Exact elastic P/SV impedance

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ABSTRACT

Several extensions of the concept of acoustic impedance (AI) to oblique incidence exist and are known as elastic impedances (EI). These quantities are constructed by heuristic integrations of reflectivity series but still involve approximations and do not represent a unique medium property. Nevertheless, for unambiguous interpretation, it is desirable to have an EI that would (1) be a mechanical property of the medium and (2) yield exact reflection coefficients at all angles of incidence. Here, such a definition is given for P- and/or SV-wave propagation in an arbitrary isotropic medium. The exact elastic P/SV impedance is a matrix quantity and represents the differential operator relating the stress and strain boundary conditions. With the use of the matrix form of the reflectivity problem, no approximations are required for accurate modeling of reflection (P/P and SV/SV) and mode-conversion (P/SV and SV/P) coefficients at all angles and for any contrasts in elastic properties. The matrix EI can be computed from real well logs and inverted from ray-parameter-dependent seismic reflectivity. Known limiting cases of P- and S-wave acoustic impedances are accurately reproduced and the approach also allows the extension of the concept of impedance to an attenuative medium. The matrix impedance readily lends itself to inversion with uncertainties typical of the standard acoustic-impedance inversion problem.

INTRODUCTION

Impedance is the key quantity used for characterizing reflection and transmission of seismic waves. For P-waves at normal incidence, the acoustic impedance (AI) is usually taken as the product of density ($\rho$) and velocity ($V$) of the medium:

$$AI = \rho V.$$  

Several extensions of this formula to oblique incidence were proposed. In these extensions, the impedance was invariably constructed as a quantity $Z$ so that the reflection coefficient $r_{1,2}$ from a boundary of two media was solely determined by the ratio of their impedances, $z_{2,1} = Z_2/Z_1$:

$$r_{1,2} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{z_{2,1} - 1}{z_{2,1} + 1}. \quad (2)$$

Consequently, impedance is always defined with an uncertainty of an arbitrary scalar, which is usually selected so that $Z$ reduces to AI at normal incidence.

All impedance measures proposed to date attempt to achieve two goals: (1) inverting the series of reflectivity values (equation 2) for a $Z$ series and (2) giving $Z$ the sense of an intrinsic elastic property of the propagation medium. The first of these goals is easily achieved by using one of the following approximations for impedance contrast:

$$z_{i+1,i} = \frac{1 + r_{i,i+1}}{1 - r_{i,i+1}} \approx 1 + 2r_{i,i+1} \approx \exp(2r_{i,i+1}), \quad (3)$$

and recursive multiplication,

$$Z_i = Z_0 \prod_{k=1}^{i} z_{k,k-1}. \quad (4)$$

However, the second goal is unachievable in such an inversion because elastic P/SV-wave reflection and conversion cannot be rigorously described by a single scalar parameter. In addition, the reflectivity-series integration (equations 3 and 4) can be unstable (e.g., Francis, 1997) and often requires regularization. Reflectivities only constrain the ratios of impedances across each interface and cumulative values of $Z$ resulting from multiplying hundreds of factors in equation 4 can drift in complex ways depending on the approximations, data, numerical noise, and the algorithms used. This drift can include variations with depth and incidence angles, which can be difficult to control. The variety of the proposed forms of elastic impedance (EI) outlined in the following section illustrates this uncertainty.

The debate of the differences between the various EI approximations might lie largely within the uncertainties of such ad hoc inte-
gration of reflectivities. All of the existing EI models rely on assumed parameter correlations and do not correspond well to real observation environments, in which deviations from background trends are typically of primary interest. Thus, the two fundamental problems of such approaches are (1) deriving the impedance from reflectivities whereas it should arise from the boundary conditions and only lead to reflectivities (e.g., VerWest et al., 2000) and (2) treating EI as a scalar quantity whereas for the welded-interface boundary conditions the impedance should be a matrix incorporating P- and SV-waves simultaneously.

Nevertheless, in this paper, I show that reflectivity at any incidence angle can be readily presented in a rigorous impedance form by emphasizing the two physical principles above. Although the new expression for EI and the reflectivity values (equation 2) are in matrix form, they still can be easily numerically evaluated and inverted similarly to the existing heuristic EIs and AI. Because of its rigorous nature, the new matrix EI is free from any assumptions and approximations, leading to accurate reflection/conversion equations at all incidence angles, and is unambiguously related to mechanical properties of the medium.

A striking general consequence from the realization that EI should be an elastic property of the rock is that along with the ray parameter, it should depend on only two elastic parameters (for example, products \(4k\) and \(\mu k\) of the Lamé parameters \(\lambda\) and \(\mu\) and the vertical wavenumber \(k\)) and not three \((V_p, V_s,\) and \(\rho)\), as in all existing EI definitions. Density \(\rho\) is part of neither Hooke’s law nor the boundary conditions. Clearly, finding a two-parameter dependence from the data should be a much easier task and the results should also be more reliable and portable. Below, after a brief discussion of what appears to be the common weaknesses of the existing EI measures, I present a new “substantive” EI definition in the form of a matrix function of AI and the \(V_s/V_p\) ratio and compare it to several known impedance solutions.

**Existing EI measures**

Connolly (1999) introduces the initial concept of EI, which is currently in broad use. This concept describes the dependence of P-wave reflectivity on the incidence angle \(\theta\),

\[
EI(\rho, V_p, V_s|\theta) = \rho V_p (1 - 4k \sin^2 \theta V_p^2 + 2 \sin^2 \theta V_s^2 - 8k \sin^2 \theta),
\]

where \(V_p\) and \(V_s\) are the P- and S-wave velocities, respectively, \(\rho\) is the density, and \(K = (V_s/V_p)^2\), which is assumed to be constant across the reflecting boundary. However, the assumption that \(K\) is constant is difficult to justify in cases of practical interest. Another serious problem with EI is that its dimensionality does not correspond to impedance and varies with \(\theta\). Variation ranges of EI with \(\theta\) also strongly depend on the measurement units (e.g., VerWest et al., 2000); therefore, it cannot be considered a constitutive property of the medium.

To work around the dimensionality problem, Whitcombe (2002) proposes a normalized EI, which is the EI divided by itself measured at some reference level and scaled by AI,

\[
EI_{N}(\rho, V_p, V_s|\theta) = \rho V_p \left(\frac{\rho}{\rho_0}\right)^{-4k \sin^2 \theta} \times \left(\frac{V_p}{V_{p,0}}\right)^{\tan^2 \theta} \left(\frac{V_s}{V_{s,0}}\right)^{-8k \sin^2 \theta}.
\]

Martins (2006) further extends this concept to the cases of weak anisotropy and Mallick (2007) studies its stability with respect to tuning and the effects of multiples. Whitcombe et al. (2002) propose an extended EI by modifying certain functional dependencies in twoterm EI in equation 6 and designing empirical rules to highlight the contributions of fluids and rock types therein. However, despite its many useful applications, EI still hardly corresponds to a mechanical property of the medium: first, because of its relative character and, second, because the integration of reflectivity is carried out along nonphysical paths of constant \(\theta\).

To relax the above limitations, Ma (2003) proposes the raypath EI (RI), which was constructed by integrating the reflectivity along an actual reflection path with ray parameter (movc) \(\rho\):

\[
RI(\rho, V_p, V_s|\rho) = \frac{\rho V_p}{\sqrt{1 - \rho^2 V_p^2}} \exp[-2(\gamma+2)V_s^2 p^2].
\]

In this equation, the \(V_s/V_p\) ratio no longer needs to be constant; however, the density is taken to be functionally related to \(V_s\), as \(\rho = \rho_0 V_s^2\) (Potter et al., 1998) and small conversion angles were also assumed, \(p V_s \ll 1\). Santos and Tygel (2004) give another form of RI:

\[
RI(\rho, V_p, V_s|\rho) = \frac{\rho V_p}{\sqrt{1 - \rho^2 V_p^2}} \exp[-4V_s^2 p^2 \rho_0].
\]

At \(p V_s \ll 1\), these approximations are equivalent and correspond to Bortfeld’s (1961) formula but they are differently extrapolated to greater \(\rho\) values. VerWest et al. (2000) and VerWest (2004) also propose two similar RI forms such as

\[
RI(\rho, V_p, V_s|\rho) = \frac{\rho V_p}{\sqrt{1 - \rho^2 V_p^2}} \exp[-4V_s^2 p^2 \rho_0].
\]
Such a property is proposed in the following sections.

**P/SV impedance matrix**

To derive the general P/SV impedance, first consider an elastic wavefield in the vicinity of a planar boundary separating two half-spaces with uniform elastic properties (Figure 1). From two components of displacement, the nonzero components of strain can be expressed as

\[
\begin{pmatrix}
\varepsilon_{zz} \\
\varepsilon_{xz}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \partial_z \\
\partial_x
\end{pmatrix} \begin{pmatrix}
u_z \\
u_x
\end{pmatrix}.
\]

(11)

From Hooke’s law, the stress tensor is given by \( \sigma_{ij} = \lambda \varepsilon_{ij} + 2 \mu \varepsilon_{ij} \), where \( \lambda \) and \( \mu \) are the Lamé constants and \( \delta_{ij} \) is the Kronecker symbol. Therefore, the two components of traction on the boundary between the two half-spaces are

\[
\begin{pmatrix}
\sigma_{zz} \\
\sigma_{xz}
\end{pmatrix} = \begin{pmatrix}
(\lambda + 2 \mu) \partial_z \\
\mu \partial_x
\end{pmatrix} \begin{pmatrix}
u_z \\
u_x
\end{pmatrix}.
\]

(12)

From this equation, the impedance operator relating the displacement components to stress can be defined as

\[
\bar{Z} = -\begin{pmatrix}
(\lambda + 2 \mu) \partial_z \\
\mu \partial_x
\end{pmatrix}.
\]

(13)

As shown below, the impedance matrix \( \bar{Z} \) allows solving the stress and displacement boundary conditions in terms of \( \bar{Z} \) alone, as in equation 2. In the absence of attenuation, this impedance is reactive (i.e., imaginary), which makes it a correct analog to the electric-line impedance. However, in acoustics, the conventional definition of impedance is a real-valued scalar quantity \( Z \), which is the ratio of normal stress to particle velocity. \( Z \) is implicitly present in this equation through the dependence of \( k \) in \( \omega \). Also, from Snell’s law and the stationarity of the wave equation, parameters \( \zeta = k \sin \theta \) and \( \omega \) are common to all converted and reflected wave modes.

For a P-wave at normal incidence (\( \theta = 0 \)), the impedance given by equation 15 becomes

\[
Z = \frac{\lambda k}{\omega} \begin{pmatrix}
1 + 2 \mu' & 0 \\
0 & \mu'
\end{pmatrix} = \rho V_P \begin{pmatrix}
1 & 0 \\
0 & \frac{V_S^2}{V_P^2}
\end{pmatrix},
\]

(16)

whose first element corresponds to the conventional P-wave impedance. If attenuation is present (i.e., \( u_{ix} \propto \exp(-i \omega t + ikz) \)), equation 13 yields (Morozov, 2009)

\[
Z = \rho V_P \begin{pmatrix}
1 + \frac{i}{2 Q_P} & 0 \\
0 & \frac{V_S^2}{V_P^2} \left( 1 + \frac{i}{2 Q_P} \right)
\end{pmatrix}.
\]

(17)

In the acoustic limit (\( \mu = 0 \)) with no attenuation but at an arbitrary incidence angle \( \theta \),

\[
Z = \begin{pmatrix}
\rho V_P \cos \theta & \rho V_P \sin \theta \\
0 & 0
\end{pmatrix}.
\]

(18)

Because in this case \( u_x/u_z = \tan \theta \), the conventional (scalar) acoustic impedance represents the ratio of pressure to velocity perpendicular to the boundary:

\[
Z_{AI}(\theta) = \frac{\sigma_{zz}}{i \omega u_z} = \left( Z \begin{pmatrix} 1 \\ \tan \theta \end{pmatrix} \right)_{\zeta} = \frac{\rho V_P}{\cos \theta}.
\]

(19)

which is the well-known AI expression valid at all angles (Brekhovskikh, 1980).

The simple matrix impedance equations 13 and 14 rigorously cover all cases of P/SV-wave propagation at any incidence angles and even for nonplanar waves. Therefore, equations 5–9 might be unnecessary; however, to use the matrix forms of \( Z \) we need to see how the reflectivity equation 2 changes in the case of oblique P/SV-waves.

**P/SV reflectivity in matrix form**

Here, I use the scattering-matrix approach by Aki and Richards (2002) to show that all formulas for reflection and transmission coefficients

\[
\begin{align*}
&u = B_0 \exp(i k \cos \theta_3 z) \\
u = B_0 \exp(-i k \cos \theta_3 z) \\
u = A_3 \exp(i k \cos \theta_3 z) \\
u = A_3 \exp(-i k \cos \theta_3 z) \\
u = A_2 \exp(i k \cos \theta_2 z) \\
u = A_2 \exp(-i k \cos \theta_2 z) \\
u = B_2 \exp(i k \cos \theta_2 z) \\
u = B_2 \exp(-i k \cos \theta_2 z)
\end{align*}
\]

Figure 1. Model geometry and notation. Black arrows indicate P-waves, gray arrows indicate SV-waves. Thick arrows indicate positive displacement directions. Factors dependent on \( z \) in the equations for P- and SV-wave displacements are shown. Second subscripts of \( k \) and \( \theta \) indicate the propagation media.
coefficients can be expressed in terms of impedance matrices of the type given by equation 13. Consider incident, reflected, and transmitted P- and/or SV-wavefields (subscripts inc, r, and t, respectively), with the following components of displacement:

$$\begin{pmatrix} u_z \\ u_x \end{pmatrix}_{\text{inc}} = U_{1,+} \begin{pmatrix} A_{\text{inc}} \\ B_{\text{inc}} \end{pmatrix},$$  \hspace{1cm} (20)

$$\begin{pmatrix} u_z \\ u_x \end{pmatrix}_r = U_{1,-} \begin{pmatrix} A_r \\ B_r \end{pmatrix},$$

$$\begin{pmatrix} u_z \\ u_x \end{pmatrix}_t = U_{2,+} \begin{pmatrix} A_t \\ B_t \end{pmatrix},$$  \hspace{1cm} (21)

where subscripts \(m = 1\) and \(2\) indicate the corresponding medium (Figure 1). In these equations, the downward and upward “propagator” matrices \(U_{m,+}(z,x,t)\) and \(U_{m,-}(z,x,t)\) for these plane waves are

$$U_{m,+}(z,x,t) = e^{-i\omega t + iz},$$

$$\begin{pmatrix} \sigma_{zz} \\ \sigma_{zt} \end{pmatrix}_w = -Z\begin{pmatrix} u_z \\ u_x \end{pmatrix}_w = -Z\begin{pmatrix} A_w \\ B_w \end{pmatrix},$$  \hspace{1cm} (22)

On the welded boundary of the two media, components of the displacement and stress are continuous,

$$\begin{pmatrix} u_z \\ u_x \\ \sigma_{zz} \\ \sigma_{zt} \end{pmatrix} = \begin{pmatrix} u_z \\ u_x \\ \sigma_{zz} \\ \sigma_{zt} \end{pmatrix},$$  \hspace{1cm} (23)

which can be written as

$$\begin{align*}
U_{1,+} \begin{pmatrix} A_{\text{inc}} \\ B_{\text{inc}} \end{pmatrix} + U_{1,-} \begin{pmatrix} A_r \\ B_r \end{pmatrix} &= U_{2,+} \begin{pmatrix} A_t \\ B_t \end{pmatrix}, \\
S_{1,+} \begin{pmatrix} A_{\text{inc}} \\ B_{\text{inc}} \end{pmatrix} + S_{1,-} \begin{pmatrix} A_r \\ B_r \end{pmatrix} &= S_{2,+} \begin{pmatrix} A_t \\ B_t \end{pmatrix},
\end{align*}$$  \hspace{1cm} (24)

with

$$S_{m,+} = Z_m U_{m,+}$$

$$= -ie^{-i\omega t}(k_{p,m} A_{\text{inc}} + 2 \mu_{m} \cos^2 \theta_{p,m} - \mu_{m} k_{s,m} \sin 2\theta_{s,m})$$

$$\pm \mu_{m} k_{p,m} \sin 2\theta_{p,m} \pm \mu_{m} k_{s,m} \cos 2\theta_{s,m}),$$  \hspace{1cm} (25)

where all matrices are taken at \(z = 0\) and subscripts \(m = 1\) and \(2\) indicate the elastic parameters in the corresponding media. The solution to equation 24 is

$$\begin{pmatrix} A_t \\ B_t \end{pmatrix} = R \begin{pmatrix} A_{\text{inc}} \\ B_{\text{inc}} \end{pmatrix},$$

$$\begin{pmatrix} A_r \\ B_r \end{pmatrix} = T \begin{pmatrix} A_{\text{inc}} \\ B_{\text{inc}} \end{pmatrix},$$  \hspace{1cm} (26)

where the reflectivity and transmission matrices are

$$R = -[S_{2,+} U_{2,+}^{-1} U_{1,-} S_{1,-}]^{-1}(S_{2,+} U_{2,+}^{-1} U_{1,+} - S_{1,+}),$$  \hspace{1cm} (27)

and

$$T = U_{2,+}^{-1}(U_{1,+} + U_{1,-} R),$$  \hspace{1cm} (28)

respectively.

These formulas give the propagator form (e.g., Aki and Richards, 2002) for all reflection, transmission, and mode-conversion amplitudes for P- and S-incident waves from medium 1.

Considering that matrices \(S_{m,+}\) are proportional to the corresponding \(Z_{m,+}\), note the similarity of equations 27 and 28 to the normal-incidence reflectivity by equation 2. This equation is independent of the ambiguity of impedance scaling and it is the same for reactive and resistive impedance definitions in equations 13 and 14.

Despite its matrix form, reflectivity equation 26 is only slightly more difficult to evaluate numerically (for example, by using MATLAB) than equation 2. At the same time, the effort invested in \(2 \times 2\) matrix computations pays off by removal of all assumptions and approximations, differences between the various EI models, and the relative and path-dependent character of EI. Approximate solutions can still arise from insufficient data when, for example, S-wave or density information is not available. However, in such cases, the approximations can still be made explicit such as the use of Gardner’s formula for density or assuming a depth-variant \(V_p/V_s\) as in equations 5–9. Such approximations can be analyzed and justified separately and not embedded in the EI definition.

Inversion

As a complete description of the medium transmission/reflection properties, the impedance matrix is a function of all three of its elastic parameters: \(\lambda, \mu, \rho\) (equation 15, with \(\rho\) entering through the \(k/\rho\) ratio). If we have matrix \(R\) combining all three recorded, reflected, and converted amplitudes from P and S incident waves, equation 27 can be inverted for \(S_{2,+}\) (in terms of its parameters \(\lambda_s, \mu_s, \rho_s\)):

$$S_{2,+} = [U_{2,+}^{-1} U_{1,-} + U_{2,+}^{-1} U_{1,+} R]^{-1}(S_{1,-} R + S_{1,+}),$$  \hspace{1cm} (29)

yielding a recursive matrix-inversion formula for \(\tilde{Z}\) that extends the scalar equation 4.

However, all P-, S-, and P/S-wave reflection coefficients are rarely available; therefore, the formal matrix inverse (equation 29) at oblique incidence can be impractical. A practical inversion procedure for \(Z\) consists of solving the forward equation 27 given the recorded P/P, SV/SV, and/or P/SV reflection amplitudes at several values of ray parameter for the elastic parameters of the medium; therefore, generally, \(Z\) can be derived by \(\lambda, \mu, \rho\) (lambda, mu, rho; LMR) inversion.

The three-parameter LMR curve fitting can be prone to instabilities (e.g., Mallick, 2007) and the impedance property \(Z\) is particularly valuable for indicating the stable combinations of these parameters. Thus, equation 15 shows that \(Z\) is a function of \(\lambda_k, \mu_k, \theta\). The sensitivity of P/P reflection amplitudes to the values of \(\theta\) in the second medium is quite weak, below approximately 35° incidence, and for P/SV amplitudes below approximately 25° incidence (Figure 2). Note that using the correct Snell’s law for the values of \(\theta_{p,s,2}\)
(Figure 1) is not important in the equations for $Z$ at such angles. Therefore, $Z$ is principally a function of parameters $\mu \mu$ or equivalently of $\mu \mu$ and $\mu \mu$. For the medium beneath the reflecting boundary, these parameters can be estimated by fitting the reflection amplitudes at a range of incidence angles. For example, considering the two amplitudes shown by gray dots in Figure 2a as $P/P$ reflectivity data, we can reconstruct the $AI$ and $\mu \mu$ values of the underlying layer (Figure 3) and thereby reproduce the entire $Z$ matrix. Notably, this matrix would also describe the effects of $P/SV$, $SV/SV$, and $SV/P$ reflections and transmissions. Also, from the shapes of the data misfit contours in Figure 3, note that approximately 10% uncertainty in $AI$ corresponds to approximately 10% uncertainty in $\mu \mu$ with a moderate positive correlation of these uncertainties.

Thus, similarly to the $AI$, matrix $P/SV$ impedance is invertible from seismic data but requires amplitude variation with offset (AVO) information. Similarly to $AI$, the single-boundary inversion step shown in Figure 3 should be continued recursively, producing a matrix $EI$ column from AVO or $P/SV$ reflectivity data. Similarly to $AI$, this recursive inverse should suffer from instabilities caused by the lack of low-frequency information, noise, and the fundamental ambiguity of the impedance. Note that the inversion is conducted synchronously for all ray-parameter values, i.e., the entire reflection/conversion $AVO$ curves are inverted for $Z$ at each depth level. To correlate the results to a well log, one would need to estimate its $AI$ and $\mu \mu$ values and compute the components of $Z$ (equation 15) instead of using the approximate equations 5–9. More detail and examples of such an inversion using real data will be given elsewhere.

**Elastic impedance**

In this section, we compare the $EI$ in matrix equations 13–15 to the approximate equations 5–9. Matrix $Z$ in equation 15 depends on the velocity/density structure and on the direction of wave propagation, but its effect on different types of waves is different. For a $P$-wave with wavenumber $k = \omega / V_p$ propagating at incidence angle $\theta$, $Z$ becomes (equations 15 and 21)

$$ Z_p = \rho V_p \left( \cos \theta \left( 1 - \frac{2 V_p^2}{V_p^2} \sin^2 \theta \right) \right) $$

and for an incident $S$-wave ($k = \omega / V_S$

$$ Z_S = \rho V_S \left( \frac{V_p^2}{V_S^2} \cos \theta \left( \frac{V_p^2}{V_S^2} - 2 \sin^2 \theta \right) \right). $$

Compared to the approximate $P$-wave RI (equations 5–9), the dependencies of the resulting matrix $EI$ components on $\theta$ are quite simple (see Figure 4). Note that the vertical-component part $Z_{ZS}$ is generally close to $RI$ at all angles (Figure 4) although all the components of $Z$ are needed to accurately describe the reflection AVO effects. Comparison to $EI$ attributes (equations 5 and 6) cannot be made because they do not represent single-medium properties. Figure 5 shows logs of the components of matrix $Z$ (equation 30) computed in a real well from CREWES Blackfoot project (CREWES, 2004). Note the similarity of the first component ($Z_{Zp}$) to other impedance measures and the differences between the four components of $Z$.

Another interesting scalar impedance property can be constructed from matrix $Z$ (see equation 19):

$$ Z_p(\theta) = \left[ Z_p \left( \frac{1}{\tan \theta} \right) \right] = \rho V_p \cos \theta \left[ 1 - \frac{2 V_p^2}{V_p^2} \sin^2 \theta \right] $$

$$ = \frac{\rho V_p}{\sqrt{1 - (pV_p)^2}} \left[ 1 - 2(pV_p)^2 \right]. $$

Its meaning is the ratio of the normal traction to normal displacement in the incident $P$-wave alone, i.e., what could be called the “$P$-wave-alone” impedance: the response of the boundary to the incident $P$-wave. This quantity corresponds to the acoustic impedance at all

![Figure 2](image-url)

Figure 2. (a) $PP$ and (b) $PS$ reflection coefficients in shale over wet- and gas-sand models from VerWest (2004). Model parameters are $V_p = 2.77 \text{ km/s}$, $V_S = 1.52 \text{ km/s}$, and $\rho = 2.29 \text{ g/cm}^3$ for shale; $V_p = 3.85 \text{ km/s}$, $V_S = 2.24 \text{ km/s}$, and $\rho = 2.34 \text{ g/cm}^3$ for wet sand; and $V_p = 3.08 \text{ km/s}$, $V_S = 1.98 \text{ km/s}$, and $\rho = 2.14 \text{ g/cm}^3$ for gas sand. The average angle is defined as $\theta_1 = \text{asin} \left[ \rho V_p \right]$, and the shaded area shows the exact solutions; solid gray lines give solutions with $P$- and $S$-wave refracted angles $\theta_2$ (see equation 15) in layer 2 (sand) replaced with their average values; and dashed gray lines correspond to $\theta_1$ taken as the same as in the shale layer. Gray dots in (a) indicate the points used in the inversion (Figure 3).

![Figure 3](image-url)

Figure 3. Illustration of inversion for $Z$ at a single boundary in shale over the gas-sand model. Axes correspond to rms misfits from two points marked with circles in Figure 2a, using the exact $Z$ (black lines in Figure 2a) and drawn at intervals of 0.001. Black, dark gray, and light gray triangles correspond to solutions obtained by using the corresponding $\theta$ dependencies shown in Figure 2a. The gray dashed line indicates the direction of correlated uncertainties in $AI$ and $\mu \mu$. 

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incidence angles. As one can see from Figure 4, in the typical velocity ratio $V_S/V_P = 0.59$, $Z_P$ only weakly varies with incidence angles and the faster reduction of $R_I$s with angles is due to the effects of $P$/SV-wave conversions. However, note that this impedance still should not be used to generate reflectivity series as in equations 5–8. These series can be accurately reproduced by using the complete matrix $Z$. It is important to realize that the meaning of the matrix $EI$ ($Z$) proposed here is different from those suggested previously (equations 5–9). The approximate impedances in equations 5–9 resulted from heuristic summations of $P$/P reflectivity series derived from approximations of the reflectivity equations (Aki and Richards, 2002) and slightly different approximations for time integrals. Transmission losses were ignored, mode conversions were only included through $P$/P AVO effects, small-amplitude and in some cases small-angle limits were considered, and stringent constraints were imposed to perform the integrations. However, the integration of reflectivity series (equations 3 and 4) is well-known for its instability (e.g., Francis, 1997); therefore, the differences between the existing $EI$ forms can reflect the biases accumulated during the reflectivity discretization and integration. VerWest et al. (2000) also note the differences from using the $Aki$ and Richards (2002) and Bortfeld (1961) approximations during such integration.

In contrast, the matrix impedance (equation 30) arises from considerations of the boundary conditions in a single medium and not from integrating a reflectivity series. Elements of this matrix arise from a simple differential operator relating the stress in the wavefield to its $P$- and $S$-wave displacements (or velocities, equation 13) and this operator is closely related to the elastic parameters of the medium. It depends on neither a specific wave type nor its propagation history through other layers. Due to its matrix form, it incorporates all the effects of $P$/SV reflections and conversions and yields the correct acoustic limit at all incidence angles. Therefore, the matrix $EI$ above should provide the most unambiguous information for interpreting elastic rock properties from seismic inversion.

**CONCLUSION**

The $P$/SV-wave impedance of an elastic medium can be uniquely described as a matrix quantity depending on its mechanical parameters and angle of incidence. Reflection, transmission, and conversion coefficients at arbitrary angles are related to matrix-impedance contrasts similarly to the acoustic cases. When using the matrix form of the reflectivity problem, no approximations are required for accurate modeling of reflection coefficients at all angles. Known cases of $P$- and $S$-wave acoustic impedance are accurately reproduced at all incidence angles and the approach also allows the extension of the concept of impedance to an attenuative medium. The matrix impedance readily lends itself to inversion with uncertainties typical to the standard acoustic-impedance inversion problem.

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