parameterized by the zero-frequency attenuation coefficient $\gamma$ as: $G(t) = t^{-\nu}e^{-\gamma t}$, where $\nu$ is still the fixed, conventional theoretical GS exponent, and $\gamma$ should be measured from the data.

In conclusion, the results presented here may even appear qualitatively obvious, yet they provide the quantitative substance to the discussion of the double GS ↔ $Q(f)$ and $Q_i ↔ Q_s$ ambiguity. We will see that when variable GS is considered, geometrical (i.e., scattering- and attenuation-free) properties of the wavefield can be modeled by using realistic velocity structures within the lithosphere. For earthquake sources located below the zone of upper-crustal reflectivity, the modelled GS tends to be systematically under-compensated, with $\gamma \approx 10^{-3} – 10^{-2}$ s$^{-1}$, in agreement with the observations (Morozov, 2008). Scattering quality factor $Q_s$ as an ad hoc compensation for inaccurate knowledge of the background structure can therefore be abandoned. Instead, in the structured background model, effective attenuation $Q_e$ can be rigorously defined as related to the intrinsic losses and scattering on heterogeneities that are smaller-scale compared to the background structure. Due to such definition, the effect of attenuation becomes automatically zero at $f = 0$, which allows clear and unambiguous differentiation of the effects of geometrical factors from those $Q_e$. However, separation of the contributions from $Q_i$ and $Q_s$ in $Q_e$ still appears infeasible with the exiting data. Finally, from a number of real-data cases considered to date (Morozov, 2008, and unpublished), $Q_e$ appears to be frequency-independent within the available data quality.
Numerical modeling

To examine the variability of geometrical effects, I performed 1D waveform modeling in several 1-D crustal and upper mantle models. As in several previous studies (Duenow, 2003; Morozov, 2004; Morozov et al., 2008), synthetic seismic sections were created by using the reflectivity method (Fuchs and Müller, 1971). The program by K. J. Sandmeier was modified to handle larger computations, parallelized, and incorporated into our data processing system (Chubak and Morozov, 2006) allowing seamless processing and plotting of the results. The computations were performed on a Linux notebook computer.

Modeling resulted in over 800-s long, 3-component synthetic records sampled at 200-ms intervals and output at 1-km intervals from near-zero to 600-km distances from the epicentres. This allowed examining the wavefield to much large offsets and avoiding any numerical wrap-around effects. The modelled frequency band was 0.2 – 2.4 Hz by using a “spike” source function suitable for spectral measurements. Sufficiently dense phase velocity spectrum from 1 to 120 km/s was automatically selected in order to avoid frequency aliasing during numerical mode summations. As in any implementation of the propagator matrix method (Aki and Richards, 2002), all $P/SV$ mode conversions and multiple reflections were accounted for in the modeling (Figure 1).

For each three-component record, a sample-by sample root-mean square (RMS) trace was formed, and its peak vector amplitude and the total trace energy were measured. The peak amplitudes were further squared, and both quantities multiplied by $R^2$ to compensate for the theoretical GS of body waves. The amplitudes were finally scaled and presented together (Figures 2 - 4).

Several velocity models were tested, most of them based on the global IASP91 model (Kennett and Engdahl, 1991) consisting of a simple three-layer crust and mantle without strong gradients and low-velocity zones (Figures 2 and 3). In order to focus on the geometrical effects, quality factors within the entire models were set large and equal $Q_P = 15000$ and $Q_S = 10000$. The densities were equal 2.8 g/cm$^3$ within the crust and 3.2 g/cm$^3$ within the mantle. In addition to the standard IASP91 model (Figure 2a), two of its
Variability of geometrical spreading

modifications were also considered: one containing a 2-km thick low-velocity sedimentary layer with the same high $Q$’s (Figure 2b) and another one with strongly attenuating sediments: $Q_P = 20$ and $Q_S = 10$ (Figure 2c). Point sources were located at 7-km depths in all models.

Because of the crustal and mantle structure, the resulting wavefields are complex and far from the idealized direct $P$- and $S$ waves (Figure 1). Note that the free-surface, Moho and intra-crustal reflections cause rapid and persistent variations of the amplitudes recorded at the surface. Nevertheless, distinct trends can be recognized in the pre-critical Moho reflection range (0 – 100 km) and beyond it (Figures 2 - 4).

In all cases, the peak amplitudes (crosses in Figure 2, middle and bottom) show an approximately $R^{-1}$ behaviour (near-horizontal slopes in Figure 2, middle and bottom), but only when averaged and considered beyond ~100-km distance ranges. Closer than ~60 – 70 km from the source, the amplitudes drop off quickly, corresponding to $\nu \approx 1.32$ for the IASP91 model and $\nu \approx 1.4 – 1.5$ for models with sedimentary layers (Figure 2). Around ~100-km hypocentral distances, near-critical Moho reflections arrive, whose GS-corrected amplitudes may rise above the level near the epicentre (in particular, for the IASP91 model, Figure 2a). At greater offsets, the $P$- and $S$-wave Moho reflections are followed by numerous multiples developing a more uniform GS at greater distances (Figure 2, bottom).

The introduction of a sedimentary layer above the source increases the near-source GS exponent from $\nu \approx 1.3$ to 1.4 (Figure 2b). Such trend of increasing pre-critical $\nu$ was also observed in other models with heterogeneities above the source. This effect was noted by Frankel et al. (1990), who explained it by waves reflecting downward by the base of the sedimentary layer. The increased attenuation within the sediments appears to somewhat increase the spreading (to $\nu \approx 1.5$; Figure 2c).

When additional reflectivity is placed below the source region, $\nu$ decreases and may drop below 1 (Figure 3). When the reflectors are located close beneath the source, the geometrically-compensated amplitudes rise monotonically from the source to about ~50 km (with $\nu \approx 0.93$), followed by a decay at larger offsets (Figure 3b). This behaviour
Varialbility of geometrical spreading resembles that observed in total-energy measurements (Toksöz et al., 1988; Mayeda et al., 1991) who attributed such increased amplitudes to backscattering. This similarity is not surprising, as the reflectors can indeed be viewed as “scatterers” returning the energy to the surface. Note that the position of the peak at ~50 km corresponds to the fixed source depth of 7 km in this modeling, and with deeper sources and reflective zone depths, the peak should accordingly move to longer offsets.

The two final numerical tests show a simple crustal model with a constant velocity gradient (Figure 4a) and a realistic cratonic model Quartz-4 derived from our studies of Peaceful Nuclear Explosions (PNEs) in Russia (Figure 4b; Morozova et al., 1999). The model in Figure 4b was also used in our previous studies (Morozov and Smithson, 2000; Duenow, 2003; Morozov et al., 2008). As expected, in the gradient-crust model, the amplitude decay curves are the simplest and show the best agreement with the theoretical \( R^{-1} \) dependence (Figure 4a). This is the only example of good agreement with the assumed theoretical GS I have found so far. By contrast, because of its greater crustal thickness, the Quartz-4 model shows a range of amplitudes decaying faster than in all IASP91-based models, with \( \nu \approx 1.7 \), followed by strong \( PmP \) and \( SmS \) onsets at ~150 km. Note that this large GS exponent is still within the range observed by Frankel et al. (1990).

**Interpretation of numerical tests**

To develop a model suitable for interpreting the numerical (as well as experimental) data, assume that there exists a “reference” Earth structure in which the GS, denoted \( G_0(t, f) \), is close to the one observed. However, real structures (observed or modelled in a specific area), still differ from the reference structure, and therefore the seismic wave amplitudes should also be different. This difference may be caused by variations in GS and by the elastic and anelastic attenuation. In a scattering-theory approximation (Chernov, 1960), the deviation of the logarithm of the reference-GS compensated amplitudes is small and proportional to propagation time \( t \):

\[
\ln \left( \frac{P(t, f)}{G_0(t, f)} \right) = -\alpha(f) t, \quad (4)
\]
Variability of geometrical spreading

where \( \alpha(f) \) is the attenuation coefficient and \( P(t, f) \) is the path factor in eq. (1). Therefore:

\[
P(t, f) = G_0(t, f) e^{-\alpha(f)t}.
\]  \( \text{(5)} \)

Further, by separating the frequency-dependent part of \( \alpha(f) \), we have:

\[
\alpha(f) = \gamma + \frac{\pi}{Q_e} f.
\]  \( \text{(6)} \)

Here, the frequency-independent term \( \gamma \) gives the true geometrical spreading within the structure:

\[
G(t, f) = G_0(t, f) e^{-\gamma t}.
\]  \( \text{(7)} \)

Parameter \( \gamma \) effectively replaces \( Q_e \) in eq. (2) without a confusing association of GS with attenuation quality. The frequency-dependent term in \( \alpha(f) \) [eq. (6)] corresponds to the effective attenuation, where \( Q_e \) becomes the quality factor. Unlike the conventional \( Q(f) \) (e.g., Aki and Chouet, 1975), this \( Q_e \) is defined so that its effect vanishes \( (Q_e^{-1} = 0) \) when no scattering or anelastic attenuation is present.

The exponential form of \( P(t, f) \) in eq. (5) leads to a straightforward measurement procedure for \( \gamma \) and \( Q_e \) by using logarithms of spectral ratios, as illustrated in detail in Morozov (2008). Parameter \( \gamma \) is obtained from the intercepts of the \( \alpha(f) \) dependencies at \( f = 0 \), and \( Q_e \) is measured from their slopes. If the \( \alpha(f) \) dependence shows significant deviations from linearity in \( f \), a frequency-dependent \( Q_e \) can be derived from its curvature.

In practice, \( G_0(t, f) \) is unknown for all frequency bands and wave types, and therefore we use the conventional \( G_0(t, f) \approx t^\nu \) approximation, in which \( \nu \) is selected differently for the different frequency bands and types of observations. In our body-wave examples (Figures 2 - 4), \( \nu \) is considered equal \( 1 \), and \( \gamma \) represents the corresponding variations of GS. The relatively small values of \( \gamma \) (Figures 2 - 4, bottom) show that this approximation is acceptable within the \( \sim 0 – 100 \)-km distance ranges. For other types of waves, \( \nu \) can be different [such as \( \frac{1}{2} \) or 0.83 for \( Lg \) (Campillo, 1990)] and \( \gamma \) would vary accordingly. Therefore, \( \gamma \) may vary with frequency bands, offset ranges, and types of...
observations in this “patchwork” approach to selecting the $G_0(t, f)$ (for more on this, see Morozov, 2008). Also, for refracted body waves ($Pn$ or $Sn$), $G_0(t, f)$ is inherently frequency-dependent (Zhu et al., 1991; Yang et al., 2007). However, this does not alter the role of $\gamma$ as the measure of GS deviations from its best-known theoretical level.

Parameter $\gamma$ was measured from our synthetic data shown in bottom plots in Figures 2 - 4 by using the following relation [see eq. (4)]:

$$\ln[E(t)t^2] = 2\ln \left[ \frac{P(t, f)}{G_0(t, f)} \right]_{f=0} = \text{const} - 2\gamma t,$$

(8)

where $f$ was set equal 0 because the effect of attenuation should be negligible. Within the data scatter, the same time dependencies of reference-GS corrected amplitudes can also be measured as:

$$\ln[E(t)t^2] = \text{const} - 2\delta \nu \ln t,$$

(9)

where $\delta \nu$ is the under-compensated GS exponent. The resulting values of $\nu = 1 + \delta \nu$ are shown in the middle-row plots in Figures 2 - 4. Note that the dependence of type (9) can only be fit beyond ~5-km offsets, apparently where the reflections from the free surface become significant (middle-row plots in Figures 2 - 4). This shows that the $t^\nu$ law represents a poorer approximation for the actual GS whereas the dependence (8) covers the modeled GS well from 0 to ~50 – 100-km distances (Figures 2 - 4). This could be mostly due to the scattering-type, $t^\nu e^{-\gamma t}$ GS law using a linear scale in $t$ without exaggerating the effect of the immediate vicinity of the source.

**Discussion**

In different types of attenuation measurements (e.g., body-wave, $Lg$, coda, or total-energy), different seismic phases and time windows are selected for measuring the amplitudes leading to eq. (5). The peak and whole-record amplitude measures shown in Figures 2 - 4 are likely most related to $P$-wave, coda, and total-energy studies. However, my goal here was not to model any specific case with sufficient accuracy (this is hardly possible!) but to demonstrate that the GS: 1) is measurable and can be modeled to some extent.
Variability of geometrical spreading

extent, 2) does not fit into any simple model like $r^{-\delta}$, 3) is variable and depends on the lithospheric structure, and in particular, 4) because of the upper-crustal reflectivity, it is typically faster than expected from isotropic, uniform-space models. These conclusions are invariant in respect to the choices of measurement time windows. In addition, note that association of a specific window (for example, 3.5-km/s window commonly selected for $Lg$) is another conventional simplification that should not replace careful identification and separation of seismic phases in the records.

For most crustal models without strong mid-upper crustal reflectivity (such as in Figures 2 and 4), the numerical modeling showed values of $\gamma > 0$. Such structures are likely to be most abundant around the world, and therefore the GS should most often be under-compensated by the body-wave correction $R^1$. In the traditional $Q(f)$ measurements based on formulas (1) and (Error! Reference source not found.), positive $\gamma$ values imply that $\eta > 0$. As I showed before (Morozov, 2008), $\eta$ approximately equals $\pi f / f_{obs} Q_e$, where $f_{obs}$ is the central observation frequency. This relation could explain widespread observations of positive and high $\eta$'s, especially at lower frequencies and in lower-$Q$ settings.

The predicted values of $\gamma$ range from -0.005 to ~0.01 s$^{-1}$ (Figures 2 - 4), and similar values were also derived from numerous observations (Morozov, 2008). It is also important to ascertain whether such GS uncertainties could be significant in attenuation measurements. A reference level of $\gamma_Q$ can be defined as such whose effect equals that of the effective attenuation, which gives $\gamma_Q = \pi f_{obs} Q_e$. With typical $Q_e \approx 1000$ and $f_{obs} \approx 1$ Hz, $\gamma_Q \approx 0.003$ s$^{-1}$, which is below most of the modelled and observed $\gamma$ values (Figures 2 - 4; Morozov, 2008, and unpublished) Therefore, in most cases, errors due to the uncompensated GS should affect the measurements of $Q$. As a simple rule, the effects of $\gamma$ are significant whenever non-zero $\eta$ values are observed (Morozov, 2008).

Interestingly, the level of $\gamma \approx 0.008$ s$^{-1}$ derived for model Quartz-4 (Figure 4b) is close to the measurements from real $Lg$ coda data in this PNE and also to independent numerical coda simulations, from which $\gamma \approx 0.0075$ s$^{-1}$ (Morozov et al., 2008). This suggests that scattered body waves at 0 – 100 km distances make key contributions to the
coda. Also, from a worldwide compilation of $S$- and $Lg$-wave results, the same level of $\gamma_D \approx 0.008 \text{s}^{-1}$ represents the threshold separating tectonically active ($\gamma > \gamma_D$) and stable ($\gamma < \gamma_D$) regions (Morozov, 2008).

Upper-crustal reflectors above the hypocenter deflect the upcoming waves and cause steeper GS (Figures 2b, c). The modeled GS increases when sedimentary layers are present above the hypocenter, and particularly layers with lower $Q$ (compare Figure 3 to 2). This suggests a potential explanation to the observed trend for $\gamma$ decreasing with the tectonic age (Morozov, 2008). Tectonically young structures are more likely to have strong velocity and attenuation contrasts within the upper crust and above the seismogenic zone (Christensen and Mooney, 1995). Due to cooling, metamorphism, and dehydration, such contrasts are likely to erode with age, leading to increasing $\gamma$ (or, alternately, $\nu$). Note that the average crustal velocity gradient does not cause any significant GS effect (Figure 4a), and therefore the upper-crustal structure is the most important for it.

The upper-crustal reflectivity also appears to be the only mechanism for reducing $\gamma$ and particularly for obtaining GS-compensated amplitudes that increase away from the hypocenter ($\gamma < 0$). To produce such an increase, the source should be located above and close to the reflective layers providing additional illumination of the pre-critical distance interval of Moho reflections (Figure 3b).

Notably, in all cases considered above, the total-energy GS is different from that of the amplitude peak. The total-energy curves show much faster amplitude decays to ~200-km distances compared to the peak amplitudes (grey diamonds and black crosses, respectively, in Figures 2 - 4). A significant portion of the energy appears to be converted from the direct waves in the vicinity of the source and propagates as faster-dissipating, likely surface-wave modes. Although this effect could be insignificant in body-wave attenuation studies where only the maximum-amplitude direct arrivals are used, such difference should be important in total-energy and coda studies where the entire wavefield contributes to the measurements. Therefore, in coda attenuation studies, neither of the empirical GS curves in Figures 2 - 4 should be sufficiently accurate (to say...
nothing of the theoretical $R^{-\nu}$ dependencies), and special modeling needs to be conducted (Morozov et al., 2008). However, attenuation measurements based on eq. (6) do not require compensation of GS, which is measured from the data (Morozov, 2008).

The above observations definitely show that the upper-crustal structure and position of the earthquake source within it are the most important factors forming the GS patterns. The GS, which is distinctly different from the theoretically assumed forms, controls the results of most frequency-dependent attenuation measurements. Therefore, it appears that the only two reasonable approaches to the assessment of the frequency-dependent $Q$ are:

1) Establish realistic GS by comprehensive modeling of the lithospheric structures in the study area.

2) Seek $Q$ measurement approaches not relying on GS compensation, such as the spectral ratio methods.

Unfortunately, approach 1) would always be prone to uncertainties related to the lack of knowledge of the structure, source mechanisms, or instrument site effects. The three-dimensional structure along the propagation path [eq. (1)] will hardly be ever known to the degree of detail required for sufficiently accurate modeling independent of $Q$ (i.e., without any knowledge about scattering and attenuation!) and by using the complete wavefield. Although very important for verification of hypotheses and interpretation, modeling could again land us in the quagmire of assumptions when used as the basis for observations.

The empirical approach 2) is by far more reliable and adequate for describing the data. Its strength is also in utilizing the fundamental property of attenuation as an energy dissipation process related to the frequency of wave oscillations. One problem with this approach is that spectral amplitude data typically show large scatters that only rarely allow determination of a frequency-dependent $Q$. For example, no indications of frequency-dependent $Q_e$ were found in several studies revisited by using this technique (Morozov, 2008, and unpublished). Nevertheless, even the best of physical theories
should recognize the limitations of the data, and perhaps frequency-independent $Q_e$ is all we can have at the moment. With ever increasing volumes and quality of seismological data, the frequency dependence of $Q_e$ should undoubtedly become measurable in the future.

**Implications, critique of the $Q(f)$ model, and its defence**

The results of this paper support the critique of the conventional $Q(f)$ models and observations started in Morozov (2008). In about thirty years of its dominance, the $Q(f)$ paradigm has deeply rooted in seismological practice, and questioning its observational validity could lead to several major implications. For example, switching from the $(Q_0, \eta)$ to $(\gamma, Q_e)$ model could give a totally different view at the absorption band model (Anderson and Given, 1982) and remove the “10 Hz transition problem” (Abercrombie, 1998; Morozov, 2008). Because of the observed frequency-independent $Q_e$, relaxation-mechanism and scattering-based theories (e.g., Liu et al., 1976; Frankel and Clayton, 1986; Fehler et al., 1988) may become much less needed. Suggestions of the power spectra of velocity fluctuations in the lithosphere decaying as $\kappa^{-m}$ (with $\kappa$ being the wavenumber and $m$ lying between 1 and 2; Aki, 1980) would become unsupported. Numerous datasets may need to be revisited and the reported $Q(f)$ values revised, whereas in many cases, the $(\gamma, Q_e)$ values can be estimated from the published $(Q_0, \eta)$ data (Morozov, 2008 and unpublished). In most cases, $Q$ values would be upward-corrected (from $Q_0$ to $Q_e$) by 2 – 20 times (Morozov, 2008). At the same time, the interpretation of frequency-independent $Q_e$ may become significantly simplified, and reliable inversions for spatially-variable $Q_e$ should become possible.

In Morozov (2008), I proposed to shift the methodological emphasis in attenuation measurements from model-based $Q(f)$ curve fitting to examining the actual data constraints. Although plotting the data first might sound trivial for an observational discipline, this is not what is commonly done today! To illustrate the role of models in supporting the $Q(f)$ concept in eqs. (1) and (Error! Reference source not found.), consider the following summary of an inversion for coda attenuation by Padhy et al. (2007), whom I highly respect for the use of advanced physics and detailed presentations:
“… we present a new method to estimate the effects of source, site, and path from inversion of coda envelopes assuming multiple isotropic scattering of acoustic waves in 3D infinite space with isotropic source radiation based on RTT [radiative transfer theory – I.M.], where we take into account the full seismogram envelope.” I cite this sentence for its quintessential clarity and conciseness representing the spirit of most papers on $Q(f)$ published today. It gives a long chain of idealized model assumptions: isotropy, uniform infinite space, only acoustic waves ($P$ or $S$, but no conversions), random (uniform) scattering as the cause of the coda, and the use of a specific method (RTT). None of these assumptions are sufficiently accurate for the given dataset, and therefore the resulting $Q(f)$ estimates remain mostly related to these assumed models and coda envelopes, but not to the Earth.

Four typical arguments are often advanced against the model in eq. (5) and in support of the scattering hypothesis and the $Q(f)$ in eq. (1). The first argument is that such models “work,” and so other models are not needed, or maybe eq. (5) is just another such model. However, their performance is typically judged by the ability to fit the data [i.e., the $P$ values in eq. (1)], which is an insufficient criterion for this type of problem. The $Q(f)$ inversion is performed for a complex set of parameters [$Q_0$, $\eta$, and sometimes source and site; Padhy et al. (2007)] but by using only limited observations (such as envelope slopes). Such inversions are known as under-constrained, and it is also known that their principal property is the ability to fit the data but not so much to determine the model parameters. The ability to fit the attenuation-coefficient data still does not prove the scattering hypothesis, despite the suggestions by Aki (1980) repeated by many others. It simply does not disprove it. The data can be fit in other (and better) ways, one of which was given by Morozov (2008). The actual determination of the $Q(f)$ model comes from “a priori constraints,” which in this case are the assumed geometrical spreading, isotropy, a simple form of the wavefield, etc. The uniqueness of the ($\gamma$, $Q_e$) model [eq. (5)] is in its unambiguously reflecting the properties of the Earth and independence on assumptions.

Second, the assumption $G(t) = t^\nu$ is sometimes called “practical” as the one needed to make the estimates of $Q(f)$ possible. An inaccurate choice of the constant $\nu$ or of the entire $G(t)$ dependence may lead to biased $Q$ estimates, but this model still remains
theoretically valid. However, such biased $Q(f)$ estimates present the structure-related GS as frequency-dependent attenuation and thereby lead to errors in interpretation. In this paper, I am only concerned with the accuracy and meaningfulness of the measured $Q$ and $\gamma$, and not in proving the frequency dependence of $Q$ or validity of theoretical models. Morozov (2008) showed that an unambiguous measurement of $Q$ and $\gamma$ is quite practical without underlying assumptions, and leads to strongly different results.

Another indirect argument in favour of the uniform-space GS + $Q(f)$ tandem is that despite their apparent character, parameters $Q_0$ and $\eta$ correlate reasonably well with tectonics, and therefore they should be meaningful (Aki, 1980). However, how does this correlation work? The transition from active to stable tectonics occurs by simultaneously increasing $Q_0$ and decreasing $\eta$, which happens to be the direction constrained by the frequency-independent part in $P(f, t)$. This is actually the direction of $\gamma$ decreasing with tectonic age, as explained in detail in Morozov (2008). Thus, once again, neither the existence nor need for $Q(f)$ are ascertained from this argument.

Finally, in applied studies such as nuclear-test monitoring, $Q(f)$ is often viewed more as a wave amplitude parameter (e.g., coda shape) than a property related to a well-defined physical process, and the ambiguity of the assumed GS is considered inconsequential (W. S. Phillips, pers. communication). In the citation from Padhy et al. (2007) above, the authors were also careful to state that they developed yet another model mostly in order to explain the observed coda envelopes. Nevertheless, in such studies, the final interpretation is still carried out by correlating $Q_0$ and $\eta$ to regions and geological structures. However, as $Q_0$ and $\eta$ trade-off with the assumed GS, they cannot be uniquely compared to geology. By contrast, the results of Morozov (2008), Morozov et al. (2008) and of this paper show that stable and apparently improved interpretations should result from using the $\gamma$ and $Q_e$ parameters, which can be measured from the data and predicted by numerical simulations.

Conclusions

Numerical modeling in several lithospheric structures shows that geometrical spreading (GS) should be complex and regionally variable, and thus impossible to
represent by using any simple theoretical dependence on the distance. After the traditional GS-compensation, residual variations in GS still remain and may lead to spurious frequency dependence of the attenuation quality factor $Q$.

Because of its irresolvable trade-off with the GS, the elastic attenuation factor $Q_s$ is interpreted as a theoretical artefact, and because of this, the use of $Q_s$ should be avoided. Instead of this parameter, I suggest using $\gamma$ in the semi-empirical GS law in the following form: $G(t) = t^\gamma e^{-\gamma t}$. This parameter directly describes the variations of GS within the distance range below the critical Moho reflection onsets (0 – 100 km). From the scattering-theory point of view, $\gamma$ also represents the zero-frequency limit of the generalized scattering coefficient: $\gamma = \alpha(f=0)$.

In most synthetic models examined, positive values of $\gamma$ were found, indicating stronger GS compared to the commonly used theoretical predictions. This could explain the positive frequency dependencies ($\eta > 0$) of the apparent $Q(f)$ which are observed in many areas and types of attenuation studies.

The upper-crustal structure and the position of the hypocenter within it determine the character of GS. Reflectors above the earthquake hypocenter tend to increase $\gamma$, and reflectors immediately below the source – to decrease it. However, with sufficiently strong reflectivity below the source, $\gamma$ may become negative and lead to geometrically-compensated amplitude peaks at 30 – 70-km hypocentral distances, which are often also attributed to $Q_s$. Such sensitivity of GS to the upper-crustal structure may explain the previously inferred decrease of $\gamma$ with tectonic age (Morozov, 2008).

Overall, frequency-dependent apparent $Q$ and other effects often attributed to $Q_s$ can be consistently explained by variable GS in realistic lithospheric structures. This conclusion likely relates not only to body-wave, but to their derivatives ($L_g$, coda, and total-energy), and also to surface waves.

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Figure 1. Vertical-component synthetic record section modeled in IASP91 lithospheric structure. Travel-time reduction velocity of 9 km/s was used for plotting. Note the complex wavefield.
Figure 2. Results of numerical simulations in: a) IASP91 model, b) IASP91 with low-velocity sedimentary layer, c) the same as b) but with \( Q_P = 20 \) and \( Q_S = 10 \) within the sedimentary layer. Top row: the \( V_P \) and \( V_S \) velocity models; Middle row: geometrical spreading within near-offset ranges, in logarithmic distance scale; Bottom row: the complete distance range in linear scale. Grey diamonds show the total recorded energy and black crosses – peak energy in two (radial and vertical) components combined. Both amplitudes are geometrically-compensated by using the theoretical \((\text{range})^2\) factor. Dashed lines labelled with \( \nu \) values indicate the approximations of geometrical spreading using the \( t^\nu \) law at near offsets, and lines with labels \( \gamma \) show the same ranges approximated by \( e^{-\gamma t} \) dependencies.
Figure 3. Modeling results with a reflectivity zone below the source at 7 km: a) reflectivity starting at 10-km depth, b) starting at 8-km depth. Geometrical compensation, lines, and labels are as in Figure 2. Note the decreased levels of geometrical spreading at near offsets ($\nu$ and $\gamma$) dropping below the theoretical $\nu = 1$ and $\gamma = 0$ in case b).
Figure 4. Modeling results for: a) a hypothetical gradient model of the crust; b) detailed, realistic structure from inversion PNE Quartz-4 in Russia (Morozova et al., 1999). Geometrical compensation of the amplitudes, lines and labels as in Figure 2.

3. Applications of the new attenuation model

3.1. Upper-mantle Love wave attenuation without $Q$ and visco-elasticity

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Abstract

The attenuation quality parameter ($Q$) is a phenomenological quantity depending on the types of observations in which it is measured and underlying theoretical models. Its meaning needs to be carefully understood during interpretation. In many existing models, $Q$ factors are defined as properties of the “time-dependent visco-elastic moduli,” and inversions are based on analytical extrapolations of the elastic parameters into the corresponding complex planes. Although these theories are rigorous and mathematically self-contained, the resulting $Q^{-1}$ values may lead to inaccurate or even unreasonable solutions when interpreted intuitively, as measures of energy dissipation. Overall, the attenuation coefficient $\alpha$ represents a significantly more robust representation of the in situ attenuation than $Q$. As an example, I show that the well-known visco-elastic solution for the long-period Love-wave $Q_L$ violates the conservation of energy and overestimates the attenuation. A new derivation is given, based on an explicit interpretation of the attenuation coefficient. The mantle $Q^{-1}$ sensitivity kernel is different from the velocity sensitivity kernel, which may have significant implications for 1D and 3D inversions for the Earth’s attenuation structure. The new $Q_L$ in the Gutenberg’s continental structure is 10-20% higher than before but shows a similar frequency dependence. The modeled Love-wave $\alpha(f)$ predicts two distinct and near-linear trends within the bands corresponding to the upper-mantle and near-crustal surface-wave modes. Unlike $Q$, the attenuation coefficient is directly and unambiguously measurable, and such piecewise-linear behavior of $\alpha(f)$ is indeed observed in a compilation of global broad-band S-wave data. Two attenuative layers (the outer core and upper mantle) and the low-$\alpha$ crust appear to shape the apparent absorption band of the Earth.

Introduction

Seismic attenuation is typically observed by indirect methods, such as using temporal amplitude decays of standing waves or spatial decays of amplitudes in propagating waves at fixed frequencies. Once the frequency dependence of the quality factor $Q(f)$ and the differences between elastic and anelastic attenuation have become recognized, many attenuation inversion methods were proposed to address these
properties. Unfortunately, with typical data paucity, differences between wave types and datasets, and the increasing complexity of inversion techniques, attenuation models have become prone of uncertainties. It also appears that the basis of the $Q$ concept and the character of its measurement in seismological studies have still received insufficient attention (Morozov, 2009a, b).

Although this may seem surprising with the routine use of $Q$ in today’s seismological studies, the very existence of a quality factor describing the energy dissipation within the Earth is neither unequivocally obvious nor necessary. For propagating waves, the parameter that truly exists and is directly measurable is the spatial attenuation coefficient (e.g., Chernov, 1960)

$$\alpha(f) = -\frac{1}{2} \frac{d \ln \left[ \frac{E(r,f)}{G(r,f)} \right]}{dr},$$

where $E$ is the wave energy density, $r$ is the travel distance, $f$ is the frequency, and $G(r,f)$ is the purely geometrical amplitude spreading that is assumed to be removed from $\alpha$. In general, this attenuation coefficient is frequency-dependent, and by denoting its limit at $f = 0$ by $\gamma^*$, one can write (similarly to Morozov, 2008):

$$\alpha(f) = \gamma^* + \frac{\pi}{Q_e^*(f)} f.$$  

Thus defined quantity $Q_e^*$ would be analogous to the quality factor of a mechanical or electrical oscillator. It is a measure of the frequency-dependent energy dissipation from the field per unit volume. However, the conventional $Q$ is defined differently, and is based on the relative amplitude drop per one wavelength $\lambda$ (Aki and Richards, 2002):

$$Q(f) = \frac{\pi}{\alpha(f) \lambda} = \frac{\pi f}{\alpha(f) V},$$

where $V$ is the incident wave velocity. The important difference of eq. (3) from (1) is that a characteristic of the incident wave (wavelength $\lambda$) is embedded in it. This makes $Q$ not entirely medium-controlled, and leads to $Q \propto \lambda^{-1}$ for long waves (Figure 1).

Spatial and temporal attenuation factors $Q^{-1}$ are usually derived for plane or
spherical waves (e.g., Sato and Fehler, 1998; Aki and Richards, 2002). In such cases, spatial uniformity makes assigning the resulting $Q^{-1}$ to the medium trivial, and $\alpha(f)$ is nearly proportional to $f$, compensating the artificial factor $\lambda^{-1}$ in eq. (3). However, clearly, this only holds for such simple wave types. Quantities (3) and (1) would also have been equally applicable to the medium if there were reasons to assume that $\alpha \propto f^{-1}$; however, there are no such reasons. Thus, unlike $\alpha(f)$, the quality factor (3) is more of a phenomenological attribute of the wave, and not of the medium itself.

Although the transformation of $\alpha(f)$ into $Q(f)$ (eq. 3) may appear natural for wave-like processes, its assumptions are neither trivial nor entirely innocent. The very use of $Q$ instead of the more general attenuation coefficient eliminates the possibility of variations of geometrical spreading $\gamma_* \neq 0$ and leads to spurious frequency-dependent $Q$ in such cases. By contrast to its equivalent in mechanics, this $Q$-based paradigm may often complicate the interpretation (Morozov, 2008, 2009a,b).

The above observation is just one illustration of the uncertainty of the notion of the “medium $Q$.” In interpreting the $Q$ values arising from various forward and inverse Earth models, it is important to keep in mind what type of quantity is being measured. Bourbié et al. (1987, Chapter 3) summarized a number of such measurements and noted that although most of them can be successfully described by the corresponding visco-elastic models, there is little agreement between the resulting values of $Q$.

As most researchers would likely agree, the general purpose of using $Q$ is for describing the rate of energy dissipation per unit volume as some important property of the medium. This is how the interpreters usually understand $Q^{-1}$ within the Earth, associating its increased values with younger tectonic ages, elevated temperatures, presence of fluids, or scattering. However, for theorists, the in situ $Q^{-1}$ is often a different quantity. Because no single parameter describing the ability of the medium to dissipate the elastic energy exists, various phenomenological proxies were proposed (Bourbié et al., 1987). In particular, in formal visco-elastodynamics, a $Q^{-1}$ is indeed defined as a local property of the medium, equal to the argument of its complex elastic modulus in the frequency domain (Anderson and Archambeau, 1964). This $Q^{-1}$ leads to refined models of wave propagation and energy dissipation (e.g., Carcione, 2007; Borcherdt, 2009);
however, this still does not mean that complex elastic moduli are indeed present within
the Earth, and that such $Q^{-1}$ is related to any geologically-meaningful properties.

In this paper, I take the first, “interpretive” view on $Q$ above, and discuss
modeling of the frequency-dependent surface-wave $Q$ by using examples from the
original surface-wave study by Anderson et al. (1965). These old results are selected for
their simplicity and also because they formed the methodological basis for numerous
subsequent inversions. The analytical and numerical predictions of the Rayleigh and
Love-wave $Q$ by Anderson et al., (1965) are hitherto unquestioned and still broadly used
(Aki and Richards, 2002). The clarity of the 1D problem allows us to focus on the
underlying paradigms, conjectures, and theoretical assumptions. These assumptions are
imbued in numerous subsequent investigations of the Earth’s 3D attenuation structure
(e.g., Gung and Romanovicz, 2004, which I use only as a representative reference to
many recent studies). However, in 3D tomographic inversion, understanding of the
fundamental principles of attenuation modeling is complicated by many additional
details, and a review of the 1D case would still be instructive. In particular, as I show
below, the visco-elastic model (albeit mixed with several other related postulates) leads
to incorrect predictions of the surface-wave $Q$. Because of the forward modeling failing
in a 1D case, it is unlikely that inversion, and particularly in 3D, should remain accurate.

The key theoretical observation facilitating both classic 1D (Anderson et al.,
1965) and modern 3D attenuation inversions (Gung and Romanovicz, 2004) is that the
sensitivity kernels $K_q$ relating the in situ surface-wave properties $q$ of to the observed
ones, $q_{obs}$:

$$q_{obs} = \int K_q(r_{obs}, r)q(r) d^3r , \quad (4)$$

are the same for $q$ taken equal to $Q^{-1}$ or the logarithm of wave velocity. This statement is
closely related to the interpretation of $Q^{-1}$ as a negative complex argument of the medium
velocity (Aki and Richards, 2002). These results are most amazing, considering that
many factors control the energy dissipation within the Earth, such as fracturing, fluid
content and saturation, viscosity, porosity, permeability, properties of “dry” friction, and
distributions of scatterers (Bourbié et al., 1987). Most of these factors are only remotely
related to the velocity. The ability to lump them all together in a cumulative medium $Q^{-1}$ suggests that only some specific wave mode is in fact considered, and its properties are being substituted for the properties of the medium.

Indeed, $Q^{-1}$ in eq. (3) is not the type of quantity to be uniquely attributed to any point in the medium, as parameter $q(r)$ in eq. (4). Values of $Q^{-1}$ are different for different waves (for example, $P$, $S$, and different inhomogeneous waves – see Borcherdt, 2009). This difference is attributed to the different elastic parameters of the medium, such as the bulk and shear moduli (Anderson and Archambeau, 1964). However, let us ask ourselves, what properties of the elastic moduli $\lambda$ and $\mu$ led to their association with attenuation? In answering this question, four fundamental hypotheses can be recognized in the pioneering studies of Earth’s attenuation of the 60’s-70’s:

H1) In the frequency domain, complex-valued elastic moduli can be used to write the anelastic equations of motion. Conceptually, complex moduli arise from the popular interpretation of attenuation as “imperfect elasticity” (Anderson and Archambeau, 1964; Aki and Richards, 2002) presenting the stress ($\sigma$) as a convolutional response to the strain-rate history, $\dot{\varepsilon}(t)$ (Dahlen and Tromp, 1998)

$$\sigma(t) = \int_{-\infty}^{t} M(t-t')\dot{\varepsilon}(t')dt', \tag{5}$$

where $M(t)$ is the generalized time-dependent visco-elastic modulus. This interpretation is also closely related to relaxation spectra (e.g., Liu et al., 1976), equivalent mechanical models, and the correspondence principle (e.g., Bland, 1960; Carcione, 2007). It was supported by extrapolating the results of creep measurements (Lomnitz, 1956) to the seismic frequencies; however, this was also done indirectly, with the use of the relaxation model (5) and equivalent models.

H2) Phase-velocity dependencies on the medium parameters can be analytically extrapolated into the complex plane in order to derive the attenuation properties (e.g., Anderson and Archambeau, 1964). This property directly led to the
variability of the forward velocity and $Q^{-1}$ kernels in eq. (4).

H3) The power-law form of the frequency dependence $Q(f) = Q_0 f^{\eta}$ is often suitable for describing the observed (apparent) and also material attenuation (e.g., Aki and Chouet, 1975; Anderson and Given, 1982).

These hypotheses form the basis of both frequency-dependent attenuation modeling and measurements and are rarely questioned today. Visco-elastic theories based on these assumptions are elegant, self-consistent and comprehensive (Carcione, 2007; Borcherdt, 2009). Nevertheless, they may only describe the reality in a limited number of cases, and the case of surface-wave $Q$ in a layered Earth appears to be not among them.

Note that the correspondence principle unquestionably describes real wave propagation only in a homogenous medium, because only in such medium there exists a parameter directly corresponding to the phase velocity (such as $V_s = \sqrt{2\mu/\rho}$ for $S$ waves, where $\mu$ is the rigidity modulus and $\rho$ is the density). This allows attributing the imaginary shift in the phase velocity to $\mu$, thereby justifying assumption H1 above. However, in a heterogeneous, attenuative medium, the phase velocity does not equal the material velocity at any point, and therefore the reasons for introducing a complex-valued material $\mu$ remain unclear. As shown in the first section of this paper, even assuming that all attenuation factors can be summarized in a single material $Q^{-1}$, they should not be attributed to $\mu$.

In the second section, I discuss the general use of complex-valued elastic medium parameters for describing attenuation. To understand the process of attenuation from first principles, I outline an approach based on Lagrangian wave mechanics. This description is well-known and commonly used in theoretical physics and exploration geophysics (e.g., Bourbié et al., 1987); however, it appears to be under-utilized in global seismology. The general conclusion from this section is that again, the elastic moduli should remain real and attenuation should not be mixed with elastic processes.

In the third section, I briefly review the model of the surface-wave $Q$ in a radially-
symmetric Earth by Anderson et al. (1965) with an emphasis on its key conjectures H1 and H2. To derive the corrected surface-wave solutions, I use the wavenumber-eigenvalue approach (Aki and Richards, 2002) which can also be related to the frequency-eigenvalue, spherical harmonic summation used in whole-Earth studies (Dahlen and Tromp, 1998; Gung and Romanovicz, 2004). The results show that hypotheses H1 and H2 lead to mantle Love-wave $Q_L$ values overestimated by 10-20% and violating the conservation of energy. The more empirical assumption H3 was discussed by Morozov (2008) in relation to several real-data observations.

The last objective of this study is to establish what types of frequency dependences of the surface-wave attenuation coefficient $\alpha(f)$ should be expected from mantle layering. Unlike $Q$, $\alpha$ is a directly measurable quantity, and the functional form of its dependence on the frequency may give important clues to the interpretation (Morozov, 2008). The principal tools of investigation are the conservation of energy, the Hamilton variational principle, and the Rayleigh-Ritz method. For brevity, I only consider the Love-wave ($SH$) case, and Rayleigh waves can be treated in a similar manner. The conclusion is that in a layered Earth, the attenuation coefficient should exhibit piecewise-linear variations with frequency. Such behavior of $\alpha(f)$ can indeed be recognized in the data, and it is shown in a compilation of broad-band $S$-wave data from $\sim 7 \cdot 10^{-4}$ to $\sim 70$ Hz.

**Phenomenological Q factor**

The $Q$ factor is a phenomenological quantity dependent on the type of observation conducted with the deformation-stress field. Quasi-static experiments (creep and relaxation, e.g., Lomnitz, 1956) first lead to the convolutional laws as in eq. (5), and in some cases, such laws could be implemented in rheological models (e.g., Carcione, 2007, sections 2.4-5). However, within the seismic frequency bands, creep and relaxation are still known only implicitly through the interpreted quality factor $Q$. In different experimental environments (wave propagation, harmonic oscillations, or free vibrations), different values of $Q$ arise from the relaxation models, and they cannot be reduced to each other (for an overview, see Chapter 3 in Bourbié et al., 1987). For example, the Biot theory for wave attenuation in saturated rock, or scattering theory (e.g., Chernov, 1960)
clearly lead to the spatial attenuation factor $\alpha$ (eq. 1) which can be converted to $Q$ by using eq. (3). However, this $Q$ is not equivalent to those determined in the experiments with resonant bars (White, 1983) and may not represent a medium property at all. Thus, from the practical standpoint, the generalized visco-elastic model (5) can satisfactorily represent and compare the various field problems, yet its significance in relation to the fundamental medium properties should not be overstated. The models can be viewed as working tools rather than an end in themselves (Bourbié et al., 1987).

Presentations of the dynamics of deformation in the literature can generally be subdivided into two groups. First, in the “axiomatic” approach usually referred to as the linear visco-elasticity (Bland, 1960; Borcherdt, 2009), a rigorous mathematical theory is constructed by starting from the constitutive law (5). This approach is broadly used in theoretical global seismology (Dahlen and Tromp, 1998). This theory does not use the Hamilton variational principle which is commonly used in mechanics but starts from differential equations of motion and boundary conditions and relies on dashpot-spring analogies for support (e.g., Carcione, 2007). Further description is entirely self-consistent and close to that of an elastic problem, from which it differs by using complex-valued elastic constants in the frequency domain. Nevertheless, this difference also results in new types of solutions (such as inhomogeneous and multiple $S$ waves), sometimes possessing peculiar properties (Richards, 1984; Borcherdt, 2009).

In the frequency-domain form of the linear visco-elastic model (eq. 5), the attenuation is described by negative phase shifts of the medium velocities $V_P$ and $V_S$, (e.g., Aki and Richards, 2002), and $Q^{-1}$ is defined as a phase shift between the complex-valued particle velocity and stress. This phase shift is further attributed to the arguments of the complex elastic moduli, whereas the possibility of an imaginary component of density is not considered (Anderson and Archambeau, 1964; Borcherdt, 2009). However, such extrapolation of Boltzmann’s (1874) after-effect theory (describing relaxation phenomena such as creep) to seismology is still phenomenological and seems to disregard some key mechanical principles. Firstly, if the attenuation is caused by friction on grain boundaries and faults, or by viscosity in pore fluids, it should affect the kinetic, and not the elastic-deformation energy. Strain energy does not dissipate into the heat by
means of some “imperfect elasticity”, but only by causing relative movements within the medium. The Lamé parameters $\lambda$ and $\mu$ describe the elastic energy stored in the field, and they should not be responsible for attenuation. Secondly, attenuation is a property of a particular wave and corresponds to an imaginary part of its wavenumber or frequency. The key idea of visco-elastodynamics can be summarized as attributing the attenuation property (namely, $Q^{-1}$) of a plane, uniform-medium $S$-wave to the local shear modulus, and those of the $P$-wave – to the local bulk modulus. This is an elegant mathematical conjecture yet not particularly necessitated by the physics. Its predictions start deviating from reality when considering inhomogeneous media and boundary conditions, which also become “anelastic” in this approach (Borcherdt, 2009). For example, when compared to the derivation based on the traditional differential wave equations, visco-elastodynamics gives opposite signs of the phase shifts for reflections from attenuation contrasts (Lines et al., 2008).

By contrast, the approach that can be called “physical” attempts building a wave-propagation model by using the traditional mechanics to describe the energy dissipation by viscous fluid flows or “dry” friction. Apparently because of its attention to fluids, this approach is more developed in exploration seismology (e.g., Bourbié et al., 1987; Carcione, 2007). The medium is described by using the Lagrangian formulation, in which the dissipation is considered separately from elasticity, and all medium parameters remain real. Notably, when certain solutions to the wave equations are considered (for example, harmonic plane waves in homogeneous media), phenomenaological complex moduli may also arise (Bourbié et al., 1987). Once again, it is important to see that such moduli combining the elastic constants with viscosity only represent properties of the specific solutions and not the physical medium itself.

The second of these approaches is far more preferable for unraveling the true physics of wave propagation, and Lagrange formulation is well-known for its depth, power, and generality. To illustrate this model of elastic energy dissipation, I only summarize its conceptual principles below; however, the analogies to the full wave problem (and also its complexity in real-world applications) are quite apparent. For more complete treatments of visco-elasticity in realistic porous media, see Bourbié et al. (1987)
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Attenuation in Lagrangian elastic-medium mechanics

In Lagrangian form, the dynamics of any mechanical system (such as the elastic field) is described by a single function of some generalized coordinates \( q \) and their time derivatives \( \dot{q} \)

\[
L(q, \dot{q}) = E_{\text{kin}} - E_{\text{el}},
\]

where \( E_{\text{kin}} \) and \( E_{\text{el}} \) are the kinetic and elastic energies. Vector \( q \) consists of any parameters describing the field (e.g., local displacements, Fourier amplitudes, or Rayleigh-Ritz coefficients below), and \( q \) and \( \dot{q} \) are treated as independent variables. The corresponding Euler-Lagrange equations of motion are given by

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \ldots
\]

(7)

In the presence of energy dissipation, these equations are modified by adding the generalized dissipative force \( Q_i^D \) (do not confuse with quality parameter \( Q \)):

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i^D,
\]

(8)

which is defined as a derivative of some “dissipation function” (sometimes called pseudo-potential) \( D \) in respect to the generalized velocity:

\[
Q_i^D = -\frac{\partial D}{\partial q_i}.
\]

(9)

For a simple mechanical analogy, consider a linear oscillator of mass \( m \). If the force of friction is linear in velocity

\[
f_D = -m \omega_0 \ddot{\theta}
\]

(10)

(where \( \omega_0 \) is the natural frequency of the oscillator), then the so-called Rayleigh dissipation function is (e.g., Razavy, 2005)
If the oscillator is driven by an external force with frequency \( \omega \), the energy \( \delta E \) dissipated in one period \( T = 2\pi/\omega \) is

\[
\delta E = \int_{0}^{T} \xi m \omega_0 \dot{r}^2 \, dt = \xi \omega_0 T E_{\text{kin}}. \tag{12}
\]

where \( E_{\text{kin}} \) is the peak kinetic energy in that period. Therefore, the quality parameter \( Q \) can be defined as

\[
Q = \frac{2\pi E_{\text{kin}}}{\delta E} = \frac{\omega}{\omega_0 \xi}. \tag{13}
\]

For oscillations near the natural frequency, \( \omega \approx \omega_0 \), and therefore \( Q \approx \xi^1 \). Note that for the oscillator, \( E_{\text{kin}} \) also equals the peak strain and the total energies; however, these quantities should be differentiated in the case of a distributed surface wave in a layered medium.

Father, with eq. (11), the equation of motion (8) becomes

\[
m \ddot{\mathbf{r}} = -m \omega_0^2 \mathbf{r} - \xi m \omega_0 \dot{\mathbf{r}} \tag{14}
\]

and its general solution to eq. (14) is \( \mathbf{r}(t) = \text{Re}[A \exp(-i \omega'_0 t)] \), where

\[
\omega'_0 \approx \omega_0 \left(1 - \frac{i \xi}{2}\right) \tag{15}
\]

is the “complex frequency” (compare to Dahlen and Tromp, 1998, Chapter 6), and \( A \) is an arbitrary complex-valued amplitude. One can therefore define a complex-valued spring constant (an equivalent to the elastic modulus)

\[
k' = m \omega'_0^2 \approx k \left(1 - i \xi\right) \tag{16}
\]

and eq. (14) takes the form of the free-oscillator equation (11):

\[
m \ddot{\mathbf{r}} = -m \omega'_0^2 \mathbf{r}. \tag{17}
\]

If an external force \( f(t) \) is considered in the r.h.s. of eq. (14), its frequency-domain solution can also be expressed by using the complex frequency alone:
\[ A(\omega) = \frac{f(\omega)}{m(\omega_0^2 - \omega^2)}. \]  

This formula resembles a single-mechanism “relaxation spectrum” (Liu et al., 1976).

Thus, expressions (6-13) give a consistent analog to the elastic-wave attenuation processes. Eqs. (15-18) shows how the complex-valued elastic moduli and relaxation spectra are obtained, but only for the specific solutions. The important difference of the wavefield case is that the energy dissipation \( \alpha \) can be measured per unit volume or travel path (Chernov, 1960; Bourbié et al., 1987), and in order to obtain a \( Q \) value, it has to be rather arbitrary converted to \( Q \) by using eq. (3).

Eq. (8) shows that attenuation is caused by the external friction forces which are not included in the Lagrangian (i.e., in \( \mu \) or \( \lambda \)). Also, eq. (11) shows that the effect of friction can nevertheless be described by an energy-like function \( D \). For example, in a porous medium (Bourbié et al., 1987, p. 69-72), the dissipation function is quadratic in filtration velocity \( w_i \) (velocity of the fluid relative to the rock matrix):

\[ D = \frac{\eta}{2\kappa} \dot{w}_i^* \dot{w}_i, \]  

where \( \eta \) is the fluid viscosity, and \( \kappa \) is the absolute permeability, which depends on the geometry of the pores. Although the equations of motion in real porous media are significantly complicated by their multi-phase nature (such as the solid rock matrix and fluid), eq. (19) shows that in general, the dissipation is dependent on \( \dot{q} \) and is due to the kinetic, and certainly not to the elastic energy.

The above shows that if taken literally, the association of friction with \( \mu \) (which is a measure of shear-strain energy) may be problematic. Instead of \( \mu \), attenuation could be related to an imaginary part of \( \rho \), although this also does not seem to be very necessary or productive. Anderson and Archambeau (1964) also discarded such possibility of \( \text{Im} \rho \neq 0 \), but because of its interpreted association with “imperfect gravity” (which was also imprecise, because \( \rho \) is the inertial mass density and not directly related to gravity). However, note that the density factor nevertheless changes in the presence of energy dissipation! Because of the presence of fluids (or likely of any relative movement of the
different components of the medium), a coupling force arises (Bourbié et al., 1987):

\[ F_{\text{inertial}} = \rho_f (1 - a) \dot{u}, \]

where \( a \geq 1 \) is the tortuosity parameter, and \( \rho_f \) is the pore fluid density. This force is proportional to the acceleration and effectively changes the inertial property of the rock (i.e., its density \( \rho \)) to \( \rho + \rho_f (1-a) \).

In summary of this section, it appears that phenomenological analogies and analytic extrapolations should be carefully scrutinized when used to infer wave attenuation within the Earth. The fundamental equation (8) and realistic multi-phase models based on the physics of dissipative processes should be much more reliable in such descriptions.

**Existing Love-wave Q(f) model in a layered Earth**

The derivation of the surface-wave \( Q(f) \) measured on the surface of a layered-Earth model by Anderson et al. (1965) (also see Aki and Richards, 2002, p. 289-291) was based on two of the heuristic conjectures above: H1) interpretation of the attenuation parameter \( Q^{-1} \) as a phase shift of the complex-valued phase velocity:

\[ \delta \ln c = -\frac{i}{2} \left( \text{spatial } Q^{-1} \right), \]

and H2) assumed analytical dependence of the surface-wave phase velocities \( c \) on the velocities of the individual mantle layers \( (V_{P,i} \text{ and } V_{S,i}) \):

\[ c = f(V_{P,i}, V_{S,i}, \rho_i), \]

where all properties also depended on the frequency \( \omega \). It was also assumed that eq. (21) could be applied to both the apparent \( Q^{-1} \) observed on the surface and to the \textit{in situ} \( Q \) (in combination with the medium velocity replacing \( c \) in eq. (21)). This allowed extrapolating the partial derivatives of phase velocities, such as \( \partial c / \partial V_{P,i} \), into the corresponding complex planes, which allowed calculating the surface-wave attenuation simply by using the phase-velocity derivatives. This approach gave for Love waves [formula (7.88) in Aki and Richards, 2002]:

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Variability of geometrical spreading
Variability of geometrical spreading

\[ \text{spatial } Q^{-1}_z = \int_0^\infty K_v Q_s^{-1} \, dz, \]  

where \( z \) is the depth, \( K_v \) is the S-wave velocity sensitivity kernel

\[ K_v = \frac{\left[ k^2 l_1^2 + \left( \frac{d l_1}{dz} \right)^2 \right] \mu}{k^2 \int_0^\infty \mu l_1^2 \, dz}, \]  

\( k \) is the wave number, \( \mu \) is the Lamé rigidity modulus, \( Q_s \) is the shear-wave attenuation quality factor of the mantle, and \( l_1 \) is the amplitude of the mode of interest. All values in eqs. (23) and (24) are depth-dependent, so that the \( SH \)-wave displacement is given by

\[ u_y(x,z,t) = l_1(z) \psi(x,t), \]  

where \( \psi(x,t) = \exp(-i\omega t + ikx) \).

However, despite its superb analytical elegance, approach (21-22) leads to significant difficulties. Its key problems are 1) in treating \( Q^{-1} \) as a fundamental medium property (such as \( \mu \)) and 2) in mixing the notions of wave velocities as parameters of the propagating medium and phase velocities of the various wave modes in it. Note that by its definition, the spatial \( Q \) corresponds to the argument of the complex wavenumber (e.g., Aki and Richards, 2002, p. 167-169):

\[ k \rightarrow k + i\alpha = k \left( 1 + \frac{i}{2Q} \right), \]  

where \( \alpha \) is the spatial attenuation coefficient. Positive signs of \( \alpha \) for all modes ensure their amplitudes decaying in the directions of propagation. Through its relation to the phase velocity \( c = \omega k \), a positive \( \text{Im} k \) corresponds to negative \( \text{Im} c \). However, in a surface wave, a single value of \( k \) is common to all depth levels, and velocities \( V_P \) and \( V_S \) in formula (24) do not serve as phase velocities for any waves. Therefore, the assumptions about negative imaginary parts in \( V_P \) and \( V_S \) similar to that of the phase velocity (21) represent heuristic extrapolations of the property of the plane-wave solution \( c = \omega k \) away from the point at which it is valid. Similarly, \( Q^{-1} \) describes the complex phase of the wave vector (eq. 26) and is not an unequivocal medium property.
Further, let us consider the hypothesis of analyticity (H2 above). Analyticity (holomorphism) is a very strong condition on a complex function, requiring that the derivatives of its imaginary part are related to those of the real part by the Cauchy-Riemann equations. Once such analyticity is assumed, the many degrees of freedom ($V_S$ and all the attenuation parameters, or at least $\alpha$) collapse to one ($V_S$), and the sensitivity kernels to $Q_S^{-1}$ and $V_S$ attain the same shapes in both 1D (Anderson et al., 1965) and 3D (Gung and Romanovicz, 2004). Inversion for $Q_S^{-1}$ or $\text{Im} \mu$ thus becomes closely related to velocity tomography. However, function (22) represents an integral transform of the layer parameters (see eq. 24), and its analyticity in respect to the functional integrands $V_{P,S}(z)$ (or equivalently, $\lambda(z)$ and $\mu(z)$) is unlikely in the general case. For example, analyticity of the dependence of the phase velocity (eq. 22) on the crustal or sedimentary-layer velocities can hardly be proven, and it is also not something that can be safely assumed for convenience. Thus, we need to avoid treating the phase velocity as a holomorphic function and attenuation – as its imaginary part, and consider them independently.

Finally, considering the properties of the resulting solution, formula (24) is also problematic, because it violates the total energy balance by exaggerating the amount of energy dissipation. Values of $Q_L^{-1}$ are presented as weighted averages of $Q_S^{-1}$ within the layers; however, the weights in the numerator of this ratio are systematically larger than those in the denominator. This flaw of expression (24) can be easily seen on an example of a two-layer model with different velocities but constant in situ $Q_S$. In such a model, assuming that the energy in each layer $l = 1,2$ can be subdivided into non-dissipating ($E_{n,l}$) and dissipating ($E_{d,l}$) parts, $E_{d,l}$ should decrease after time $t$ by the same factor for both layers: $E_{d,l} \propto (1 - \lambda) = \exp(-\omega Q_S^{-1} t)$. Consequently, the total relative energy dissipation should not be faster than $\lambda$:

$$\frac{\partial E}{E} = \frac{\lambda (E_{d,1} + E_{d,2})}{E_{n,1} + E_{d,1} + E_{n,2} + E_{d,2}} \leq \lambda.$$

Therefore, spatial attenuation of $Q_L^{-1} = Q_S^{-1}$ should be expected if all mechanical energy dissipates in this process, and $Q_L^{-1} < Q_S^{-1}$ if only a part of it (such as the kinetic energy argued for above). However, formula (24) predicts $Q_L^{-1} > Q_S^{-1}$ on the surface, showing...
dissipation of more energy than present in the field, which is unrealizable.

**Love-wave Q(f) from energy-balance constraints**

An alternate expression for the effective spatial $Q_L$ can be derived directly from the energy balance considerations. Consider an $SH$ surface-wave field of form (25) in a layered, isotropic, and lossy medium, in which the horizontal wavenumber contains a positive imaginary term according to eq. (26). Its time-averaged kinetic energy density can be given as (Aki and Richards, 2002, section 7.3)

$$\langle E_{kin} \rangle = \left( \frac{1}{2} \rho \dot{u}_i \dot{u}_i^* \right) = \frac{\rho \omega^2}{4} l_i^2,$$  \hspace{1cm} (28)

where complex conjugation (denoted by the asterisk) is used to account for the complex-valued wavefield amplitudes in eq. (25). The corresponding average elastic energy density is

$$\langle E_{el} \rangle = \left( \frac{1}{2} \lambda \epsilon_{kk} \epsilon_{nn}^* + \mu \epsilon_{ij} \epsilon_{ij}^* \right) = \frac{\rho \omega^2}{4} \left[ \mu k^* l_i^2 + \left( \frac{dl_i}{dz} \right)^2 \right],$$ \hspace{1cm} (29)

and the total energy

$$\tilde{E} = \int_0^\infty \langle E_{kin} + E_{el} \rangle dz = \frac{1}{2} \left( \omega^2 I_1 + k k^* I_2 + I_3 \right),$$ \hspace{1cm} (30)

where the energy integrals are

$$I_1 = \frac{\rho u_i^*}{2} \int_0^\infty \rho l_i^2 dz, \quad I_2 = \frac{\rho u_i^*}{2} \int_0^\infty \mu l_i^2 dz, \quad \text{and} \quad I_3 = \frac{\rho u_i^*}{2} \int_0^\infty \mu \left( \frac{dl_i}{dz} \right)^2 dz. \hspace{1cm} (31)$$

The total elastic energy contained in a normal mode equals its kinetic energy (Aki and Richards, 2002, p. 284)

$$\omega^2 I_1 = k k^* I_2 + I_3.$$ \hspace{1cm} (32)

As argued above, $E_{kin}$ can be considered as the source of energy dissipation, and for weak attenuation, its loss is continuously being replenished from the potential energy through eq. (32). For horizontal surface-wave propagation (i.e., because of the common
factor $uu^*$ in eqs. (31)), both $E_{kin}$ and $E_{el}$ at any depth should thus decrease with travel distance $x$ as $\propto \exp(-2\alpha x)$. The total dissipation is a sum of energy losses at each depth:

$$\frac{d\tilde{E}}{dx} = -\int_0^\infty \frac{d\langle E_{kin} \rangle}{dx} dz = \int_0^\infty 2a(z)\langle E_{kin} \rangle dz,$$

(33)

where $\alpha(z)$ is the local spatial $S$-wave attenuation coefficient at depth $z$. In the absence of information about the specific mechanisms of attenuation (e.g., the effects of fluids), we can consider the spatial attenuation coefficient $\alpha$ as the substantive property responsible for $S$-wave energy dissipation. We can also approximate $\alpha(z)$ by considering a plane $S$ wave with frequency $\omega$ and wavenumber $k$ traveling in a medium with propagation velocity $V_S(z)$ and spatial attenuation $\alpha(z)$. Although this wave does not satisfy the equation of motion ($\omega k \neq V_S(z)$, i.e., it represents a vibration forced by the adjoining layers), its energy dissipation over distance $L$ still equals (Figure 1)

$$\frac{\delta E(L)}{E} = -2\alpha(z)L.$$  

(34)

Considering that the wavelength is usually much longer than the size of the dissipating volume $L$, and also that $E_{kin}$ is responsible for dissipation, $\alpha(z)$ should be nearly independent of $k$. However, the $S$-wave quality factor of the medium $Q_S^{-1}$ is usually defined as the dissipation rate per wavelength $\lambda$ of a freely traveling wave at the same frequency, and therefore (e.g., eq. 5.86 in Aki and Richards, 2002):

$$Q_S^{-1}(z) = -\frac{\delta E(\lambda)}{2\pi E} = -\frac{2V_S(z)\alpha(z)}{\omega}.$$  

(35)

Therefore, $\alpha(z)$ can be estimated from $Q^{-1}(z)$ parameters, which are commonly presented in attenuation models (e.g., Gung and Romanovicz, 2004):

$$\alpha(z) = \frac{\omega Q_S^{-1}(z)}{2V_S(z)}.$$  

(36)

Note a similar transformation for the temporal equivalent of $\alpha$ in Morozov (2008).

Finally, from eqs. (33) and (35), the total attenuation coefficient becomes
\[ \alpha = -\frac{d\tilde{E}}{Edx} = \int_0^\infty K_\alpha \alpha dz = 2k \int_0^\infty K_\omega Q_S^{-1} dz, \]  

(37)

where \( K_\alpha \) and \( K_\omega \) are the \( \alpha \)- and \( Q_S \)-sensitivity kernels, respectively:

\[ K_\alpha = \frac{2k}{E} \left\langle E_{\text{kin}} \right\rangle, \]  

and

(38)

\[ K_\omega = \frac{\omega}{2E} \frac{\left\langle E_{\text{kin}} \right\rangle}{V_S}. \]  

(39)

Note that \( K_\omega \) differs from the velocity kernel \( K_\nu \) in eq. (24). Finally, the spatial Love-wave \( Q_L^{-1} \) can be obtained from \( \alpha \):

\[ Q_L^{-1} = \frac{2\alpha}{k} = \int_0^\infty K_\omega Q_S^{-1} dz. \]  

(40)

By its definition, and unlike formula (24), expression (37) automatically preserves the sum of the total propagating and dissipated energies.

Note that the introduction of attenuation \((\alpha > 0)\) also slightly shifts the phase and group velocity spectra. To see this, consider the variational principle for finding the dependence of \( l_1(z) \) on the depth (Aki and Richards, 2002)

\[ \delta \int_0^\infty \left\langle L(u, \dot{u}) \right\rangle dz = \frac{1}{2} \left( \omega^2 \partial l_1 - kk^2 \partial l_2 - \partial l_3 \right) = 0, \]  

(41)

where \( L(u, \dot{u}) \) is the Lagrangian density of the elastic field, and \( l_1, l_2, \text{ and } l_3 \) are the energy integrals defined in eqs. (31). For a fixed \( \omega \), the absolute value of the corresponding wavenumber \(|k|\) is obtained by solving the eigenvalue problem of eqs. (32) and (41). However, integrals (31) only depend on \( \alpha \) via a common factor \( uu^* \), and therefore \(|k|\) is independent of attenuation. Consequently, with non-zero attenuation, the real part of the wavenumber decreases as

\[ \text{Re } k = \sqrt{\left|k\right|^2 - \alpha^2} \approx \left|k\right| \left(1 - \frac{1}{8Q^2}\right), \]  

(42)

which corresponds to a negligibly small phase-velocity \((c = \omega k)\) dispersion due to attenuation (Anderson et al., 1965). From its stationary-phase definition and the
variational principle (eqs. 32, 41), the group velocity remains real and changes accordingly (cf. Aki and Richards, 2002):

\[ U = \frac{\delta \omega}{\delta \text{Re} k} = \frac{\text{Re} k}{\omega} \frac{I_2}{I_1} \]  

(43)

Numerical model of Love-wave Q_L

The Rayleigh-Ritz method provides efficient numerical solutions of the eigenvalue equations (32) and (41) (Wiggins, 1976). By approximating the functional form of \( l_1(z) \) in terms of some basis functions \( \phi_i(z) \)

\[ l_1(z) = \sum_{i=1}^{N} m_i \phi_i(z), \]  

(44)

where coefficients \( m_i \) comprise a discrete model vector \( \mathbf{m} \), integral equations (32) are transformed into a matrix eigenvalue problem

\[ \mathbf{k} \mathbf{k}^* \mathbf{m} = \mathbf{A}_2^{-1} (\omega^2 \mathbf{A}_1 - \mathbf{A}_3) \mathbf{m}, \]  

(45)

where the energy matrices are

\[ A_{1,ij} = \frac{uu^*}{2} \int_0^e \rho \phi_i \phi_j dz, \quad A_{2,ij} = \frac{uu^*}{2} \int_0^e \mu \phi_i \phi_j dz, \quad \text{and} \quad A_{3,ij} = \frac{uu^*}{2} \int_0^e \mu \frac{d\phi_i}{dz} \frac{d\phi_j}{dz} dz. \]  

(46)

Earth-flattening corrections (Aki and Richards, 2002) can be incorporated in integrals (46) in order to account for the Earth’s sphericity. By solving this eigenvalue problem, all possible values of \(|k|\) and the corresponding eigenfunctions (44) are obtained, from which the attenuation spectra (37) can be calculated.

As an example, in the Gutenberg continental Earth model (Table 1), 45 cubic polynomial basis functions give a convenient decomposition for \( l_1(z) \) (Figure 2; Wiggins, 1976). With the shear-wave \( Q \) values from the attenuation model MM8 (Anderson et al., 1965; Table 1), formula (37) yields the apparent frequency-dependent Love-wave \( Q \) (Figure 3). Note that these \( Q \) values are consistently higher than those from the presently used formula (24) (gray dashed line in Figure 3; Anderson et al., 1965; Aki and Richards, 2002). This difference is significant and dependent on the underlying velocity and \( Q \)
distributions within the upper mantle. Therefore, the discrepancy in expression (24) should affect the 1D inversion for mantle $Q$ values, and 3D inversions based on the equivalence of the $Q^{-1}$ and velocity sensitivity kernels should also be prone to similar biases.

**Functional forms of Love-wave attenuation coefficient**

In most attenuation observations, the directly observed measure of energy dissipation is the temporal or spatial attenuation coefficient, such as $\alpha$ in eq. (37). This quantity linearly depends on $f$ in many observations (Morozov, 2008). Unfortunately, $\alpha$ is typically converted to $Q$ in data presentations, which leads to strong apparent $Q(f)$ frequency dependences. For comparison of the various models, it is therefore convenient to write all attenuation results in the form of the temporal attenuation coefficient $\chi(f) = \pi f Q^{-1} = \alpha(f)c$ (this quantity was denoted $\alpha(f)$ in Morozov, 2008).

In the linear $(f, \chi)$ coordinates of Figures 4b and c, one can see that the wave modes below $\sim 0.025$ Hz and above $\sim 0.03$ Hz show distinct trends characterized by sharply different intercepts $\gamma$ (called “geometrical attenuation” in Morozov, 2008) and the “effective attenuation” $Q_e$ values:

$$\chi(f) = \gamma + \frac{\pi f}{Q_e}. \quad (47)$$

Unlike $Q_e^*$ in eq. (2), this $Q_e$ is dimensionless and has the meaning normally attributed to $Q$. For mantle Love-wave modes ($f < 0.025$ Hz), $\gamma \approx -1.5 \cdot 10^{-5}$ s$^{-1}$ and $Q_e \approx 130$, and for the transition into the crust ($f > 0.03$ Hz), $\gamma \approx 7.2 \cdot 10^{-4}$ s$^{-1}$ and $Q_e \approx -900$. Note that the first of these $Q_e$ values corresponds to the average $Q$ of the sub-crustal mantle layers in the model (Table 1), whereas the negative second $Q_e$ is related to the attenuation decreasing with frequency as the wavefield gets progressively concentrated within the high-$Q$ crustal layers. In both cases, $Q_e$ shows that attenuation mostly occurs within the uppermost mantle.

As Morozov (2008) suggested, the “cross-over” frequency $f_c = |\gamma Q_e|/\pi$ is an important property of the lithospheric structure. For frequencies below $f_c$, the geometrical
effects (i.e. the changes of wave modes with frequency) on attenuation become dominant. In our model (Figure 4a), $f_c \approx 6 \cdot 10^{-4}$ Hz for the “mantle” part of the spectrum, and $f_c \approx 0.2$ Hz for the “near-crustal” part. Therefore, sharply variable $Q(f)$ below ~50-sec periods (Figure 3) should not be attributed to a rheological absorption band. This variation of the apparent $Q(f)$ is “geometrical,” i.e. related to the variations of the depth penetration of the surface wave (i.e., by using the electromagnetic analogy, to the variations of the of the “skin layer” thickness).

**Discussion**

Although derived for an old and relatively simple 1D case, this study can have significant implications for modern, 3D attenuation investigations. First, it shows that the classic 1D case is still not closed. The 10-20% overestimation of $Q^{-1}$ in the benchmark 1D solution suggest that similar errors may be present in the sophisticated 3D modeling schemes which inherit the same conceptual background. The noted violation of the energy balance originates from the fundamental assumptions of the method that and is also likely to affect the more recent studies.

Second, if the attenuation is no longer viewed as the complex-plane rotation (as in eq. 21) of the medium velocity, it may require special analysis similar to the one presented here. The $Q^{-1}$ sensitivity kernel becomes different from velocity sensitivity (Figure 5), showing that inversions for attenuation structure should no longer be similar to velocity tomography. For example, note that the observed 60-sec Love-wave $Q_L^{-1}$ is most sensitive to the near-surface, where particle velocities (and therefore friction and pore/fault fluid flows) are the fastest (solid black line in Figure 5). This is different from the phase velocity, which is most sensitive to the depth near the base of the crust, where the elastic strain is the strongest (dashed black line in Figure 5). Although this difference appears natural, it is contrary to the traditional assumption of $Q^{-1}$ responding to the same structures as the velocity (see eq. (24) and also Aki and Richards, 2002; Gung and Romanovicz, 2004).

Third, if the attenuation is treated separately and medium parameters are viewed as real-valued, the complex-moduli visco-elastodynamics becomes significantly limited
in its scope. Surface-wave problems of the type discussed here become solved differently. Another example is the problem of the anelastic acoustic impedance, whose complex phase is positive (Morozov, 2009c) and opposite to that predicted by using the complex medium velocity (eq. 21) in the standard formula \( Z = \rho V \). Thus, it appears that significant deviations from the “axiomatic” visco-elastodynamics are found in heterogeneous media. Therefore models, inversions, and interpretations based on formal visco-elasticity may need to be revised when applied to real-Earth problems.

Unfortunately, the distinction between the traditional model based on hypotheses H1–H3) and the one presented here lies in the realm of physical methodology which rarely receive adequate attention. The difference is in whether we explain the attenuation by time-retarded elasticity (5) in some type of “effective medium” or look for the actual mechanisms of energy dissipation. If only considering simple wave processes, equivalent models and relaxation spectra can always be derived from, for example, pore fluid properties (Bourbié et al., 1987; Carcione, 2007). However, the number of such tractable problems is very limited and basically amounts to homogenous-media cases.

Elastodynamics should not be formulated to only describe some particular form of solutions. The Lagrangian formulation appears to be the most general and reliable, and it allows solving a variety of physical problems. The solution for Love-wave \( Q \) in eq. (38) is also not a final and complete solution but more of an ad hoc approximation based on the requirement of energy balance and assumption about local similarity to S-wave dissipation at any depth level. A complete solution would require modeling of crust/mantle fluid properties, porosity, permeability, and fracturing, which is not available at present. However, it is clear that all these properties cannot be reduced to a single velocity-like behavior.

As this study shows, the \textit{in situ} \( Q(f) \) likely does not exist in most useful cases, but the attenuation coefficient \( \alpha(f) \) represents a useful and measurable physical quantity. Analysis of the typical \( \alpha(f) \) dependencies may provide useful guidance in correlating and cleaning up attenuation data prior to inversion for \textit{in situ} \( \alpha(f) \). For example, in controlled-source seismology, interpretation of the mutually-correlated “travel-time branches” is performed prior to any inversion, providing critical insights in the structure of the
wavefield and model. To accomplish such interpretation, a physically viable “empirical” model of the data is needed (such as the concept of “head-wave” travel times). From the above analysis, it appears that in Love-wave $\alpha(f)$ inversion, the linear parameterization (47) is significantly closer to the character of the forward and inverse problems than the commonly used $Q(f)$ power law. Note that Dainty (1981) and Padhy (2005) also utilized a similar parameterization to describe the attenuation of high-frequency scattered $Lg$ and $S$ waves. Numerical seismic coda modeling (Morozov et al., 2008) also illustrated the linear behavior of coda $\chi$ with frequency.

The choice of data parameterization is consequential in inversion and interpretation. Not surprisingly, the attenuation model (47) works for a broad variety of attenuation data, sometimes causing dramatic changes in interpretation compared to $Q(f)$ (Morozov, 2008). For example, Figure 1a shows broad-band $S$-wave $Q^{-1}$ data from Anderson and Given (1982) and Abercrombie (1998) in $1000Q^{-1}(f)$ and $[-\chi(f)]$ forms (insets). Note that in place of the band-like apparent $Q^{-1}(f)$, $\chi(f)$ reveals three segments of linear trends at frequencies corresponding to predominantly normal-mode, surface-wave, and crustal body-wave frequencies. This is a purely an observation without any theory implied, but recognition of such linear $\chi(f)$ bands may be very significant for understanding the structure of the Earth. These three bands have distinct $\gamma$ and $Q_e$ values labeled in Figure 1b, where another useful way for presenting the same data by subtracting the measured geometrical backgrounds is also shown:

$$Q_e^{-1} = \frac{\chi(f) - \gamma}{\alpha f} = Q^{-1}(f) - \frac{\gamma}{\alpha f}. \quad (48)$$

In this form, the ranges of constant $Q_e$ values can be identified as horizontal segments, so that the potential frequency dependence of $Q_e$ can also be assessed. As one can see, $Q_e$ drops sharply near $\sim 0.008$ Hz and strongly increases across the broad data gap from $\sim 0.04$ Hz to $\sim 1$ Hz (Figure 1b).

All three of the observed $Q_e$ values are positive and should be related (yet not equal) to the averaged in situ $Q$ of the layers containing the wave modes dominant within each of these bands. The two lower-$Q_e$ bands could correspond to the effects of the outer
core and uppermost mantle, and the high-$Q$ band – to the crust. Notably, the values of $\gamma$ are small and negative for the surface waves and higher and positive for crustal body waves. These crustal $\gamma$ values also correlate with tectonic types and crustal structures, as discussed by Morozov (2008).

**Conclusions**

Interpretations of the Earth’s attenuation models may be ambiguous because of the differences between the intuitive understanding of $Q$ (as a measure of local energy dissipation) and its formal definition (as the complex argument of the effective modulus) accepted in visco-elastodynamics. In addition, modeling of the $Q$ structure within the Earth and its inversion are based on strong theoretical assumptions such as analyticity and the similarities of the attenuation and velocity sensitivity kernels, which may also be inaccurate or incorrect.

As an example, the expression for Love-wave $Q_L$ observed on the surface of a layered mantle model (Anderson et al., 1965) was found to violate the conservation of total energy. This was attributed to the interpretation of attenuation as imaginary part of the elastic moduli. This interpretation is discussed in detail and found inadequate in problems involving heterogeneous media.

The Love-wave $Q_L$ solution was corrected based on the explicit energy-conservation principle, resulting $Q$ in the combined Gutenberg and MM8 models was 10-20% higher than in the previous model but showing similar apparent frequency dependence. The attenuation sensitivity kernels were found to be different from the corresponding velocity kernels. These results may have significant implications for 1D as well as for recent 3D inversions for the Earth’s attenuation structure.

Love-wave attenuation coefficient $\alpha(f)$ exhibits distinct, linear variations with frequency within two frequency bands corresponding to the upper-mantle and near-crustal wave modes. Contributions from all mantle layers in the resulting $\alpha(f)$ show near-linear variations with frequencies, suggesting that at least in surface-wave studies, quasi-linear trends in $\alpha(f)$ should be expected and could provide important clues to interpreting mantle attenuation. As an example, three such trends are recognized in a broad-band
Variability of geometrical spreading

References


Tables

Table 1. Gutenberg’s layered continental structure model (Aki and Richards, 2002) with $Q_S$ values from model MM8 (Anderson et al., 1965)

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Variability of geometrical spreading
Variability of geometrical spreading

Figures

Figure 1. Energy dissipation problem (intrinsic or scattering) in eqs. (1-3) and (34). For typical sizes of dissipating volumes $L \ll \lambda$, the attenuation coefficient $\alpha = \delta E/E$ should depend on the frequency but be independent of $\lambda$. 
Figure 2. Basis functions used for modeling Love waves in Gutenberg Earth model (Table 1): a) functions normalized by $d\phi/dz = 1$ at layer boundaries; 2) functions normalized by $\phi = 1$ at the boundaries.
Figure 3. Love-wave $Q_L$ calculated in the combined Gutenberg/MM8 model by using expressions (37) (labeled “This study”) and (24) (labeled “ABA,” after Anderson et al., 1965).

Figure 4. Attenuation coefficient $1000 \chi$ in Gutenberg/MM8 model: a) plotted in log-log form, b) in linear scales, c) derivative $1000 \partial \chi / \partial f$ emphasizing the two $Q_e$ levels. Thin lines in plot b) indicate the interpreted linear trends. Labels as in Figure 3.
Figure 5. Normalized distributions of the kinetic ($E_{kin}$, solid lines), elastic ($E_{el}$, dashed lines) and total ($E$, dotted lines) energy density for the fundamental Love-wave modes at 60-sec (black) and 30-sec (gray) periods. Note that the $E_{el}$ curve also represents the velocity sensitivity kernel $K_V$, and $E_{kin}$ – the $Q^{-1}$ sensitivity $K_Q$ in eq. (4).
3.2. Geometrical-attenuation and low-Q band model for the Earth

**Summary**

Frequency-dependent seismic attenuation coefficient ($\alpha(f)$) yields a consistent,
assumption-free, and stable description of attenuation which has several advantages over the traditional quality factor ($Q(f)$). In particular, it allows model- and assumption free measurement of geometrical compensation and frequency dependence. By contrast to what is commonly thought, geometrical compensation of different wave types is variable and not sufficiently accurate, and the resulting errors may lead to spurious frequency dependencies of $Q(f)$.

Here, global attenuation data compiled by Anderson and Given (1982) and Abercrombie (1998) are analyzed in the $\alpha(f)$ form. The values of $\alpha$ at $f = 0$ show that surface-wave data are slightly over-corrected, and short-period body-wave data – strongly under-corrected geometrically. The most fundamental observation is that $\alpha(f)$ shows clear linear dependences on $f$ within three frequency bands. Therefore, the “effective” $Q_e$ within each band is frequency-independent, and relaxation mechanisms or band-selective scattering are not required for their explanation. The bands are interpreted as associated with relatively localized layers of increased attenuation within the core, mantle, and the crust. The gap in data coverage from 0.020 Hz to 1 Hz is interpreted as the “structural complexity” band, in which the transition from predominantly surface-wave to body-wave propagation occurs and no geometrical compensation model is known.

**Introduction**

Frequency-dependent attenuation has been observed in numerous seismological observations at 0.001 – 100 Hz. Usually, the attenuation is described by using the concept of “quality” factor, and the observed frequency-dependent $Q_{obs}^{-1}(f)$ values exhibit a band-limited increase (Figure 1). Combined with the associated velocity dispersion, this range of increased $Q_{obs}^{-1}(f)$ leads to the absorption band model of the Earth (Jackson and Anderson, 1970; Liu et al., 1976; Anderson and Given, 1982; Doornbos, 1983; Butler, 1987). Frequency-dependent quality factors and dispersion were also studied in exploration seismic environments (e.g., Sun and Milkereit, 2007).

The observed complex $Q_{obs}(f)$ dependence on frequency is commonly linked to the in situ frequency-dependent $Q(f)$, for which a power-law form is usually used (Aki,
\[ Q(f) = Q_0 \left( \frac{f}{f_0} \right)^{\eta}. \]  

Here, \( f_0 \) is some reference frequency commonly equal 1 Hz, and \( Q_0 \) is the quality factor at that frequency. Above \( \sim 1 \) Hz, \( Q(f) \) typically increases with the frequency \((\eta > 0; \text{Figure 1})\) and forms the upper flank of the upper-mantle absorption band (Aki, 1980; Anderson and Given, 1982). The steep drop in \( Q^{-1} \) from 1 to \( \sim 10 \) Hz was also described as the “10-Hz transition problem” making correlations of \( Q \) between surface- and body-wave phases difficult (Abercrombie, 1998). Note that steep \((\eta \approx 1)\) \( Q(f) \) dependencies are commonly attributed to scattering on larger scatterers (Dainty, 1981). However, large-scale scatterers may be difficult to differentiate from lithospheric structures, making application of scattering terminology to such cases uncertain. This particularly applies to the use of quality factor \( Q \) to describe scattering (Aki, 1980). Additional ambiguities of \( Q(f) \) model (Error! Reference source not found.) are discussed in section Background below.

The goal of this note is to show that the attenuation data in Figure 1 can be described consistently and without the above problems of model (Error! Reference source not found.) by combining variable geometrical effects with “effective” \( Q_e \) comprising intrinsic attenuation and scattering on small-scale heterogeneities. The approach that I take below is to completely abandon the use of \( Q \) for scattering, drop the assumed power-law form (Error! Reference source not found.), make no assumptions about the “background” model, and focus on adequately representing the observed data. In particular, instead of assuming accurate geometrical-spreading compensation from general theoretical considerations (as it is commonly done), we will measure the quality of this compensation from the data.

The resulting model may appear somewhat simple-minded yet it also draws several important, and also somewhat revealing conclusions: 1) the geometrical spreading is variable and can rarely be compensated as well as it is usually assumed; 2) three broad frequency bands can be identified in the Earth, within which the attenuation is...
coefficients are linear in $f$, and therefore the corresponding $Q_e$ is frequency-independent (gray dashed lines in Figure 1); 3) these bands can be associated with two relatively localized layers of increased attenuation (the inner core and asthenosphere) and the high-$Q_e$ crust. The stable and frequency-independent parameters allow consistent interpretation and comparisons between different wave modes and study areas. A simple, semi-heuristic theory is advanced to support the conclusions above. A more extensive theoretical treatment and several additional case examples are given in Morozov (submitted to JGR and GJI, 2008).

Background

Inversion for the in situ $Q$ of the Earth is a difficult procedure imbued with conventions, uncertainties, and model assumptions. The observed $Q_{obs}$ data are scarce and variable in quality, and they represent averages of $Q^{-1}$ values along long mantle paths. Because of the nature of these measurements, uncertainties in $Q_{obs}^{-1}$ values often reach over 30%. Spectral-ratio based and frequency-dependent $Q$ measurements using compensation of geometrical spreading may be difficult to reconcile. It is well known that parameters $Q_0$ and $\eta$ in relation (Error! Reference source not found.) depend on the underlying assumptions about the geometrical spreading, and both of them vary widely when the geometrical spreading models are changed (e.g., Aki, 1980; Abercrombie, 1998). As a result, the frequency-dependent $Q(f)$ in eq. (Error! Reference source not found.) is often referred to as the apparent $Q$ (Aki, 1980) and may be somewhat tricky in interpretation. The functional forms of $Q(f)$ vary significantly for different wave types in the same area. The values of $Q_0$ and $\eta$ may vary strongly depending on the assumed regionally-constant or variable geometrical spreading. In global modeling, a variety of data of variable reliability has also to be combined, and regional biases in surface- and body-wave data coverage add additional uncertainties. At the same time, introduction of frequency- and depth-dependent models (Anderson and Given, 1982) gives additional degrees of freedom and parameter trade-off that may be difficult to constrain.

With scarce data and complex inversion techniques, the initial data presentation
and underlying models is critical and may have a strong impact on the resulting interpretation. For example, although the frequency-independent, layered $Q$ models such as PREM (Dziewonski and Anderson, 1981) describe the observations well (Figure 1), the principal motivation and value of the frequency-dependent ABM model (Anderson and Given, 1982) is in its attractive insights into the possible elastic relaxation mechanisms in mantle rocks (Liu et al., 1976). For this study, it is also particularly important to see the origin of the power-law $Q(f)$ model (Error! Reference source not found.) has motivated hundreds of studies examining and correlating values of $Q_0$ and $\eta$ in numerous tectonic settings and wave types.

Note that apart from being merely a convenient functional form, the $Q(f)$ model (Error! Reference source not found.) was justified by Aki (1980) by considering scattered coda waves (also cf. Aki and Chouet, 1975). The principal advantage brought by this model was in describing scattering by using $Q$, i.e., similarly to the intrinsic attenuation. However, interpretations of the observed $Q(f)$ often extend beyond scattering and address intrinsic rheological rock properties (e.g., Jackson and Anderson, 1970; Liu et al., 1976). On the other hand, theoretically, the use of $Q$ for scattering is unnecessary and may become misleading, because scattering is physically quite dissimilar to dissipation for which the intrinsic $Q$ is defined (Morozov, submitted to JGR). The use of scattering $Q$ also always introduces poorly constrained “background” models.

In the presence of scattering, a reliable alternative to $Q$ is the scattering coefficient $\alpha(f)$, which is commonly used in the scattering theory (e.g., Chernov, 1960). By using $\alpha(f)$, many published $Q(f)$ results can be re-interpreted, often revealing no need for a frequency-dependent in situ $Q$ from the observations (MGJI). A whole-Earth attenuation example of this kind is presented here.

**Model**

In a variety of seismic measurements, the recorded time- ($t$), space- ($x$), and frequency- ($f$) dependent amplitude can be approximated as:

$$A(f, x, t) = S(f)G_0x^{-\nu}e^{-\alpha(x, f)}R(x, f).$$  \hspace{1cm} (2)
where $S(f)$ is the source spectrum, $R(x, f)$ is the instrument and recording site response, and $\nu$ is some selected exponent depending on the wave mode being analyzed and whether the measurements are being done in the time or frequency domain (for example, in $L_g$ studies, $\nu$ may be $0, \frac{1}{2}, 0.83, 1$, or variable; cf. Campillo, 1990; Frankel et al, 1990). In this equation, $x$ can be the source-receiver distance, three-dimensional receiver coordinate, station number, or even time gate position in coda measurements. The site ($R(x, f)$) and geometrical ($G_0 x^\nu$) factors may vary accordingly in these cases.

The exponential factor $e^{-\alpha(x, f)y}$ in eq. (2) includes all path effects outside of the theoretical geometrical spreading $G_0 x^\nu$ and corresponds to the intrinsic attenuation and weak scattering (Chernov, 1960). However, note that an uncertainty or regional variation in $\nu$ can also be absorbed in this factor, and so it can incorporate the uncompensated geometrical spreading (for a discussion of its relation to the commonly used scattering quality $Q_s$, see MJGR). Also note that for waves propagating in a uniform isotropic half-space, the frequency-independent part of $\alpha(x, f)$ is related to the mean free path (Sato, 1978): $L = V / \alpha(x, f=0)$, where $V$ is the propagation velocity.

Further, we approximate $\alpha(x, f)$ by retaining only the first two terms in its Taylor series in $f$:

$$\alpha(x, f) = \gamma(x) + \kappa(x) f = \gamma(x) + \frac{\pi}{Q_e(x)} f.$$  (3)

This approximation, in the form of an alternate $Q^{-1}(f)$ law (defined by setting $\alpha(f) = \pi Q^{-1}(f)$), was used by Dainty (1981) and recently by Padhy (2005). Note that in the standard approach, the compensation of geometrical spreading is assumed to be accurate, and therefore $\gamma(x)$ is set equal to 0, and $Q_e(x)$ in eq. (3) is replaced with $Q(f)$ taken in form (Error! Reference source not found.). However, in real data, ~10 % variations in $\nu$ appear common (MJGR), and the resulting bias in $\alpha(x, f=0)$ gets mapped as strong frequency dependencies in $Q(f)$. From a number of case examples world-wide, values of $\gamma$ are typically positive and range from about $2 \cdot 10^{-4}$ to $6 \cdot 10^{-2}$ s$^{-1}$ in several types of lithospheric studies, leading to significant spurious values of $\eta > 0$ in eq. (Error! Reference source not found.) (MGJI and MJGR).
Parameters $\gamma$ and $\kappa$ (and for data with sufficient offset coverage, $\nu$ as well) can be measured directly from waveform data by using spectral ratios normalized to 1-sec lag times (MGJI). Where appropriate (e.g., in $Pn$ and $Sn$ studies), for data with adequate azimuthal and frequency coverage, parameters $\nu$ and $\gamma$ can be made azimuthally- and $\nu$–frequency-dependent (Yang et al., 2007). Also, when required by the data, higher terms in $f$ can be considered, leading to frequency-dependent $Q_e$ and/or $\gamma$.

The “effective” attenuation parameter ($Q_e$) in equation (3) denotes the frequency-dependent effects of the intrinsic attenuation and scattering on random, small-scale heterogeneities (i.e., Rayleigh- or Mie-type scattering). The corresponding amplitude decays are proportional to the numbers of wave oscillations and not only to the travel distances. To the leading order in $f$, such effects can be characterized by a frequency-independent $Q_e^{-1}$, although rheological relaxation mechanisms (Liu et al., 1976) and small-scale scattering heterogeneities are also possible and may lead to frequency-dependent $Q_e^{-1}$.

When raw spectral-ratio data are not available, $\alpha(x, f)$ can often be estimated from the reported $Q^{-1}(f)$ values by multiplication by $f$: $\alpha(f) = \pi f Q^{-1}(f)$. This multiplication removes much of the spurious frequency dependence in $Q(f)$ caused by non-zero values of $\gamma$. Although such transformation does not allow proper data weighting and error estimation, it still revealed simple, linear $\alpha(f)$ relations in all cases I analyzed so far (Morozov, GJI, 2008; Morozov, submitted to JGR, hereafter referred to as MGJI and MJGR, respectively). In the following, I apply this approach to the broad-band global $Q^{-1}$ data from Figure 1.

**Global $\alpha(f)$ and geometrical attenuation**

Figure Error! Reference source not found. shows the data of Figure 1 in the $\alpha(f)$ form separately within the two frequency sub-bands. For comparison, the Rayleigh- and Love-wave $Q(f)$ dependencies modeled by Anderson et al. (1965) in a layered frequency-independent $Q$ model MM8 and transformed into $\alpha(f)$ are also shown. As one can see, three ranges of linear $\alpha(f)$ patterns of type (3) can be recognized: 1) $0 - 0.007$
Variability of geometrical spreading

Hz, with $\gamma \approx -0.1 \cdot 10^{-4}$ s$^{-1}$ and $Q_e \approx 125$, b) $0.005 - 0.020$ Hz, with $\gamma \approx -0.35 \cdot 10^{-4}$ s$^{-1}$ and $Q_e \approx 97$, and 3) $f \geq 1$ Hz, where $\gamma \approx 2.5 \cdot 10^{-2}$ s$^{-1}$ and $Q_e \approx 1170$. The lowest-frequency band above can potentially be subdivided into two sub-ranges, but we consider this inconclusive with the present data. Given the data scatter, there seems to be no evidence for frequency-dependent $Q_e$ (systematic line curvatures) within these ranges, but the changes between ranges occur sharply. Also note that the modeled $Q_R$ and $Q_L$ dependencies (dashed lines) give a poorer match to the data above 0.005 Hz than the interpreted $\alpha(f)$ trends (Figure Error! Reference source not found.).

The frequency ranges above generally correspond to the different types of observations and/or different depth sampling by seismic waves. These waves should also have different geometrical compensation. From their low negative values of $\gamma$, we see that normal-oscillation data and lowest-frequency Love waves (range (1) above) are best-compensated geometrically, and the surface waves within frequency range (2) are somewhat over-compensated (Figure Error! Reference source not found.). By contrast, the short-period local earthquake data are significantly under-compensated ($\gamma > 0$; Figure Error! Reference source not found.). This is not surprising, as crustal body waves include numerous diving and reflected modes that are most remote from the assumed uniform geometrical spreading in a boundless isotropic medium (for more on this, see MGJI).

Thus, the $S$-wave $Q_e$ values considered at increasing frequencies form an “absorption band,” although in a different sense from Anderson and Given’s (1982). $Q_e$ drop from 125 for core modes to 97 within the upper mantle and rise again to 1170 within the crust (Figure 1). As with all apparent $Q(f)$ measurements, the resulting $Q_e$ are significantly higher than the corresponding $Q_0$ (e.g., compare $Q_e \approx 1170$ and $Q_0 = Q(1$ Hz) $\approx 100$ for range (3) in Figure 1).

The wide frequency gap from $f \approx 0.02$ to $0.5 - 1$ Hz is poorly sampled, and $\alpha(f)$ dependence changes dramatically across it (compare Figures Error! Reference source not found. b and c). This gap can be termed as the “structural complexity band,” in which a transition from surface-wave type to body-wave geometrical compensation occurs, with
values of $\nu$ in eq. (2) changing from 0.5 to 1. Because this frequency range contains virtually no data, we can only speculate that values of $\gamma$ and $Q_e$ could likely be also measurable within this band, provided an appropriate frequency- and region-dependent geometrical correction is also found.

**Layered low-$Q_e$ Earth model**

The observed linearity of $\alpha(f)$ within three frequency bands (Figure Error! Reference source not found.) is a very strong evidence and cannot be incidental, thereby suggesting a “three-layer” Earth attenuation model of some kind. Such linearity also appears to be a most interesting and common observation invariably present in different types of lithospheric attenuation studies (MGJI; MJGR; Morozov et al., submitted to BSSA). It is therefore important to see what could be the general cause of such linearity. My interpretation is that, somewhat similarly to high-velocity contrasts in refraction seismics, linear $\alpha(f)$ patterns should be caused by the presence of distinct low-$Q$ layers within the mantle, as explained by the following simple considerations.

In a heterogeneous Earth, the observed $\kappa$ factor in eq. (3) should be dominated by the effects of regions of highest attenuation $Q^{-1}$ along the ray path:

$$\kappa = \pi Q_e^{-1} = \frac{\pi}{T} \int Q^{-1} \, dt,$$

where $T$ is the total travel time. Because $Q$ values within the mantle (and also within the crust) vary by at least an order of magnitude, relatively thin low-$Q$ layers may be responsible for the observed values of $\kappa$, as long as the corresponding waves sample these layers. Therefore, the two of the observed three frequency bands of low $Q_e$ (Figure Error! Reference source not found.b) could correspond to the waves sampling, respectively: 1) the low-$Q$ parts of the inner core; and 2) the asthenospheric low-velocity and low-$Q$ zone (Doornbos, 1974; Dziewonski and Anderson, 1981). Note that our $Q_e$ values (Figures 1 and Error! Reference source not found.) are close to those of the lowest-$Q$ layers within these depth ranges in PREM model. Considering the uncertainty of the various reference models, such as the suggestion by Montagner and Kennett (1996) that the lower-mantle $Q$ could be much larger than in PREM model, strong $Q^{-1}$ contrasts
within the lower mantle and core appear likely. Also note that from detailed controlled-source studies on continents, the second of these zones may be localized to within ~60 - 100 km in thickness, exhibit structures related to the tectonics (Morozov et al., 1998), and also cause seismic scattering (Thybo and Perchuc, 1997). Similarly to high-velocity layers in refraction seismics, the low-\(Q\) layers may effectively be best-constrained while masking other areas from attenuation imaging.

In order to dominate the average \(\kappa\) in (4), the quality factor \(Q_{\text{low}}\) within the low-\(Q\) layer should be lower than the corresponding observed \(Q_e\):

\[
Q^{-1}_{\text{low}} \approx Q^{-1}_e \frac{T}{T_{\text{low}}} \left[ 1 - \frac{Q_e}{Q_{\text{back}}} \left( 1 - \frac{T_{\text{low}}}{T} \right) \right],
\]

where \(T_{\text{low}}\) is the travel time within the low-\(Q\) layer, and \(Q_{\text{back}}\) is the quality factor within the background mantle or core. The difference of \(Q_{\text{low}}\) values from \(Q_e\) increases if \(Q_{\text{back}}\) is high. This allows using \(Q_e\) within the 0 – 0.005 Hz frequency band as an indicator of the lowest-\(Q\) level within the inner core, scaled by the \(T_{\text{low}}/T\) factor in eq. (5). However, detailed modeling is needed to confirm the above conjectures quantitatively and to constrain the depths of these low-\(Q\) layers and the scaling factor.

The above argument was made in terms of ray paths traversing the low-\(Q\) layers, which mostly applies to body waves. However, it applies to the general wave case as well. Consider the following functional integral over all configurations of elastic field \(\phi\) (Feynman and Hibbs, 1965):

\[
Z = \int D\phi e^{iS[\phi,\dot{\phi}]},
\]

where \(S[\phi,\dot{\phi}] = \int d^3x L(\phi, \dot{\phi})\) is the action, \(L\) is the Lagrangian density, and dot denotes the time derivative. Attenuation is included in \(S\) through an imaginary contribution to the Lagrangian. The functional integral (6) represents the wavefield as constructed by the interference of all possible field configurations (denoted by \(D\phi\) in eq. (6)). The solution to the wave equation is given by the stationary point in integral (6), at which \(\delta S[\phi,\dot{\phi}]/\delta \phi = 0\). Thus, field configurations covering the high-\(Q^{-1}\) zones should make the strongest frequency-dependent contribution to \(S\), thereby making \(Q_e\) related to the \(Q\) of
the attenuative zone, as shown in eq. (5). By changing the frequency range (i.e., switching from the normal-oscillation to surface-wave mode), the stationary field distribution eventually starts covering another low-\( Q \) zone, and the observed \( \kappa \) changes (Figure Error! Reference source not found.b). Finally, the third, high- and constant-\( Q_e \) zone ((3) in Figure Error! Reference source not found.c) is isolated by virtue of body waves no longer sampling the low-\( Q \) depths within the mantle.

**Conclusion**

Transformation of global \( Q \) data from Anderson and Given (1982) and Abercrombie (1998) into the form of attenuation coefficient \( \alpha(f) \) reveals three frequency bands within which the \( \alpha(f) \) is linear with frequency. The values of \( \alpha \) at \( f = 0 \) show that only the normal-mode and long-period surface-wave data are accurately corrected for geometrical effects. By contrast to what is commonly assumed, surface-wave data at 0.005 – 0.020 Hz are somewhat over-corrected, and short-period body-wave data – strongly under-corrected geometrically, which leads to spurious \( Q(f) \) dependencies.

From the \( d\alpha(f)/df \) derivatives, frequency-independent “effective” \( Q_e \) is determined for the three frequency bands. The bands are interpreted as related to two relatively localized layers of increased attenuation (within the inner core, with \( Q_e \approx 125 \), and asthenosphere, for which \( Q_e \approx 97 \)) and the crust, whose average \( Q_e \) is estimated as \( \sim 1170 \).

Although the proposed model clearly requires refinement by full modeling and inversion, the observed linearity of \( \alpha(f) \) in \( f \) is well-established directly from the data. Combined with a broad range of other examples that presented earlier (Morozov, GJI, 2008, and submitted to JGR; Morozov et al., submitted to BSSA), this linearity shows that frequency-independent \( Q \) may be not as pervasive as it is often thought. In fact, in all examples I have considered to date, no indications of \( Q(f) \) were found.

In respect to the \( \alpha(f) \) attenuation interpretation method, its stable and frequency-independent parameters allow modeling, consistent interpretation, and comparisons between different wave modes and study areas. The approach is applicable to many other seismic studies using surface, body, guided, and coda-wave records.
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Figure 1. Observed $1000 \cdot Q_{\text{obs}}^{-1}$ for spherical oscillations, Rayleigh- and $P$-waves (gray diamonds), torsional oscillation, Love-, and $S$-waves (black diamonds) compiled by Anderson and Given (1980), and borehole $S$-wave data from Abercrombie (1998) (black circles). Dotted gray lines with labels $Q_e$ summarize the interpretation of this paper (see text and Figure Error! Reference source not found.).
Figure 2. The same data as in Figure 1 in the attenuation coefficient (-\(\alpha(f)\)) form:

a) lower-frequency \(S\)-wave data; b) \(P\)-wave data; c) and d): data from plots a) and b), respectively, with linear “move-out” terms of 0.03\(f\) added to emphasize the breaks in \(\alpha(f)\) slopes; e) short-period \(S\)-wave data. Note the interpreted linear segments (solid lines), with (\(\gamma; Q_e\)) values given in the labels. Dashed lines in plots a)-d) show the Love (labeled \(Q_L\)) and Rayleigh wave (\(Q_R\)) dependencies modeled by Anderson et al. (1965) and re-calculated into -\(\alpha(f)\).
3.3. Frequency dependence of coda Q: Mapping in Siberia using nuclear-explosion profiles

Summary

In Part I of this paper (Morozov et al., 2008) we showed that short-period coda attenuation of regional arrivals can be interpreted in terms of the geometrical spreading ($\gamma$) and effective attenuation ($Q_e$) parameters. Taken together, they represent the generalized attenuation coefficient: $\alpha(f) = \gamma + \pi f / Q_e$. This model removes the commonly observed ambiguity of the frequency-dependent $Q(f)$, and parameter $\gamma$ becomes most useful for interpretation. Modeling of $P$-wave coda at regional distances and analysis of two Peaceful Nuclear Explosions in Russia in terms of this concept showed a remarkable agreement, particularly in the values of $\gamma$.

Here, we apply the same parameterization to the inversion of spatially-variable surface-consistent $Lg$ coda parameter $\alpha(f)$ from five PNE profiles in Siberia. Minimalistic model parameterization and the Constrained Simultaneous Iterative Reconstruction Technique (SIRT) are used for unbiased inversion. The resulting map of the geometrical parameter ($\gamma$) is in excellent agreement with both modeling and inversion in Part I. The values of $\gamma$ differentiate between the crustal tectonic types within the study area and are also in agreement with a summary of over forty worldwide studies (Morozov, 2008). In addition, scattering amplitudes are found to be highly variable and strongly correlate with geological structures. Coda $Q_e^{-1}$ values also show pronounced correlations with geological structures, although these values may also be somewhat overestimated compared to two detailed PNE point studies in Part I.
**Introduction**

Geophysical characterization of the Earth’s lithosphere broadly consists of descriptions of its velocity, density, and attenuation structures. Combinations of these parameters are used in a broad range of applications, from providing structural and rheological constraints for geodynamics to event location, discrimination of seismic events, calibration and regionalization. For the first two of these parameters, a variety of precise methods have been developed, but *in situ* attenuation remains notoriously difficult to measure and interpret. In particular, while quality parameter \( Q \) often showing strong frequency dependence, characterization of such frequency-dependent attenuation \( Q(f) \) may be ambiguous and dependent on model assumptions and inversion methods.

Morozov (2008) recently proposed a new technique for measuring attenuation without the use of the ambiguous \( Q(f) \) and illustrated it on a number of examples. In this study, we employ this method to invert coda \( Q \) values from five Peaceful Nuclear Explosion (PNE) profiles in Russia and to map the frequency-dependent attenuation across a broad area of Siberia (Figure 1).

Different types of attenuation (body, surface, \( Lg \), and coda waves) are often presented in the following power-law form of frequency-dependence:

\[
Q(f) = Q_0 \left( \frac{f}{f_0} \right)^\eta,
\]

where \( Q \) is the apparent quality parameter, and \( f_0 \) is some reference frequency often taken to equal 1 Hz (Aki, 1980). However, as argued by Morozov (2008) and also in Part I of this paper (2008; hereafter referred to as Part I), parameters of such a power-law dependence may be strongly influenced by the observation process, and in particular by the assumptions made about the geometrical spreading of scattered (in the case of coda \( Q \)) or other waves. Parameters \( Q_0 \) and \( \eta \) trade off with each other through the uncertainty of the assumed geometrical spreading. The geometrical spreading (GS), in its turn, cannot be considered constant within large areas, and \( \sim 10\% \) uncertainties in it can account for much of the observed frequency dependence of the apparent \( Q \) (Morozov,
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2008). Unconstrained variations of GS also make it difficult to correlate attenuation parameters from different regions, and also to link them to lithospheric properties. In seismic calibration and nuclear test monitoring studies, uncertainties in $Q_0$, $\eta$, and geometrical spreading are particularly undesirable, because they prevent from construction of consistent, seamless models of large areas and complicate magnitude calibration.

Numerical waveform coda modeling performed in Part I showed that for a layered 1-D model with a constant, frequency-independent crustal attenuation, the observed coda $Q(f)$ exhibited a strong dependence on frequency. Although such dependence is often observed in coda $Q$ and $Lg$ $Q$ data (e.g., Der et al., 1986; Campillo, 1987, 1990; Frankel, 1990; Mitchell and Cong, 1998; Benz et al., 1997; Mitchell et al., 1997; McNamara et al. 1996; McNamara, 2000; and Erickson et al., 2005), this modeling shows that it may relate to the crustal structure (particularly to layering) and not necessarily to its rheology, fracturing, or fluid content. Similar observations were made earlier by Anderson et al. (1965) for long-period surface waves and by Mitchell (1991) for $Lg$ waves in the Basin and Range. Strictly speaking, because the values of $Q$ and $\eta$ trade off with the imprecisely known velocity/density structures, these parameters are not necessarily be associated with attenuation properties and may not reliably correlate with geological structures or with the variations of the physical state of the lithosphere. However, the potential association of the widespread observations of frequency-dependent $Q(f)$ with variations of the structure still has not been examined, and values of $Q$ are typically interpreted as related to rheology, physical state, or fluid content of the lithosphere (e.g., Mitchell et al. 1997).

In practice of $Q(f)$ interpretations, and in particular in $Lg$ coda $Q$ studies (such as coda magnitude calibration), the sensitivity of coda $Q$ to the frequency is usually overcome by referencing the coda $Q(f)$ dependence to some common frequency level, such as 1 Hz, in all comparisons (e.g., Mitchell et al. 1997; Phillips et al., 2004). However, this still does not resolve the problem of $Q(f)$ ambiguity, as such referencing process depends on the velocity structure through parameter $\eta$, making the resulting values less portable. Note that because of the inherent dependence of the results on
unconstrained GS, changes in inversion approaches sometimes also lead to updated values of \( Q_0 \) and \( \eta \) (for example, indicated by Xie et al. (2006) in relation to Mitchell et al. (1997) or Xie (2002) in relation to McNamara et al., (1996)). Thus, the interplay of the assumed forms of GS and power-law \( Q(f) \) in dependence (1) creates a difficult environment for measurements and interpretation of attenuation.

In many coda \( Q \) and \( Lg \ Q(f) \), and particularly in nuclear test monitoring studies, \( Q \) is treated as a sort of coda-shape or spectral-amplitude shape parameter. When used strictly for matching the observed time-frequency dependent amplitudes, the above model works even with imprecise or even arbitrary GS. However, the ability of matching the amplitudes still does not prove that the propagation model is correct and can be used to constrain the crustal and mantle properties. This problem is well-known in the inversion theory – under-constrained (or over-parameterized) models tend to fit the data perfectly while leaving ambiguities in the solution (Menke, 1989). When interpreting such solutions and comparing them to each other, it is important to avoid using parameter combinations that are related to the unconstrained degrees of freedom. Unfortunately, both \( Q_0 \) and \( \eta \) in eq. (1) are such parameters, which co-vary with the unconstrained GS.

To resolve this difficulty of intermixed GS and attenuation effects and the resulting apparent character of \( Q(f) \), we proposed (Morozov, 2008, and Part I) to incorporate an empirical geometrical spreading parameter \( \gamma \) in the equation for coda amplitude decay:

\[
A(t, f) = A_0(f) t^{-\nu} e^{-\alpha(f)t} = A_0(f) t^{-\nu} e^{-(\gamma + \kappa f)t},
\]

where \( \kappa = \pi/Q_e \), and \( \nu \) is the theoretical GS exponent corresponding to the type of scattered wave modes that is considered as predominant. Arguments for form (2) based on the scattering theory were given by Morozov (unpublished); however, this expression can also be viewed as a purely empirical approximation (Morozov et al., 2006; Morozov, 2008). The subscript “e” in \( Q_e \) indicates the “effective” character of the quality parameter, in which we do not differentiate between the intrinsic attenuation and small-scale (Rayleigh- and Mie-type) scattering. Note that parameter \( \kappa \); and consequently also \( Q_e \), can be frequency-dependent. However, no indications of such frequency dependence

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were found in several published examples of $Lg$, and body-wave data (Morozov, 2008), and also in surface-wave and normal-oscillation data (Morozov, unpublished).

In parameterization (2), the two parameters ($Q_0$, $\eta$) of the power law (1) are replaced with three: $\nu$, $\gamma$ and $\kappa$. Within typical errors of spectral observations, there is little room for seeking frequency-dependent $\gamma$ or $\kappa$, and they can therefore be considered frequency-independent. Morozov (2008 and unpublished) re-examined several surface-, body-, $Lg$-, and coda-wave studies and showed that model (2) was invariably applicable to all of them. In most cases, this resulted in significant increases of the reported values of $Q$.

Parameter $\nu$ significantly trades off with $\gamma$, and therefore it is acceptable to use amplitudes $A(t,f)$ corrected for geometrical spreading with fixed $\nu$ values determined from theoretical considerations, as it is commonly done in $Q(f)$ studies. In such cases, parameter $\gamma$ describes the spatial variations of uncompensated geometrical spreading. In the case of regional $Lg$ coda $Q$ of this study, we use a model of cylindrically-propagating coda waves generated within the near-surface. In this case, the spreading area of scattering surrounding the source and/or receiver compensates the geometrical spreading of coda waves, and $\nu$ can be set equal to 0 (Morozov and Smithson, 2000). However, this is only a simplified model, and, for example, diving and reflected scattered waves, deeper, or multiple scattering would change the $t^\nu \approx 1$ law, and make it regionally variable. Our objective in this study is to measure these variations, in the form of a spatially-variant $\gamma$.

Parameter $\gamma$ was found to correlate with lithospheric types and ages; in particular, the value of $\gamma_D = 0.008$ s$^{-1}$ appeared to clearly separate the tectonically stable ($\gamma < \gamma_D$) and active areas ($\gamma > \gamma_D$; Morozov, 2008). In addition, numerical modeling of the regional $P$-wave coda conducted in several crust/mantle models of the East European Platform (Part I) showed that the modeled $\gamma$ was in a close agreement with the measurements from PNE data, and coda $Q$ could be related to the S-wave crustal $Q_S$ (in this study, we used proportional crustal $P$- and S-wave $Q$ values: $Q_P = 2Q_S$). Therefore, it is particularly interesting to look for correlations of $\gamma$ with geological structures across a large
contiguous area like the PNE profiling area of this study.

From the view point of the classical scattering theory (e.g., Chernov, 1960, p.53; Nikolayev, 1968), parameterization (2) is merely a return to the traditional frequency-dependent attenuation coefficient $\alpha(f)$ instead of the “scattering quality” parameter $Q_s$ introduced by Aki (1980). Dainty (1981) also gave a theoretical justification for expression (2) as describing scattering of 1-30-Hz $S$-wave on larger scatterers. In coda $Q$ studies, such scattering could occur by means of diving, reflected, and mode-converted waves propagating through the crust. Note that such processes mostly involve deterministic effects, such as ray bending in crustal velocity gradients and reflections from major lithospheric structural contrasts, and therefore the use of $Q_s$ leads to undesirable connotation of a random scattering process and loss of correlation with the structure. Therefore, instead of using $Q_s$, we prefer using the uncompensated GS, which simply represents the frequency-independent part of the attenuation coefficient: $\gamma = \alpha(f=0)$.

Preliminary analysis of two PNEs (Figure 1) in Part I suggested that $\gamma$ values could be remarkably stable, even in the case of surprisingly strong frequency dependence ($\eta \approx 1$) within the Siberian Craton. The observed values of $\gamma$ also agreed well with those derived from waveform synthetics computed for the study area. We argued that $\gamma$ could potentially provide a stable and transportable criterion for correlating the observations of attenuation with crustal tectonic types.

In this paper, we further apply the approach arising from parameterization (2) to mapping the short-period attenuation structure in central Siberia by using several PNE datasets (Figure 1). Detailed descriptions of the PNE profiles and geological setting of the study area were recently given by Li et al. (2007 and submitted to BSSA) and are not repeated here. The resulting $\gamma$ and $Q_e$ maps show good stability and correlations with velocity structures and tectonics. Because the parameters being mapped are directly related to crustal and upper mantle properties, they should also provide a good basis for comparisons to other studies and for incorporation into forward modeling.
Data

The dataset consists of log-amplitude coda amplitude readings from the Peaceful Nuclear (PNE) explosion records from five Deep Seismic Sounding profiles in Russia (Figure 1). Figure 2 shows an example of transverse-component (relative to the source-receiver direction) record from PNE Kimberlite-3 (labeled K3 in Figure 1) located near the edge of the Siberian Craton. Note the high density of recordings (10-15 km spacing) and the differences in the wavefield propagating within and beneath the West Siberian Basin (west of the PNE) and the Siberian Craton (Figure 2). This PNE was also used in detailed point studies in Part I.

The data were carefully edited by removing poor and clipped records, and all regional arrivals were identified and picked. The $Lg$ distance ranges and time windows were selected interactively from all PNE record sections. Coda windows started 20 sec after the picked $Lg$ onsets and extended for 50 – 100 sec. Pre-$Lg$ noise windows (i.e., the $P$- and $S$-wave coda; cf. Morozov and Smithson, 2000) were also picked. By using these windows, $Lg$ to pre-$Lg$ amplitude ratios were calculated, and traces with these ratios below 1.1 were discarded.

Edited records were further band-pass filtered within four overlapping frequency bands of 1 – 2, 1.5 – 3, 2 – 4, and 3 - 5 Hz. Within each band, three-component trace envelopes were formed, and $d\ln(\text{Amplitude})/dt$ derivatives and their standard error estimates were measured by using the “robust fit” technique in Matlab. The resulting values of frequency-dependent log-amplitude coda slopes were saved in a database which was used in the subsequent analysis.

Surface-consistent ($\lambda, \gamma, Q_c$) inversion method

The power and utility of the $\alpha(f)$ model is in its direct link to the observations (Morozov, 2008). In the case of regional coda attenuation, coda $\alpha_c(f)$ simply becomes the negative time derivative of the logarithm of coda amplitude (2):

$$\alpha_c(f) = -\frac{d\ln A(t, f)}{dt},$$

(3)
which can be measured by fitting the coda envelopes in each individual record. Because of the source-receiver reciprocity, the observed coda amplitude (2), and consequently $\alpha_c(f)$, contain contributions from scattering near the source and receiver. Spatial separation of these two areas and the corresponding differences in their $\alpha(f)$ values could be significant for mapping, particularly in the case of the long-range PNE sources. Thus we need to invert the observed $\alpha_c(f)$ for spatial variations in $\alpha(f)$. To accomplish this, we use the surface-consistent approach, and represented $\alpha(f)$ as functions of the coordinates and frequency. The surface-consistent model implies that for collocated source and receiver, the values of $\alpha(f)$ should be the same, and for spatially close locations, $\alpha(f)$ are close.

To derive a model for the observed $\gamma$ and $Q_e$, we use an approximation for event energy envelope by Morozov and Smithson (2000), in which the intensity (energy density) of the primary event was described by a short pulse of parabolic shape arriving at time $t_0$, with amplitude $P_0$ and duration $\tau$. The coda of this arrival was described by exponentially decaying amplitude following the pulse:

$$P_{\text{coda}}(t, \lambda, \alpha) = \begin{cases} \lambda(f)P_0 e^{-2\pi(f-t_0)^2/\tau^2}, & t < t_0, \\ \lambda(f)P_0 e^{-2\alpha(f)}, & t \geq t_0, \end{cases}$$

(4)

where $\lambda(f)$ is the scattering efficiency factor (the relative coda intensity at the time of the primary event). In general, this parameter may also depend on the frequency.

From PNE Quartz-4 records (Q4 in Figure 1), the duration of the primary onsets was estimated as $\tau = 1.25$ s for the teleseismic $P$, and $\lambda = 0.22$ for all events (Morozov and Smithson, 2000). The relative coda intensity at the time of the primary event was $\lambda\tau \approx 0.27$ (equation (4)), which can be considered as significantly smaller than 1. Therefore, the total wavefield intensity at the receiver:

$$P_{\text{SR}}(t) = [P_{\text{primary}}(t) + P_{\text{coda}}(t, \lambda_S, \alpha_S)]* [P_{\text{primary}}(t) + P_{\text{coda}}(t, \lambda_R, \alpha_R)],$$

(5)

(quantities with subscripts $S$ and $R$ correspond to the source and receiver locations, respectively, and the asterisk denotes time convolution) can be approximated to the first order in $\lambda$:  

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\[ P_{SR}(t) = P_{primary}(t) + P_{primary}(t) + P_{primary}(t) + \left[ P_{coda}(t, \lambda_S, \alpha_S) + P_{coda}(t, \lambda_R, \alpha_R) \right]. \]  

(6)

The resulting coda intensity is given by the second term in this expression:

\[ P_{SR,coda}(t) \propto \lambda(x_S, y_S, f) e^{-2\alpha(x_S, y_S, f) t} + \lambda(x_R, y_R, f) e^{-2\alpha(x_R, y_R, f) t}, \]

(7)

where we also assumed surface consistency of \( \lambda \). From eq. (2), \( \alpha(x, y, f) \) is further approximated by a linear function of frequency:

\[ \alpha(x, y, f) = \gamma(x, y) + \kappa(x, y) f, \]

(8)

Taking into account the source and receiver site effects, the coda intensity (7) is further approximated as:

\[ P_{SR,coda}(t) \approx \zeta(f) e^{-2\alpha t}, \]

(9)

where \( \zeta(f) \) is a time-independent amplitude normalization factor of the specific seismic record. With these approximations, eq. (9) leads to the following system of non-linear equations:

\[ d_{n,t,f} = \lambda_S(f) e^{-2[\alpha(x_S, y_S, f) - \alpha_t]} + \lambda_R(f) e^{-2[\alpha(x_R, y_R, f) - \alpha_t]} - \zeta_n(f) = 0 \]

(10)

for all \( t, f, \) and record numbers \( n \). Solving equations (10) is equivalent to minimizing the following objective function:

\[ \Phi(\gamma, \kappa, \lambda, \zeta) = \frac{1}{2} \sum_f \sum_n \int [\lambda_S(f) e^{-2[\alpha(x_S, y_S, f) - \alpha_t]} + \lambda_R(f) e^{-2[\alpha(x_R, y_R, f) - \alpha_t]} - \zeta_n(f)]^2 dt, \]

(11)

where \( T_c \) is the coda observation time window. The summation in eq. (11) takes place over all records and frequencies of interest, and the minimization is performed in terms of parameters \( \gamma, \kappa, \lambda(f) \), and \( \zeta(f) \). In this study, to improve the stability of the inverse, we did not consider the dependence of \( \kappa, \lambda \), and \( \zeta \) on frequency, and solved for variations of \( \gamma, \kappa, \) and \( \lambda \) as functions of spatial coordinates only.

To define the spatial inversion grid for parameters \( \gamma, \kappa, \) and \( \lambda \), care needs to be exercised in order to not over-parameterize the model excessively. To achieve an
“minimalistic” parameterization, we defined a spatial grid with increments of 3° in both latitudes and longitudes covering the entire area of profiling. Only cells containing the actual source or receiver points (Figure 1) were included in the inversion, which resulted in 53 cells. For each of these selected grid cells, a bilinear basis function (finite element) \( \psi_i(x,y) \) was defined (\( x \) was the longitude and \( y \) is the latitude, \( i \) is the cell number), such that \( \psi_i(x,y) = 1 \) in its center and \( \psi_i(x,y) = 0 \) at the centers of all other cells. “Blocky” basis functions typical in travel-time tomography (with \( \psi(x,y) = 1 \) within the entire cell) were also tried, with no significant differences in the results.

With properly defined basis functions, the following identity holds at any point \((x, y)\):

\[
\sum_{i} \psi_i(x, y) = 1,
\]

where the summation includes all cells. With the help of this functional basis, values of \( \gamma \), \( \kappa \), and \( \lambda \) at any point \((x, y)\) can be expressed through the corresponding values \((\gamma_i, \kappa_i, \lambda_i)\) at the nodes; for example, for \( \gamma \):

\[
\gamma(x, y) = \sum_{i} \psi_i(x, y)\gamma_i.
\]  (13)

Within the class of functions given by the finite-element functional basis (13), the objective function (11) becomes a non-linear function of model parameters: \( \Phi(\gamma_i, \kappa_i, \lambda_i, \zeta_i) \). However, unconstrained degrees of freedom are still present in this parameter space, such as an arbitrary scaling of parameters \( \lambda \) and \( \zeta \) in eq. (11) and source-receiver trade-off caused by non-uniform sampling of the model. These degrees of freedom are typically removed by regularization, which is performed by adding constraint terms to the objective function (11), for example:

\[
\Phi(\gamma_i, \kappa_i, \lambda_i, \zeta_k) = \Phi(\gamma_i, \kappa_i, \lambda_i, \zeta_k) + \sum_k w_k \Psi_k(\gamma_i, \kappa_i, \lambda_i),
\]  (14)

where functions \( \Psi(\gamma, \kappa, \lambda) \) penalize various undesirable types of instabilities. However, this method of regularization requires a non-trivial selection of weights \( w_k \) and biases the solution from minimizing the objective function (11) and satisfying equations (10). Most
importantly, properties of the resulting constraints may be difficult to assess in order to evaluate their effects on the interpretation. Therefore, a simpler regularization scheme with readily interpretable constraints would be preferable.

In order to obtain a stable solution without unbiased from the minimum of \( \Phi(\gamma_i, \kappa_i, \lambda_i, \zeta_n) \), we utilized the iterative nature of the SIRT (Simultaneous Iterative Reconstruction Technique) solver that we employed. After each iteration, the solution was corrected to satisfy the following criteria:

1) \( \lambda_i \geq 0 \) and \( \kappa_i \geq 0 \) at all points. These constraints correspond to non-negative scattering amplitude and \( Q \), respectively. Note that unlike \( \kappa_i \), \( \gamma_i \) can take both negative and positive values.

2) \( \sum_i \lambda_i = N_{\text{grid}} \), where \( N_{\text{grid}} \) is the total number of grid points involved in the inversion. This constraint was enforced by re-scaling all values of \( \lambda_i \) and removed the scaling invariance of the minimum of expression (11).

3) \( |\lambda_i - \lambda_i^{\text{smooth}}| < \beta \lambda_i^{\text{smooth}} \), where \( \lambda_i^{\text{smooth}} \) is the smoothed value of \( \lambda_i \) derived by averaging the adjacent points (and excluding the \( i \)-th one), and \( \beta \) is the tolerance parameter selected equal 0.05 in our inversion. This criterion guaranteed that adjacent cells differ by no more than \( \beta \) in terms of \( \lambda_i \). Similar constraints were implemented for \( \gamma_i \) and \( \kappa_i \).

4) Additionally, similar constraints, for example: \( |\lambda_i - \lambda_i^{\text{apriori}}(x, y)| < \beta^* \lambda_i^{\text{apriori}}(x, y) \), could be added to keep the solution in the vicinity of some \textit{a priori} model. Such constraints could be useful to remove instabilities at isolated points near the circumference of the model, or for extrapolation of the solution outside of the area.
of coverage. The *a priori* model can be parameterized at user-specified sets of support points between which the values of $\lambda^{\text{apriori}}(x, y)$ are linearly interpolated by using a Delaunay triangulation. However, we did not use such constraints in the solution presented below.

To invert for $\gamma_i$, $\kappa_i$, $\lambda_i$, and $\zeta_n$ numerically, we used the SIRT method by applying it directly to equations (10). For a small parameter perturbation, these equations become:

$$d_{f,n,t} = d_{f,n,t}^0 + \delta d_{f,n,t} =$$

$$= d_{f,n,t}^0 + \left[\delta \lambda_s - 2t(\delta \gamma_s + f \delta \kappa_s)\right] e^{-2(a_y-a_y,y)\gamma_s} +$$

$$\left[\delta \lambda_R - 2t(\delta \gamma_R + f \delta \kappa_R)\right] e^{-2(a_y-a_y,y)\gamma_R} - \delta \zeta_n$$

$$= 0. \quad (15)$$

Quantities $\delta \gamma$, $\delta \kappa$, and $\delta \lambda$ evaluated at the source and receiver locations are further expressed as combinations of the corresponding values at the nearest grid nodes by using equations (13). By combining all parameter perturbations $\delta \gamma_i$, $\delta \kappa_i$, $\delta \lambda_i$, and $\delta \zeta_n$ into one model vector $p$, the resulting equations form a linear system:

$$A\delta p - B = 0, \quad (16)$$

which was over- or mixed-determined (Menke, 1989; 7986 data points and 885 unknowns in our inversion). Its approximate solution in the least-squares sense is:

$$\delta p = (A^T A)^{-1} A^T B, \quad (17)$$

where $T$ denotes the matrix transpose. In SIRT (and related back-projection methods), the diagonally-dominant $A^T A$ matrix is replaced in this equation with its diagonal, whose inverse can be easily calculated:

$$\delta p \approx c \left[\text{diag}(A^T A)^{-1}\right] A^T B. \quad (18)$$

This method requires storage of only two vectors: $A^T B$ and the diagonal of $A^T A$, and consequently it can be efficiently applied to very large problems. In our study, the inverse (18) was calculated during iterative scanning through the entire $s_c(f)$ dataset, terminated
when model updates became sufficiently small. Between the iterations, “trimming” operations were applied to the model in order to make it satisfy the constraints above. Our final model (Figure 3) resulted in ~90% data error reduction from the starting model with $\gamma = 0$, $\kappa = 0$, and $\lambda = 1$. To suppress parameter oscillations during this process, a scaling factor $c < 1$ ($c = 0.5$ in our inversion) was applied during stepping (18) and was gradually reduced when data error increases were detected.

**Results**

The resulting maps of $\lambda$, $\gamma$, and $\kappa$ within the area of $Lg$ coda $Q$ data coverage from five PNE profiles are shown in Figure 3. To produce continuous maps, we used linear interpolation within a Delaunay triangulation constructed on the centers of the 53 grid cells that were used in the inversion (Figure 1). This explains the shapes of some of the features located near the edges of the coverage area (such as the Baikal Rift zone and the Urals; Figure 3).

As expected, the scattering amplitudes, $\lambda$, show strong variations which remarkably well correlate with tectonic areas (Figure 3a). High $\lambda$ values are found in the Uralian and in the Sayan-Baikal (labeled BR in Figure 3b) fold belts which could potentially be related to stronger surface topography and complex crustal structures due to folding. High scattering amplitudes are also present in the western part of the Siberian Craton. By contrast, the eastern part of the Siberian Craton (the Minusisk-Aldan High; Figures 3a and b) and most of the West Siberian Basin show low $\lambda$.

The geometrical spreading exponent, $\gamma$, also correlates with tectonics (Figure 3b). Note that the level of $\gamma_D = 0.008$ s$^{-1}$ (purple color in Figure 3b, marked) separates the stable cratonic areas (with $\gamma < \gamma_D$) from the tectonically active Baikal Rift, for which $\gamma$ is distinctly higher than $\gamma_D$. As we illustrated by numerical modeling in Part I of this study, this difference could be caused by generally higher level of contrasts within the tectonically active crust, potentially with presence of relatively thin attenuative zones.

As also suggested in Part I, within the stable parts of the study area, $\gamma$ varies only moderately, from about 0.004 to 0.006 s$^{-1}$ (Figure 3b).
The effective surface-consistent coda $Q_e^{-1}$ values, defined as $Q_e^{-1} = \kappa / \pi$, also show distinct correlation with tectonic structures and reasonable stability within the study area (Figure 3c). Coda $Q_e$ is low in the Baikal Rift zone ($Q_e \approx 100$) and near the Urals ($Q_e \approx 130$) and higher within the Siberian Craton ($Q_e \approx 300$). Within the eastern part of the West Siberian Basin, the attenuation is the lowest ($Q_e \approx 1000$). However, although indicating reasonable correlation with the geological structures, the values of $Q_e^{-1}$ appear somewhat over-estimated (could be to a factor of about 2) compared to detailed measurements performed on two individual PNEs in Part I of this paper. Potential reasons for such overestimation will be considered in Discussion below.

**Discussion**

Considering the average expected values of $Q_e \approx 500$ and $\gamma \approx 0.006$ s$^{-1}$, the “cross-over” frequency $f_c = |\gamma Q_e / \pi$ equals approximately 1 Hz. The cross-over frequency was defined by Morozov (GJI, 2008) as the frequency at which the coda amplitude decay related to attenuation ($Q_e$) equals that related to residual geometrical spreading (GS; $\gamma$). For $f >> f_c$ the effects of attenuation dominate the observations, and for $f << f_c$ – the geometrical spreading should be predominant.

For DSS PNE data, the useable frequency band extends from approximately 0.5 to 3-4 Hz, and this band was used in the measurements shown in Figure 3. Thus, we are working in the vicinity of $f_c$, and neither $\gamma$ nor $Q_e$ (i.e., $\kappa$) can be neglected. In cratonic areas with high $Q_e \approx 1000$, the effects of GS become progressively more important. Note that the same also applies to most attenuation studies using $L_g$ and short-period body-wave recordings. Disregard of uncompensated GS in such cases results in strong positive dependence of the apparent $Q$ on frequency (Morozov, 2008).

Although indicating generally good correlation with tectonics, the effective coda $Q_e^{-1}$ values may be somewhat over-estimated in Figure 3c. This could likely be caused related by the residual trade-off present in the model. With the regularized SIRT inversion followed by spatial interpolation, comprehensive analysis of the inversion uncertainties and their presentation are non-trivial tasks which be considered in future studies. At this stage, we can only suggest several factors that could influence the

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reliability of $Q_e^{-1}$ results (Figure 3c).

The general difficulty of measuring log-amplitude spectral slopes is well known. Spectral amplitudes often show scalloping which makes measurements of spectral slopes problematic. In our case, $\kappa$ effectively measures the double time-frequency slope of the logarithm of time-variant spectral amplitudes [see eq. (2)]:

$$\ln[\kappa^*A(t,f)]=-\gamma t - \kappa f t.$$  \hspace{1cm} (19)

Thus, as a second derivative in $t$ and $f$, it appears that $\kappa$ could be most affected by data noise and parameter trade-off. Further, because of the nature of $\kappa$ being small and non-negative, “leaking” of the noise and inversion uncertainties would likely increase its values.

Another factor that was not considered in this study but that could cause increased values of $Q_e^{-1}$ is the frequency dependence of scattering amplitudes $\lambda$. If scattering amplitudes decrease with frequency, such a decrease could trade-off with increasing $Q_e^{-1}$, and vice versa. In addition, the $Lg$ coda in PNE data is located on top of the codas of the preceding $P$-wave arrivals (Morozov and Smithson, 2000), which also contribute to an apparent increase of $Q_e^{-1}$ by steepening the coda decay with time. Once again, unfortunately, the uncertainties of parameters $\lambda$, $\gamma$, $\kappa$, and $\zeta$ can only be studied by using stochastic error analysis and by utilizing the full coda waveform information, by contrast to using a database of previously acquired coda slope values described above.

Finally, with some doubts in potentially over-estimated values of $Q_e^{-1}$ (Figure 3c), the values of $\lambda$ and $\gamma$ appear to be reliable (Figures 3a and b). They represent lower-order parameters and correlate well with the tectonics. In particular, the map of $\gamma$ (Figure 3b) represents the most important result of this study. The map of $\gamma$ quantitatively correlates with numerous other worldwide studies summarized in Morozov (2008) and almost perfectly corresponds to numerical modeling and analysis of two selected PNEs in Part I of this paper. It appears that mapping of spatially-variable $\gamma$ definitely needs to be considered whenever $Q$ measurements are attempted, particularly in extended areas with contrasting crustal and lithospheric structures. As shown above, once $\gamma$ is incorporated,
Only frequency-independent $Q_e$ remains to be measurable in present-day seismic data. However, both $\gamma$ and coda $Q_e$ can be measured assumption- and ambiguity-free, and these quantities represent useful basis for interpretation. As shown in Part I, both of these quantities can also be modeled by realistic numerical waveform simulations, which allows deriving useful links of $\gamma$ and $Q_e$ to the lithospheric structure and in situ crustal $S$-wave attenuation.

Conclusions

In Part I of this paper (Morozov et al., 2008) we demonstrated that short-period coda attenuation of regional arrivals can be interpreted as a combination of uncompensated geometrical spreading ($\gamma$) and effective attenuation ($Q_e$). Taken together, these parameters represent the generalized attenuation coefficient: $\alpha(f) = \gamma + \pi f/Q_e$, which in many cases can be measured directly from the data. Parameter $\gamma$ removes the common ambiguity of the $Q(f)$ and is most useful for correlating different study areas and tectonic types. Modeling of $P$-wave coda at regional distances and analysis of two Peaceful Nuclear Explosions in Russia in terms of this concept showed a remarkable agreement, especially in $\gamma$ values.

Here, we applied the same parameterization to the inversion of spatially-variable surface-consistent $Lg$ coda parameters from five PNE profiles in Siberia. The resulting map of the geometrical parameter ($\gamma$) is in excellent agreement with both modeling and inversion in Part I, and also with a summary of over forty worldwide studies (Morozov, 2008). In addition, scattering amplitudes were found to be highly variable and strongly correlate with geological structures. Coda $Q_e^{-1}$ values also showed pronounced correlation with geological structures, although these values also appeared somewhat overestimated compared to the detailed point studies in Part I.

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Data and Resources

The nuclear explosion data used in this paper were acquired by the Deep Seismic Sounding program in the former Soviet Union in 1977 – 89. These records are currently available through IRIS archives (http://www.iris.edu). Modeling and data analysis were performed using the IGeoS package (http://seisweb.usask.ca/igeos).
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Figures

Figure 1. Sources (red circles) and receivers (blue triangles) with measured frequency-
dependent coda $d\ln(\text{Amplitude})/dt$ derivatives. Black stars are the PNEs, and small black triangles – 3-component receiver locations. Only $3^\circ$ grid cells containing sources or receivers were used in the inversion. Two PNEs used in Part I are indicated: Q4 - Quartz-4, and K3 – Kimberlite-3 (see also Figure 2).

Figure 2. Vertical-component record from PNE Kimberlite-3 (K3 in Figure 1), filtered within 0.5 - 8.0 Hz pass-band. The regional phases (labeled $P$, $Pg$, $S$, and $Lg$) are clear and observed to far source-receiver distances. Note the difference between the branches of $Lg$ in two directions from the PNE.
Figure 3. PNE Lg coda $Q$ inversion results: a) coda scattering amplitude $\lambda$; b) uncompensated geometrical spreading $\gamma$; 2) effective coda attenuation $Q_e^{-1} = \kappa/\pi$. In plot b), note the level of $\gamma_D = 0.008$ s$^{-1}$ which was suggested to discriminate between stable and active tectonic areas (Morozov, 2008). Major tectonic regions are indicated in plot b): BR – Baikal Rift; SC – Siberian Craton; MAH – Minusinsk-Aldan High; WSB – West Siberian Basin. PNE profiles
are labeled in plot a).
Appendices


In a recent article, Mukhopadhyay et al. (2008; referred to as MSMK below) used the $Q_c(f) = Q_0 n^\eta$ law (although using symbol ‘$n$’ for $\eta$ here) to measure the frequency-dependent coda attenuation in ~60-sec codas recorded from Chamoli earthquake in India. The resulting $Q_c(f)$ values were found to be strongly and positively dependent on both the frequencies and coda lapse times, which were the main points of the paper. The authors further related coda attenuation to scattering, and the lapse-time dependence – to a decrease of scattering with depth.

My concern about this article is that although based on well-established techniques (e.g., Aki and Chouet, 1975), it also lacked critical evaluations of the results and analysis of the limitations of the underlying assumptions. A complex set of assumptions was stated or implied, but their validity and impact on the results were not discussed. However, it appears that most of the presented conclusions were due to such inaccurate assumptions. In particular, the following points could be worth considering:

1) **Validity and sufficient accuracy of $r^1$ geometrical spreading** [eq. (1) in MSMK].
   The $r^1$ law cannot be perfectly valid for all areas, and uncertainties in this spreading have a major impact on the results (see below).

2) **Relation of $Q_c$ measured in the coda to the in situ properties of the crust and mantle.** Frequency-dependent coda $Q_c$ is really not much more than a coda shape parameter, and its association with distributions of scatterers within the lithosphere is subject to debate. $Q_c(f)$ is an “apparent” quantity (Aki, 1980), which means that both $Q_0$ and $\eta$ change when the geometrical spreading assumption is made differently. Thus, presenting $Q_0$ and $\eta$ as attenuation parameters at depth (Fig. 4 in MSMK) is misleading.
In addition, representing the bottom of the scattering ellipsoid (Figs. 4 and 5 in MSMK) as the “depth of investigation” is incorrect. If scattering decreases with depth, as suggested by the conclusions (and this should generally be true), the lowermost point of the ellipsoid [eq. (3) in MSMK] should have the smallest effect on $Q_c$. Also, the bottom part of the scattering ellipsoid has the smallest area for collection of scattered energy. Therefore, the bulk of scattered waves should originate closer to the surface for all lapse times. Thus the depth of the bottom of the scattering ellipsoid is not the depth of scattering, and the dependence of $Q_c$ on lapse times is unlikely caused by the increasing sampling depth. Perhaps the term “depth” should be replaced with “pseudo-depth”, as it is customary in resistivity imaging.

3) **Relation of $Q_c$ to scattering.** Because of the apparent character of $Q_0$ and $\eta$, $Q_c$ also cannot be unambiguously related to in situ scattering or differentiated from the intrinsic attenuation. As I show below, the data can be successfully explained without any scattering and intrinsic $Q$ at all.

4) **Adequacy of the uniform isotropic scattering model.** The underlying uniform half-space model with only $S$ waves propagating along straight rays and undergoing single scattering (Sato, 1977) is very useful but represents only a crude approximation when used to measure variations of scattering with depth. Conclusions drawn from using various models should be commensurate with their degrees of accuracy.

Unfortunately, this is just one of a number of studies in which curve-fitting seems to overshadow the analysis of the physics of coda excitation and of the character of attenuation measurements. I expect that the authors’ argument could be that at present, one has to make such kinds of approximations, and that many similar studies have been done by others. This is quite true; however, it would also be good to critically evaluate the effects of such assumptions and make conclusions that would be the least vulnerable.
Given the complexity of the problem and the limitations of current knowledge, a complete solution could be difficult; however, simple physical constraints still can be exercised to verify the validity of the results. The following two results from MSMK are easily testable and also most questionable:

1) All of the resulting $Q_c$ values (Fig. 3 in MSMK) are proportional to the frequency or increase even faster. However, note that for $Q = \alpha \omega = 2\pi \alpha f$, the basic amplitude decay formula [eq. (1) in MSMK] becomes:

$$A(f \mid t) = A_0(f)^{-\frac{t}{2\alpha}} \exp \left[ -\frac{t}{(2\alpha)} \right],$$

in which the path effect (i.e., the one related to $t$) does not depend on the frequency. This is not attenuation, but geometrical spreading! Therefore, whenever a near-proportional $Q(f)$ is detected, uncompensated geometrical spreading should be strongly considered. By extrapolating this observation, uncompensated geometrical spreading could also be suspected whenever frequency-dependent $Q$ is found.

2) All of the measurements in MSMK show $\eta \geq 1$ (Table 2 and Fig. 4 in MSMK). These values are too high compared to the theoretical predictions, as also noted in the paper. In general, scattering theory does not support values of $\eta > 1$, and therefore, observation of such values could point to some of the four assumptions above failing.

Noting these two points could help the authors to reconsider the data analysis and interpretation procedure.

Having finished with criticisms, I will now make an alternate suggestion for interpreting the data used in MSMK. Ironically, it only takes undoing one final processing step in order to see the assumption-free picture. In Figure 1a, I reproduce the authors’ $Q_c$ data (Table 1 and Figure 3 in MSMK). From this plot, $Q_c$ appears to quickly increase with the frequency, and this was the conclusion drawn by the authors. However, note that $Q_c$ was in fact obtained from the source- and path-uncompensated log-amplitude
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$\log \left[ \frac{A(f \mid t)}{A_0(f)} t^{-1} \right] = -\alpha(f) t$,  

(2)

in which $\alpha(f)$ was equated with $\pi f Q_c(f)$ by the authors. Quantity $a(f)$ in eq. (1) is the standard attenuation coefficient calculated after geometrical compensation. Thus, by reconstructing $\alpha(f)$ back from the values of $Q_c$ in Figure 1a (i.e., calculating $\alpha = \pi f / Q_c$), we find that $\alpha(f)$ actually only very weakly depends on the frequency for all five time lapse windows (Figure 1b).

In the $\alpha(f)$ plots, $Q$ should correspond to the frequency derivatives: $\pi / Q = d\alpha / df$, and therefore, the data presented in MSMK show little evidence of attenuation. The slopes of the two lines in Figure 1b suggest $Q_c = \infty$ at 10-s and $Q_c \approx 10000$ at 30-s lapse times. By contrast, the intercepts $a(f=0)$ are positive and decrease with lapse times. These intercepts measure the uncompensated geometrical spreading and show that the data are under-compensated by the $t^{-1}$ geometrical correction. Such under-compensation appears to be common in body-wave measurements, and values of $\gamma = \alpha(0) \approx 0.02 - 0.06 \text{ s}^{-1}$ are also typical for areas of active tectonics (Morozov, 2008). A comparison of Figures 1a and b shows that the strong frequency-dependent apparent $Q$ was created by multiplying the nearly-constant uncompensated geometrical spreading by the frequency: $Q_c \approx f / \pi \alpha$.

Under-compensated geometrical spreading means that coda energy decays faster than as $t^{-1}$ in the absence of attenuation. For local earthquake recordings, this may indeed be so, as indicated by Frankel et al (1990) from observations and modeling. In local coda measurements, faster coda decay could also come from scatterers concentrating at shallow depths [as opposed to uniformly distributed in Aki and Chouet’s (1975) approximation].

Decreasing $\alpha(f=0)$ values with increasing lapse times are also not surprising and are commonly observed. Coda envelopes often show flattening-out slopes (i.e., increasing apparent $Q_c$) with increasing lapse time, caused by the extinction of quicker-decaying scattered modes. In the case of local earthquake data, such modes could be the...
near-source or receiver reflections and surface waves. However, this varying slope is still a coda property, and its attributing to the subsurface requires caution.

The \(-\alpha(f)\) plot (Figure 1b) also shows that the amplitudes below \(\sim 5\) Hz are systematically reduced for all time lapse values. The reason for this is unclear and could deserve some investigation; however, this could be related to over-estimated spectral signatures of the earthquakes \(A_0(\omega)\) at these frequencies.

In conclusion, the authors of MSMK gave a very clear, detailed, and well-documented presentation of the data and inversion procedure which provided a good basis for discussion. Because the authors followed an already well-rooted paradigm of \(Q(f)\) measurement and interpretation, the arguments above may suggest that this paradigm requires modifications. I hope that this example demonstrated that looking at the original spectral amplitude data (Figure 1b) before dividing them by the frequency in order to produce the apparent \(Q_c\) (Figure 1a) yields important insights in the data. Most importantly, the observations above are completely free from the geometrical-spreading and coda-model assumptions, and thus the resulting measurements should be transportable and useful in interpretation. Several other examples from a broad range of attenuation studies are given in Morozov (2008).

References


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Figure 1. a) $Q_c$ versus frequency for several lapse times (symbols given in the legend), as in Figure 3 in MSMK; b) the same data in $\alpha = \pi/Q_c$ form [values of $-\alpha$ are plotted, to correspond to eq. (2)]. Note that $\alpha(f)$ is positive, practically independent of the frequency, and decreases with lapse times. The range of possible $\alpha(f)$ slopes is indicated by straight lines; however, both of them correspond to $Q_c$ exceeding 10000.