Seismic attenuation without $Q$ – II. Intrinsic attenuation coefficient

Igor Morozov

Department of Geological Sciences, University of Saskatchewan, Saskatoon, SK S7N 5E2 Canada

igor.morozov@usask.ca

Summary

As argued in Part I of this study, the quality factor ($Q$) generally does not represent a consistent property of the Earth’s medium. By contrast to $Q$, the attenuation coefficient (here denoted $\chi$) can be viewed as such a property. According to the different mechanisms of attenuation, $\chi$ can be subdivided into geometrical spreading (GS), intrinsic attenuation, and elastic scattering. However, when considering a realistic (variable and measured) GS, scattering becomes indistinguishable from the other two mechanisms of attenuation in practical observations. In the context of its observations and inversion, $\chi$ can also be subdivided into zero-frequency and frequency-dependent parts. For traveling waves, the observed apparent attenuation coefficients can be represented by ray-path integrals of the corresponding intrinsic properties.

Derivations of $\chi$ are given for two end-member theoretical models: 1) refraction in smoothly varying structures, and 2) plane-wave reflectivity at normal incidence. In both cases, the geometrical attenuation coefficient is frequency-independent and related to the variations in wavefront curvatures and averaged squared reflectivity, respectively. The examples show that the reported frequency-dependent $Q$ can often be related to significant geometrical-attenuation effects and not to a frequency-dependent rheology or scattering. These models also show that...
the zero-frequency attenuation limit can be used as a measure of GS variations, as suggested in several recent studies. The non-geometrical part of $\chi$ is directly related to elastic-energy dissipation (including short-scale scattering) within the Earth’s medium. These models form the basis for quantitative attenuation interpretation in realistic Earth structures.

**Key words:** attenuation coefficient; geometrical spreading; reflectivity; scattering; wave front; $Q$; seismic attenuation.

**Introduction**

In several recent papers (Morozov, 2008, hereafter referred to as M08; Morozov, 2009a, and in press I) and also in Part I of this study (Morozov, in review I, hereafter Part I), I argued that the conventional description of seismic attenuation based on the quality ($Q$) factor is insufficiently based on mechanical principles but relies on several theoretical conjectures, analogies, and often inaccurate assumptions. In consequence, the resulting $Q$ data and models may be prone of spurious frequency dependences and lead to ambiguous and overly complex interpretations. Instead of using the $Q$ paradigm, I suggested returning to another well-known description, which is the attenuation coefficient in either its spatial ($\alpha$) or temporal ($\chi$) forms. While removing the assumptions involved in the definition of $Q^{-1}$ and the corresponding uncertainties and artifacts, this description provides a simple and reliable basis for data analysis (M08; Morozov, in press I) and offers several far-reaching empirical generalizations (Morozov, in review II).
In this paper, I continue this analysis and present theoretical treatments of the attenuation coefficients caused by variations of geometrical spreading and reflectivity along the wave propagation path. Specifically, we’ll be seeking explanations for the exponential form of the wave-amplitude path factor inferred in M08 from heuristic data analysis:

$$\delta P = e^{-\chi t}.$$  \hspace{1cm} (1)

In this expression, $\chi$ is the observed (apparent) temporal attenuation coefficient, and $\delta P$ denotes the seismic amplitude corrected for source, receiver, and reference geometrical-spreading effects. Note that expression (1) represents the starting point of most attenuation studies, in which a specific form for the frequency-dependence of $\chi$ is usually utilized:

$$\chi(f) = \frac{\pi f}{Q(f)}.$$  \hspace{1cm} (2)

However, this form assumes that $\chi \to 0$ when $f \to 0$, which is too restrictive and in most cases incorrect, and this approximation often results in $Q(f)$ values that are nearly proportional to $f$. As argued in Morozov (2009a), such replacement of $\chi$ with $Q^{-1}$ leads to several problems of the $Q$-based model of attenuation. To avoid such problems, we do not use the restrictive model (2) and simply view $\chi(f)$ as an arbitrary function. In empirical data analysis, we start with isolating its zero-frequency limit $\gamma$ by denoting $\chi(f) = \gamma + f\kappa(f)$. As illustrated on multiple data examples (M08; Morozov, 2009a, in press I, and in review II) and numerical modeling of mantle Love waves in Part I, the dimensionless parameter $\kappa(f)$ turns out to be frequency-independent for most wave types and frequency bands. For comparisons to the conventional terminology, it can be transformed into an “effective” quality factor $Q_e = \pi'\kappa$. 

3
Note that the apparent $\chi$ is also closely related to the parameter $t^*$ often used in body-wave attenuation studies (e.g., Der and Lees, 1985): $\chi = \pi t^*/t$. Similarly to $t^*$, $\chi$ can be represented by a path average of the corresponding “intrinsic attenuation coefficient” $\chi_i$

$$\chi = \frac{1}{t} \int_{\text{path}} \chi_i d\tau.$$  

(3)

This new quantity ($\chi_i$) combines the local variations of geometrical spreading, scattering, and intrinsic attenuation. Of these three factors, the intrinsic attenuation is the one that definitely requires a frequency-dependent $\chi_i$ (compare to eq. 2). As argued in detail in Morozov (in press I), the other two factors can be separated only by making additional simplifications, such as frequency-independence of the geometrical spreading. The difficulty of their separation is related to the fundamental ambiguity in the definitions of the geometrical spreading and scattering. However, in many practical cases (including this paper) separation of these quantities is not required, and $\chi_i$ can be treated as a single medium property.

A most important observation from eqs. (1) and (3) is that $\delta P$ represents a path integral which can be rendered in either temporal or spatial form:

$$\delta P = \exp \left( - \int_{\text{path}} \chi_i d\tau \right) = \exp \left( - \int_{\text{path}} \alpha_i ds \right),$$  

(4)

where $s$ is the “ray” path length, and $\alpha_i$ and $\chi_i$ are is the corresponding spatial and temporal intrinsic attenuation coefficients. This shows that variations of geometrical spreading, scattering, and attenuation have similar characters and are accumulated over wave propagation paths. The exponential form possesses important general properties and similarities to ray-, wave-, and quantum-field mechanics. Below, after a brief overview of these properties, we derive
expressions for the intrinsic attenuation coefficients for two important end-member cases: 1) ray-theoretical limit in a smoothly variable medium, and 2) plane-wave reflectivity at normal incidence. These examples illustrate the relation of variable geometrical spreading and short-scale reflectivity to the intrinsic attenuation coefficient.

The analysis below shows that in the absence of intrinsic attenuation, the resulting cumulative attenuation coefficient \( \chi \) is non-zero and depends on the refracting or reflecting structures within which the propagation takes place. For refraction, the geometrical attenuation coefficients can be positive (corresponding to defocusing) or negative (focusing). However, for reflectivity, the geometrical attenuation coefficients are always positive. When transformed into the conventional quality-factor form by using eq. (2), such positive geometrical attenuation leads to values of \( Q \) quickly increasing with frequency. These predictions explain the causes of positively frequency-dependent \( Q \) observed in many datasets. Such \( Q(f) \) dependences were interpreted as spurious and related to inaccurate treatment of geometrical attenuation (M08; Morozov, 2009a, and in press I).

**Nomenclature of attenuation factors**

Equations (4) are very general and principally only state that energy decay \( \delta E \) during a short propagation time \( \delta t \) is proportional to \( \delta t \) and also to the current wave energy \( E \). This represents the perturbation- (scattering-) theory approximation, which is clearly valid for all processes involved in seismic wave attenuation, such as geometrical spreading, elastic scattering, rheological relaxation, and various forms of anelastic energy dissipation. In the same perturbation-theory sense (i.e., for small \( \delta E \)), contributions of all these processes in \( \alpha_i \) and \( \chi_i \) should be additive, and thus we can talk about attenuation coefficients associated with each of
them individually. However, separation of these contributions in observable quantities may be subtle and difficult. To approach such separation, we need to define and clarify the relevant terminology first.

The geometrical spreading (GS) is the most fundamental property which is either explicitly or implicitly present in all descriptions of attenuation, and it receives the most emphasis in the ongoing methodological debate (Xie and Fehler, 2009; Morozov, 2009b and in press I; Mitchell, in press; Xie, in press). Nevertheless, this quantity is also among the most uncertain in attenuation studies. Usually, the GS is described as the effect of elastic wave energy spreading within an expanding wavefront. However, wavefronts exist in practically no cases of interest, because they are destroyed by multi-pathing, triplications, reflections, mode conversions, and dispersion in heterogeneous structures. Only a few analytically tractable solutions exist, typically in uniform, isotropic models. Practical GS models often represent empirical generalizations (e.g., Frankel et al., 1990; Zhu et al., 1991) and disagree with the spreading-wavefront model. Yet again, overly simple functional forms, such as \( G(t) = t^\nu \), are often assumed for such models, making the concept of GS theoretically undefined and empirically inaccurate.

For a general and useable definition of GS, I propose the following: GS is the effect of background structure on seismic amplitudes in the absence of intrinsic attenuation and small-scale scattering. In terms of conventional terminology, the geometrical limit is attained by setting \( Q^{-1} = 0 \) everywhere within the structure. The key in this definition is the “deterministic” nature of the structure, meaning that GS is only one aspect of its more complete description. GS is therefore variable, unknown, and represents one of the most important goals of attenuation
analysis (M08). It can be described by a general space-time and frequency dependence of wavefield amplitudes, \( G(r, t, f) \), without regard for a any specific propagation model.

Contrary to the conventional assumptions of GS being “reasonably” closely approximated by simple theoretical models such as \( G(t) = t^\nu \), note that the errors of this approximation can absorb the entire frequency-dependent \( Q \) signal in most cases (M08; Morozov, in press I). Because of the lack of information about the 3D structure of the lithosphere, accurate modeling of realistic \( G(r, t, f) \) appears practically intractable. Nevertheless, GS can be easily measured and modeled by means of the attenuation-coefficient technique described here.

By contrast to the true GS, the background GS (BGS) is the best-known theoretical approximation to the attenuation- and scattering-free amplitudes, denoted \( G_0(r, t, f) \). It can be derived by theoretical or numerical modeling, or by empirical matching of observed amplitudes. The only requirement to it is that the \( G/G_0 \) ratio should be close to 1 for the time ranges of interest. This quantity is corrected for when calculating the path factor (1), and consequently the corresponding attenuation coefficient \( \chi_{BGS} = 0 \).

The geometrical attenuation (GA) coefficient represents the deviation of the true GS from its background model: \( \chi_{GA} = \chi_{GS} - \chi_{BGS} = \chi_{GS} \). Combined with the dissipation (\( \chi_d \)) and scattering (\( \chi_s \)) coefficients, \( \chi_{GA} \) comprises the total intrinsic attenuation coefficient in eq. (3): \( \chi_i = \chi_{GA} + \chi_d + \chi_s \). Scattering is the most difficult to define rigorously, because it is associated with “random” structural variations. In the existing treatments (Sato and Fehler, 1998), scattering is only considered in respect to featureless, uniform-space, isotropic backgrounds. With progressive increase of detail recognized in the background structure, scattering effects are eliminated and become absorbed by the empirical GS. On the other hand, for any approximation of the structure,
the residual short-scale scattering effects become indistinguishable from those of \( \chi_d \) if no special assumptions are made about the latter. For these reasons, I suggested that scattering attenuation (\( \chi_s \)) is observationally intractable (M08; Morozov, 2009a, and in press I). Nevertheless, scattering attenuation in uniform isotropic media is also the best-studied theoretically (Sato and Fehler, 1998) and provides many insights into the mechanisms of elastic energy dissipation. By including \( \chi_s \) in \( \chi_d \), the problem of decomposition of the intrinsic attenuation coefficient becomes:

\[
\chi_i = \chi_{GA} + \chi_d.
\]

Assuming that equations (3) can be inverted for \( \chi_i \), frequency-dependence provides the best clues for separating \( \chi_{GA} \) and \( \chi_d \) in this quantity. The zero-frequency attenuation coefficient, denoted \( \gamma_i \) in this paper, provides a good approximation for \( \chi_{GA} \). This is due to the approximate frequency-independence of \( \chi_{GA} \), as follows from the theoretical examples in this paper. Similarly to \( \chi_i \), \( \gamma_i \) is related to the corresponding observable (apparent) \( \gamma \) by eq. (3). Parameter \( \gamma \) (and therefore \( \gamma_i \)) correlates with tectonic structures (M08) and is predictable by elastic waveform modeling (Morozov et al., 2008, and Morozov, in press I), which shows that it is indeed dominated by GS.

From mechanical arguments in Part I, energy dissipation \( \chi_d \) should equal 0 at zero frequency, as also suggested by its traditional definition: \( \chi_d = \pi f Q^{-1} \). This \( \chi_d = O(f) \) character follows from dissipation occurring from the kinetic-energy part of the Lagrangian, which is itself proportional to \( f^2 \). Although a frequency-dependent ‘\( Q^{-1} \)’ in the above expression is expected, it should contain no singularity canceling the leading factor ‘\( f \)’ in \( \chi_d \). Unfortunately, the traditional \( Q_s \) associated with scattering often contains such, and even stronger singularities (Sato and
which intermix $Q$, with GS and make the whole description unstable (Morozov, 2009a).

Finally, to complete our nomenclature, once $\chi_d|_{f=0} = 0$, then $\kappa_i = \chi_d / f$ becomes a useful non-geometrical, or energy dissipation factor of the medium. If GS is frequency-independent, then $\kappa_i = (\chi_i - \gamma_i)/f$ combines the effects of intrinsic material absorption and scattering at scale-lengths not accounted for by the deterministic structural (and GS) model. For an analog to the conventional ‘$Q$’ terminology, parameter $Q_i = \pi \kappa_i$ can be used.

Finally, to complete our nomenclature, note that ratio $f_c = |\gamma_i|/\kappa_i$ represents the “cross-over frequency” (M08) at which the attenuation levels from the geometrical and dissipation/scattering mechanisms equal each other. For frequencies exceeding $f_c$, dissipation mechanisms dominate attenuation, and the apparent $Q$ usually shows weaker frequency dependence. For $f < f_c$, GS effects dominate, and $Q$ are strongly frequency-dependent (M08). Seismic albedo (Wu, 1985) can be derived from this property, as a ratio of the geometrical and total attenuation coefficients:

$$B_0 = \frac{\gamma_i}{\chi_i} = \frac{1}{1 + \frac{f}{f_c}}. \quad (5)$$

Note that as discussed above, we treat the geometrical attenuation ($\gamma_i$) as “scattering” (\(\pi Q_s^{-1}\)) in this context. However, similarly to $Q$, the use of this quantity may be complicated by its embedded frequency dependence, and $f_c$ should still be preferable for characterizing the relative contributions of the geometrical and dissipation factors.
Path-integral form of attenuation

Exponential path integrals (for example, “Feynman integrals”) similar to those in eq. (4) are broadly used for describing wave propagation in quantum mechanics and field theory. Such integral form is characteristic for “propagators” and arises from their fundamental factorization property requiring that some type of a “product” of two propagators also represents a propagator. For example, Green’s functions $G(x_1,x_2)$ of some field possess this propagator property, which in this case reads:

$$G(x_1,x_2) = \int G(x_1,x)G(x,x_2)d^3x.$$  \hspace{1cm} (6)

The dynamic ray theory (Červený, 2001) also illustrates the origins of exponential form (4). In this theory, factorization (6) arises from the multiplicative property of ray propagator $\Pi$ (eq. 4.4.86 in Červený, 2001), which in logarithmic form is:

$$\ln \Pi(R,S) = \ln \Pi(R,\tilde{Q}_N) + \sum_{i=N}^{1} \gamma(\tilde{Q}_i,\tilde{Q}_{i-1}),$$ \hspace{1cm} (7)

where $S$ is the source, $R$ is the receiver, and $Q_i$ and $\tilde{Q}_i$ are the incidence and emergence points at the $i$-th interface respectively (Figure 1), and

$$\gamma(\tilde{Q}_i,\tilde{Q}_{i-1}) = \ln \left[ \Pi(\tilde{Q}_i,\tilde{Q}_{i}) \Pi(Q_{i},Q_{i-1}) \right].$$ \hspace{1cm} (8)

In our notation (eq. 4), $dP$ corresponds to $\Pi(R,S)$, and $\gamma(\tilde{Q}_i,\tilde{Q}_{i-1})$ – to $\int \chi_i d\tau$, where the integral is taken from point $Q_{i-1}$ to $Q_i$ along the ray.
Partitioning of $\Gamma(\Omega_i, \Omega_{i-1})$ into the interface- and path-related factors in eq. (8) shows that both reflection/conversion and ray-bending effects influence the values of $\gamma_i$. Below, we derive exact expressions for such effects in two end-member examples:

1) Normal-incidence reflectivity, showing that $\chi_i$ is proportional to the gradient of the acoustic impedance;

2) Refraction in a medium with smoothly varying velocities. This example will show that $\chi_i$ is also related to the variations of wavefront curvature (i.e., to perturbations of the traditional geometrical spreading).

**Normal-incidence plane-wave reflection limit: $\chi_i$ and reflectivity**

To understand the relation of the in-situ attenuation coefficient to the properties of the medium, it is instructive to analyse its properties in a simple 1D medium. For plane-wave propagation, the theoretical geometrical spreading factor equals 1; however, reflections in heterogeneous medium cause deviations from this value. Because transmission coefficients can be completely described by the reflection coefficient series, the geometrical part of the attenuation coefficient should also be expressed through reflectivity. In fact, as shown below, the geometrical attenuation coefficient equals half the average of squared reflection coefficient.

To begin, consider a boundary between two layers of acoustic impedances $Z_{j-1}$ and $Z_j$ (Figure 2). The specific expression for impedance depends on the local properties of the medium, wave type, and the angle of its incidence on the boundary. For a $P$ wave or $S$ wave in an attenuative medium at normal incidence, $Z$ is complex-valued:
\[ Z = \rho V \left(1 + \frac{i}{2Q_i}\right), \]  
(9)

(Lines et al., 2009; Morozov, 2009c), where \( \rho \), \( V \), and \( Q_i^{-1} \) are the mass density, wave velocity, and parameter of intrinsic attenuation, respectively. Note that although I generally argue that \( Q \) cannot be considered as a medium property (Part I), this parameter is retained here for convenience of comparison to the current terminology. Whenever a ‘\( Q^{-1} \)’ is mentioned, this quantity should be understood as ‘\( \chi_d / \pi \)’, where \( \chi_d \) is the energy dissipation coefficient above. This terminology is acceptable as long as a single wave type (e.g., a plane wave) and frequency are being considered. Also note that the sign of \( \text{Im}Z \) in eq. (9) is opposite to the one arising from the visco-elastic theory, which assumes negative values of \( \text{Im}V \) (e.g., Aki and Richards, 2002).

At oblique incidence, \( P \) and \( S \) waves interact upon reflection and conversion, and \( Z \) becomes a matrix quantity (Morozov, in press II); nevertheless, the general conclusions below remain valid.

Considering for simplicity the normal-incidence case and denoting the displacement in the incident wave by \( u \), the displacements in the reflected and transmitted waves become \((- R_i u)\) and \( T_i u\), respectively (Figure 2), where, \( R_i \) is the reflection coefficient,

\[ R_j = \frac{Z_j - Z_{j-1}}{Z_j + Z_{j-1}}, \]  
(10)

and \( T_j = 1 - R_j \) is the transmission coefficient,

\[ T_j = \frac{2Z_{j-1}}{Z_j + Z_{j-1}}. \]  
(11)

The corresponding energy transmission coefficient is:
\[ T_{E,j} = \frac{Z_j}{Z_{j-1}} T^2 = \frac{4Z_{j-1}Z_j}{(Z_j + Z_{j-1})^2}, \]  

(12)

and the energy reflection coefficient equals \( R_{E,j} = 1 - T_{E,j}. \)

For small impedance contrasts, these coefficients become:

\[ R_j = \frac{1}{2} \delta_j (\ln Z), \]  

(13)

\[ T_j = 1 - \frac{1}{2} \delta_j (\ln Z), \]  

(14)

\[ T_{E,j} = 1 - \frac{1}{4} \delta_j (\ln Z)^2 = 1 - |R_j|^2, \]  

(15)

were \( \delta(\ldots) \) denotes the contrast of the corresponding parameters across the \( j \)-th boundary.

Switching to a continuous \( Z(t) \) description, the impedance contrasts over an infinitesimal propagation time interval \([t, t+\delta t]\) can be considered small, and therefore from eq. (15),

\[ \ln T_E \approx -\sum_{j=1}^{N} |R_j|^2 = -\int_{t}^{t+\delta t} |r|^2 \, d\tau, \]  

(16)

where \( r(t) \) is the root-mean square (RMS) density of reflectivity.

Equation (16) only gives the transmission loss caused by reflections on the boundaries passed by the wave between propagation times \( t \) and \( t + \delta t \). The intrinsic medium attenuation over the same time interval leads to an additional energy decay:

\[ \ln T_E \approx -\int_{t}^{t+\delta t} |r|^2 \, d\tau - 2f \int_{t}^{t+\delta t} \kappa \, d\tau, \]  

(17)
where $\kappa_i$ is the non-geometrical attenuation factor.

Transmission coefficients combine multiplicatively with propagation time, and consequently, their logarithms are additive. Therefore, for a wave traversing $N$ boundaries in finite propagation time $t$, the energy density $E(t)$ is (Figure 2)

$$E_N = E_0 \prod_{j=1}^{N} T_{E,j} = E_0 \exp \left[ \sum_{j=1}^{N} \ln T_{E,j} \right], \quad (18)$$

or in terms of the continuous reflectivity function $r(t)$:

$$E(t) = E(0) \exp \left\{ -\int_{\tau=0}^{t} \left[ |r|^2 + 2\kappa_i f \right] d\tau \right\}. \quad (19)$$

This expression shows that the logarithm of the transmitted energy loss is a path integral:

$$\ln E(t) - \ln E(0) = -\int_{0}^{t} \left[ |r|^2 + 2\kappa_i f \right] d\tau, \quad (20)$$

and consequently the temporal attenuation coefficient is

$$\chi_i = -\frac{1}{2} \frac{d \ln E(t)}{dt} = \frac{|r|^2}{2} + \kappa_i f. \quad (21)$$

The corresponding spatial attenuation coefficient equals $\alpha = \chi/V$:

$$\alpha_i = \frac{|r|^2 + \kappa_i}{2V} = \frac{|r_{\text{spatial}}|^2}{2} + \frac{\kappa_i f}{V}. \quad (22)$$

Note the difference between the temporally- and spatially-averaged RMS reflectivities denoted by $r$ and $r_{\text{spatial}}$, respectively.
Thus, in the case of 1D acoustic-wave propagation, the geometrical attenuation coefficient equals half of the corresponding path-averaged squared reflectivity. As path-averaged properties, $\alpha$ and $\chi$ can be evaluated over finite propagation-time intervals, and therefore they can be time-dependent.

Note that when $\kappa_i = 0$, the resulting $\alpha_i$ or $\chi_i$ become associated with the geometrical attenuation (more precisely, geometrical forward scattering in this case). In the approximation considered here (normal incidence and absence of multiple reflections), these geometrical $\alpha$ or $\chi$ are independent of the frequency and length of the incident wave.

If multiple reflections are present, frequency-dependent effects (tuning) should arise even in the geometrical limit. Such effects should likely have the form of resonance peaks rather than a continuous trend with frequency. Interestingly, such peaks can be identified in local-earthquake coda $\chi(f)$ data (Figure 3). Along with the linear $\chi(f)$ trends discussed in detail in M08 and Morozov (in press I, and in review II), these measurements show characteristic “spectral scalloping,” with peaks and troughs located at approximately 1.0-1.5, 6, 12, and 24 Hz. Consistent pattern of these variations in different areas, as well as near 1-octave separation suggest that these amplitude variations could represent resonant oscillations caused by common crystal structures. Most likely, these oscillations are caused by layering within sedimentary rock sequences.

**Smooth-medium limit: $\chi_i$ and wavefront curvature**

In the absence of interfaces and caustics, the geometrical spreading (GS) is caused by variations in the waveform curvature (Figure 1). In the dynamic ray theory, this curvature is measured by the trace of wavefront curvature matrix, $H = \frac{1}{2} \text{tr} \ K$, which is obtained from second
derivatives of the travel-time field $T$ in respect to the wavefront-orthonormal coordinates $y_k$ (eq. 4.6.15 in Červený, 2001):

$$K_{ij} = V \frac{\partial^2 T}{\partial y_i \partial y_j},$$  \hspace{1cm} (23)

where $V$ is the wave velocity. Curvature $H$ is related to the ray-theoretical GS by the following differential equation (eqs. 4.10.28-29 in Červený, 2001):

$$H = L^{-1} \frac{dL}{ds},$$  \hspace{1cm} (24)

where $L$ is the GS denominator and $s$ is the ray arc length. The solution to this equation relating $L(R)$ at the receiver to $L(S)$ at the source is

$$L(R) = L(S) \exp \left( \int_S^R H ds \right),$$  \hspace{1cm} (25)

which again has the exponential path-integral form of the path factor in eq. (4). Ratio $G = L(S)/L(R)$ represents the desired GS factor, which equals $G_0 \delta P = G_0 e^{-\alpha s}$, where $G_0$ is the background approximation for the GS in eq. (1). In the presence of intrinsic attenuation $\kappa_i$, path factor becomes

$$\delta P = \frac{1}{G_0} \frac{L(S)}{L(R)} \exp \left( -f \int_S^R \frac{\kappa_i ds}{V} \right),$$  \hspace{1cm} (26)

and consequently the spatial attenuation coefficient is

$$\alpha_i = H - \ln G_0 + \frac{\kappa_i}{V_i} f,$$  \hspace{1cm} (27)
with the corresponding equation for $\chi_i$. This expression shows that for smoothly refracting waves, $\alpha_i$ contains a frequency-independent “geometrical” part ($H-\ln G_0$), which equals the difference of the actual wavefront curvature from the one predicted by the GS law selected as the background reference.

**Discussion: separation of geometrical spreading, scattering, and dissipation**

Considering the separation of the three contributions to the observed attenuation coefficients $\alpha$ and $\chi$, the models above show that it can be based on their frequency dependence. For both sparse and multiple-free (“white”) reflectivity and smoothly bending rays, the observed geometrical attenuation ($\chi_{Q^{-1}=0}$) is frequency-independent, and consequently,

$$\chi_{Q^{-1}=0} = \chi_{f=0} = \gamma.$$

As suggested in M08 from empirical data observations, by isolating the frequency-dependent term, we can define the dimensionless “effective attenuation quality” factor $Q_e = \kappa/\pi$ (see Part I), where:

$$\kappa = \frac{\chi(f) - \gamma}{f}.$$  \hspace{1cm} (28)

For frequency-independent geometrical attenuation (as in the examples of this paper), $\kappa$ is directly related to the intrinsic attenuation within the Earth (compare to eq. 3):

$$\kappa = \frac{1}{t} \int_{path} \kappa_i d\tau.$$  \hspace{1cm} (29)

A similar expression relates the observed zero-frequency attenuation to its intrinsic counterpart:
\[ \gamma = \frac{1}{t} \int_{path} \gamma_i d\tau. \]  

(30)

According to the conventional terminology, the attenuation-coefficient expressions (21) and (27) can also be recast in terms of some cumulative “medium \( Q \):”

\[ Q^{-1} = \frac{\kappa}{\pi f} = \frac{\gamma}{\pi f} + Q_i^{-1}, \]

(31)

However, interpretation of this quantity literally as an “attenuation factor” may be deceptive. Although neither of the models used in this paper contains dependences on the incident-wave frequency, \( Q^{-1} \) is nearly always frequency-dependent. In particular, if we set \( Q_i^{-1} = 0 \), the geometrical attenuation (first term in the r.h.s. of eq. 31) leads to the “scattering \( Q \):”

\[ Q_s^{-1} = \frac{\gamma}{\pi f}, \]

(32)

Although the concept of \( Q_s \) was used in many studies (e.g., Dainty, 1981), this definition still appears problematic because it misrepresents geometrical spreading as “random scattering.” In consequence, the corresponding spurious dependence \( Q_s \propto f \) develops. Although such \( Q(f) \) dependences are reported in nearly every issue of seismological journals, they result primarily from the definition of parameter \( Q_s \). In reality, such dependences might simply mean that the geometrical attenuation is positive (M08; Morozov, 2009a, and in review I). Because of their contamination by geometrical effects, the values of \( Q \) determined from formula (31) at 1 Hz are often 20-30 times lower than those corresponding to the true intrinsic attenuation (M08). The degree of this discrepancy depends on the ratio of the dominant observation frequency \( f \) to the
“cross-over” frequency $f_c$ discussed in the first section. In low-frequency observations ($f/f_c < 1$) the under-estimation of $Q_i$ is particularly strong.

The use of $Q_s$ to describe wave scattering inherits all the general problems of using $Q$ for seismic waves, which are discussed in Part I. In addition, the examples above demonstrate that $Q_s$ is particularly inadequate and may be misleading when used to replace $\chi$ and $\alpha$. The “$Q_s$” terminology would describe bending rays, gradual variation of $|r|^2$, or a deterministic impedance-contrast (reflectivity) series as stochastic processes of “random scattering” in an otherwise uniform background structure. This picture is obscure, by far incomplete, and distracts attention from the true, first-order effects of the large-scale structure. Attenuation should be measured after the background structure is correctly accounted for, and $Q$ should not mask any inaccurate knowledge of this structure.

On the other hand, the information available about the background structure is always approximate, the structure itself is variable, and therefore its effects cannot be accurately accounted for by any modeling. When using the attenuation-coefficient methodology, the resulting values of $\alpha_i$ (or $\chi_i$) inverted from amplitude-attenuation data correctly represent the deviations of either the geometrical spreading (eq. 27) or reflectivity (eq. 22) from the background levels assumed in constructing the corrected path factor (1). Formal separation of these two factors may be difficult, because, for example, a “smoothly varying” medium in reality always contains a large number of short-scale contrasts in mechanical properties. However, the smooth-medium, ray-theoretical approximation is still useful in the zones of relatively low reflectivity, such as the crystalline crust between the sedimentary cover and the Moho. Thus, for example, in local-earthquake coda attenuation studies (e.g., Aki, 1980; Morozov, in press I) the observed variations in $\chi_i$ should likely be related to ray bending within the crust and reflectivity
near the Moho and within the sedimentary layers. Note that indeed, the upper-crustal reflectivity can explain predominant observations of positive $\chi$ (and consequently positive $Q(f)$ dependencies) in lithospheric studies (M08; Morozov, in review II).

Finally, note that as argued in Part I, despite its old history and established use in seismology, “quality factor” still represents an inadequate terminology when applied to describing an elastic medium with attenuation. Symbols $Q_e$ and $Q_i$ have little to do with the “quality” of oscillators embedded in the medium, and they were used in this paper only as a tribute to the current terminology. These parameters simply represent the frequency-dependent parts of the observed and intrinsic attenuation coefficients, respectively. In fact, as it can be seen from the original use of $Q$ in seismology (e.g., Knopoff, 1964), $Q$ always enters all expressions for observable quantities through a combination with frequency, $\pi f Q^{-1}$, which is the frequency-dependent part of the attenuation coefficient. The only significant difference of the present approach consists in noting that the attenuation coefficient also has a zero-frequency limit ($\gamma$) related to structural effects, such as ray-bending or reflectivity. Re-combining $Q^{-1}$ and $f$ into $\chi$ removes the ambiguities of the $Q$-based model and leads to a simpler and more reliable interpretation.

**Conclusions**

Unlike the $Q$ quantity examined in Part I of this study, the attenuation coefficient represents a consistent way for describing the attenuation properties of the Earth’s medium. For traveling waves, this description leads to exponential path-integral formulation similar to the perturbation and scattering theory. The attenuation coefficient can be subdivided into contributions from geometrical spreading, intrinsic attenuation, and elastic scattering. However,
when considering a realistic (variable) geometrical spreading, scattering becomes indistinguishable from the other two mechanisms of attenuation in practical observations.

From the viewpoint of its observations and inversion, attenuation coefficient can be subdivided into the zero-frequency and frequency-dependent parts. In several practical cases, the zero-frequency attenuation can be interpreted as a measure of the variable geometrical spreading, called “geometrical attenuation.” The non-geometrical part of the attenuation coefficient is directly related to the elastic-energy dissipation (including short-scale scattering) within the Earth’s material.

In two theoretical models, the geometrical attenuation coefficient is shown to be frequency-independent and related: 1) to the averaged squared reflectivity for plane wave waves at normal incidence; 2) to the variations of wavefront curvature for waves refracting in smoothly varying structures. In both cases, the often-reported frequency-dependent $Q$ is shown to be related to significant geometrical-attenuation effects and not to a frequency-dependent rheology or scattering. These models form the basis of quantitative interpretation of attenuation in realistic Earth structures.

**Acknowledgments**

I thank Anton Dainty for his encouragement at the early stages of this research. Many stimulating discussions with Bob Nowack, Michael Pasyanos, Paul Richards, Scott Phillips, and Jack Xie, and comments by Brian Mitchell and Bill Walter have helped in improving the argument. This research was partly supported by Canada NSERC Discovery Grant RGPIN261610-03.
References


Morozov, I.B., 2009b. Reply to “Comment on ‘Thirty Years of Confusion around ‘Scattering


Morozov, I.B., in review II. Attenuation coefficients of Rayleigh and *Lg* waves, *J. Seismol*.


Xie, J., in press. Title not yet finalized – debate of Morozov’s “On the causes of frequency-dependent apparent seismological *Q’*, *Pure Appl. Geophys*.

Figures

Figure 1. Factors comprising the ray propagator in a layered medium (eq. 7). Geometrical spreading is related to the ratio of wavefront curvatures (grey dashed lines) at the receiver \((R)\) and source \((S)\).

Figure 2. One-dimensional plane-wave reflection-transmission problem. Solid lines are reflectors, dashed lines – incident-wave wavefronts at times \(t\) and \(t + \delta t\), respectively. Multiple reflections are ignored.
Figure 3. Attenuation coefficients derived from local-earthquake coda data by Aki (1980). Labels indicate seismic stations: PAC – central California, OIS – western Japan, TSK – central Japan, and OTL – Hawaii. Note the interpreted linear $\chi(f)$ trends (dashed lines with $Q_e$ values in labels) and spectral amplitude oscillations common to all four areas (grey block arrows).