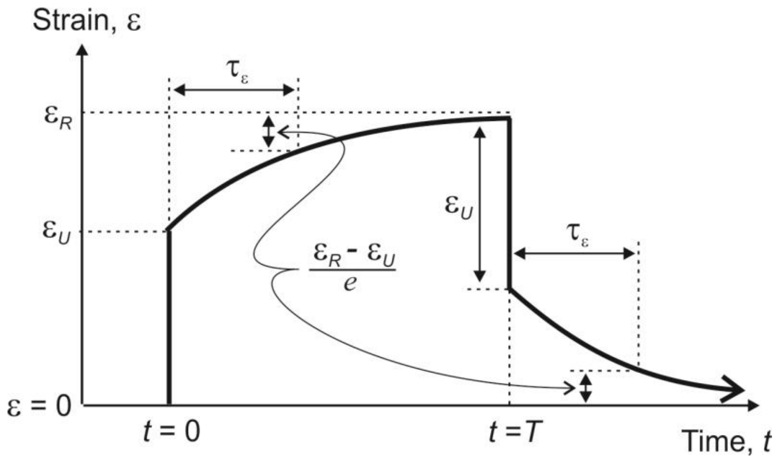
Lecture 2: The Viscoelastic Model (VE)

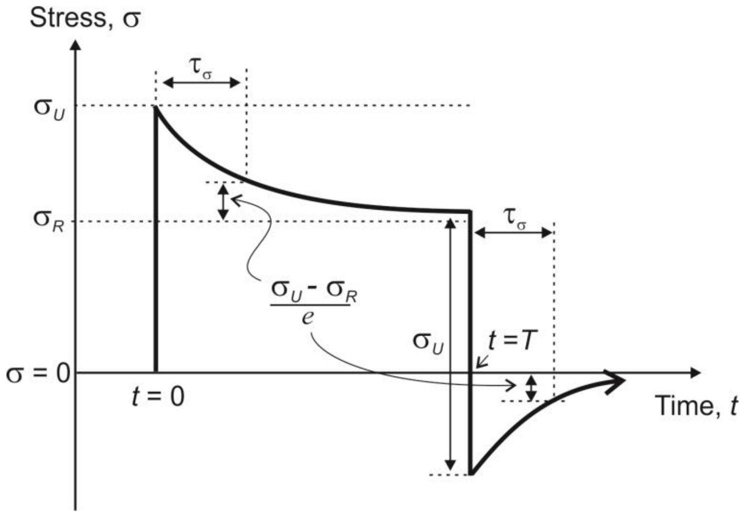
- Mathematical principles:
  - Boltzmann’s after-effect
  - Correspondence principle
- Phenomenology: creep, modulus and velocity dispersion, Q
- Integral and time-differential strain-stress relations
- Causality and Kramers-Krönig relations
- Kinetic equation
- Zener’s and related equations
- Commonly used ‘linear solids’ models
  - Time-domain responses and empirical-modulus and attenuation spectra
- Interpretation of frequency-dependent attenuation and dispersion spectra
- Problems, limitations, and applicability of VE model

- Reading: Chapter 2 in the text

The problem

- The goal of the viscoelastic model is to provide equations describing linear relaxation experiments for stress and strain shown below (plots from Lecture 1)
- This goal is achieved by constructing **phenomenological stress-strain relations dependent on time.** They describe the behavior of deformation (variation of strain and stress with time) without considering the physics of deformation.
- This means that the stress-strain relations are obtained from pure mathematical principles and without consideration of physics
    - Later, I will criticize this model for this lack of physics, but now, let us just consider it





Mathematical principle #1: Boltzmann's (linearity) principle

- Boltzmann's principle is simply a linear relation between strain and stress time functions
- This principle means that if measure a time-dependent stress $\sigma(t)$ and strain $\varepsilon(t)$ in some deformation, then one of these time functions can be derived from the other, and vice versa

$$\sigma(t) \leftrightarrow \varepsilon(t)$$

- Linearity means that if we have two experiments in which we measure strain functions $\varepsilon_1(t)$ and $\varepsilon_2(t)$ and the corresponding stress functions $\sigma_1(t)$ and $\sigma_2(t)$, then arbitrary linear combinations of these functions also give valid experiments:

$$\varepsilon(t_1) = c_1\varepsilon_1(t_1) + c_2\varepsilon_2(t_1) \quad \text{and} \quad \sigma(t_2) = c_1\sigma_1(t_2) + c_2\sigma_2(t_2)$$

Mathematical principle #2: Correspondence principle

- What is the difference between an elastic and anelastic material?
- The correspondence principle answers this question in the following way:
  1) When anelasticity is "turned on", the elastic modulus and compliance become **functions of time**:

$$M \quad \Rightarrow \quad M(t) = M_U \delta(t) + \tilde{M}(t)$$

$$J = \frac{1}{M} \quad \Rightarrow \quad J(t) = J_U \delta(t) + \tilde{J}(t)$$

However, note that this transition '$\Rightarrow$' is actually impossible to achieve or verify. There exist no 'elastic' analogs for real anelastic materials.

  2) Instantaneous stress-strain relations become **convolutional relations containing all preceding times**:

$$\sigma(t) = M\varepsilon(t) \quad \Rightarrow \quad \sigma(t) = \int_{-\infty}^{t} M(t-\tau)\varepsilon(\tau)\,d\tau$$

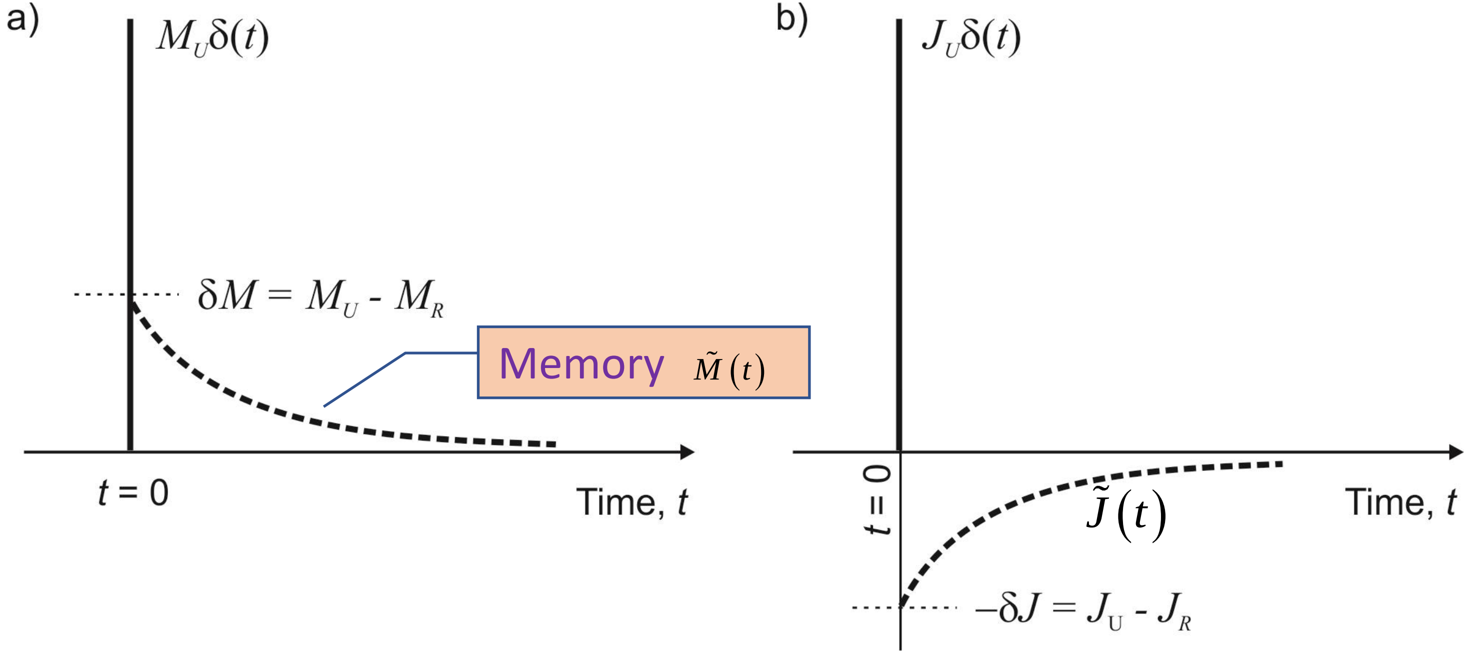
$$\varepsilon(t) = \frac{\sigma(t)}{M} \quad \Rightarrow \quad \varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\sigma(\tau)\,d\tau$$

Modulus and compliance functions

- The VE modulus $M(t)$ and compliance function $J(t)$ consist of a singular (instantaneous, elastic) response at $t = 0$ and extended “memory” responses at $t \geq 0$
- The memory functions interpolate between the “unrelaxed” response at $t \to 0$ and “relaxed” response at $t \to \infty$

$$\boxed{M(t) = M_U \delta(t) + \tilde{M}(t)}$$

$$\boxed{J(t) = J_U \delta(t) + \tilde{J}(t)}$$



Frequency domain

- Fourier transform presents the signal $u(t)$ by a superposition of harmonic oscillations:

$$u(t) = \frac{1}{2\pi}\int d\omega e^{-i\omega t} U(\omega) \qquad \Leftrightarrow \qquad U(\omega) = \int d\omega e^{i\omega t} u(t)$$

(inverse) (forward)

- In the frequency domain, convolutional strain-stress relations are simple ($\omega = 2\pi f$):

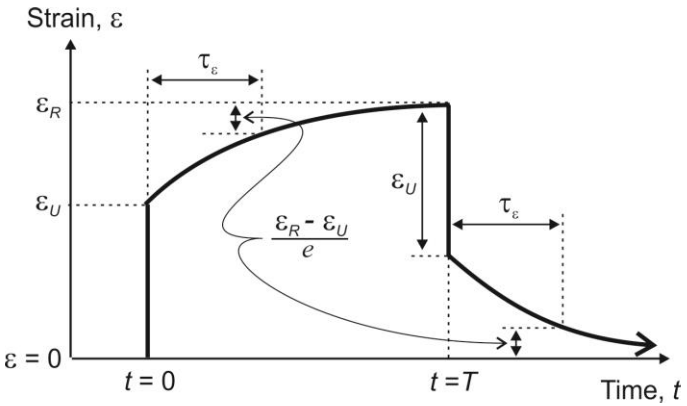
$$\sigma(f) = M(f)\varepsilon(f) \quad \text{and} \quad \varepsilon(f) = \frac{\sigma(f)}{M(f)}$$

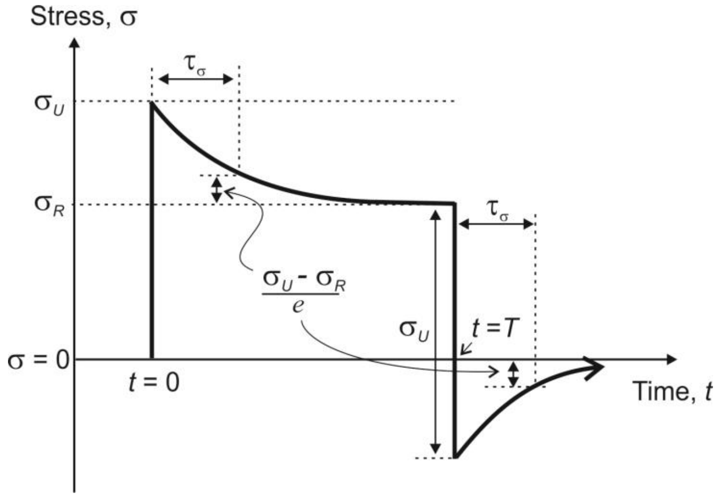
- *Q*-factor is one of the most important measured quantities.
  It only exists in the frequency domain:

$$Q(f) = -\frac{\operatorname{Re} M(f)}{\operatorname{Im} M(f)}$$

Phenomenology

- There are two types of observations which are explained by the viscoelastic model (VE):
  1) Time-domain – creep (transient deformation)
  2) Frequency-domain – modulus (or wave velocity) dispersion and attenuation
- The key observation is that in rock deformation, there is always some characteristic delay ("relaxation") time τ or some characteristic frequency $f = 1/\tau$
- For example, note the delay times τ_ε (strain relaxation time) and τ_σ (stress relaxation time) in static loading experiments discussed before:





Note that the overall periodic variation of **strain is delayed relative to stress**

Phenomenology #1: Creep

- The basic phenomenology addressed by the viscoelastic model is creep, which is the time-delayed deformation observed in static loading experiments
  - The viscoelastic theory simply assumes that this transient behavior is also contained in all other types of deformation
- Creep is usually described by giving the time-dependent creep function $\phi(t)$. This function is the relative deviation of strain from elastic deformation, measured in an experiment with static stress loading $\sigma = \sigma_0 \theta(t)$:

$$\frac{\varepsilon(t)}{\sigma} = \frac{1}{M_U}\left[1+\phi(t)\right]$$

- Examples:

$$\phi(t) = M_U\left(\frac{t}{\eta} + \beta t^n\right)$$

Andrade law
Characteristic time
(for small t):

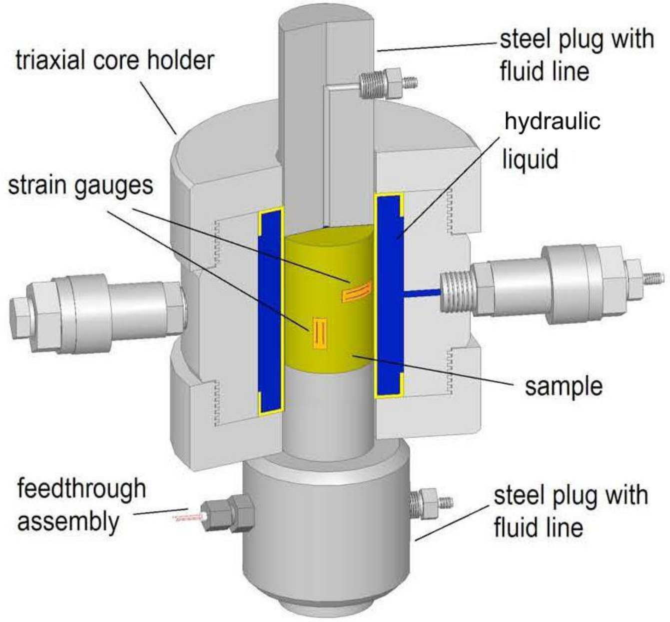
$$\tau = \frac{\eta}{M_U}$$

$$\phi(t) = q\ln\left(1+\frac{t}{\tau}\right)$$

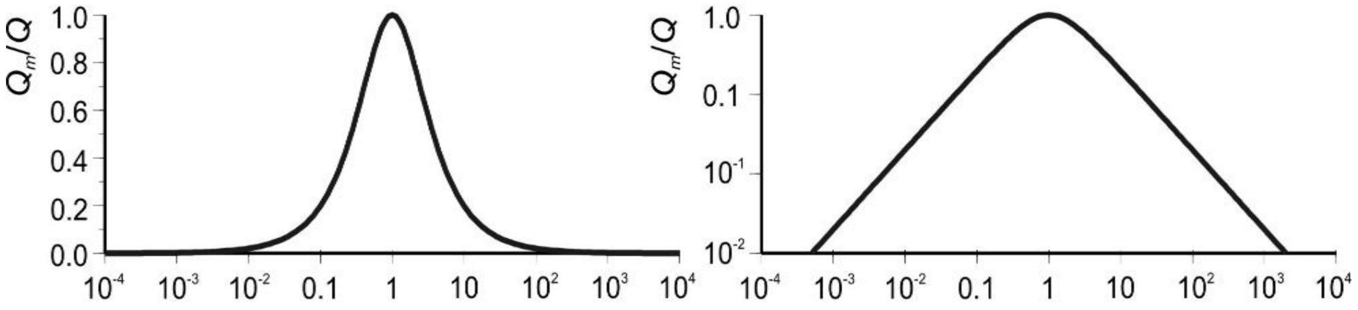
Lomnitz's model
From this model, the idea of Q-factor for materials seems to have started

Phenomenology #2: Absorption (phase-lag) peak or band

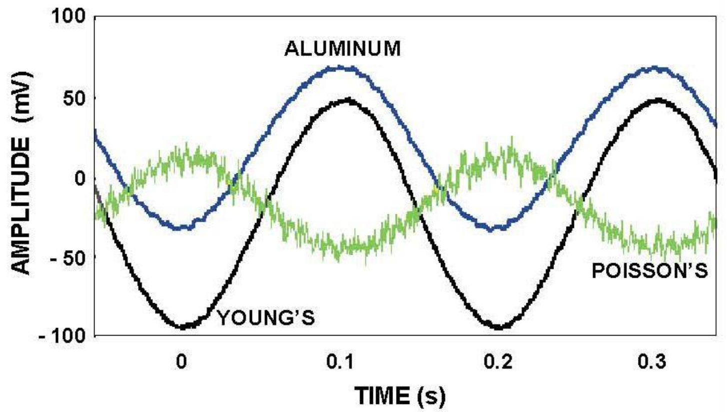
- The typical observation is the “absorption peak” and “modulus dispersion band”
- The typical absorption peak predicted by the VE model (“standard linear solid”, or “Zener’s body”) looks like this:



(linear scale in Q^{-1}) (logarithmic scale in Q^{-1})



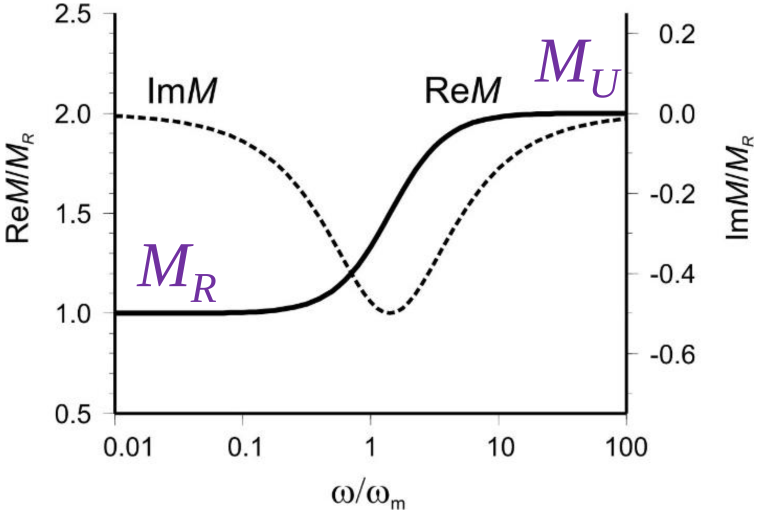
$$Q^{-1} \overset{\text{def}}{=} \arctan\left(\text{stress-strain phase lag}\right)$$



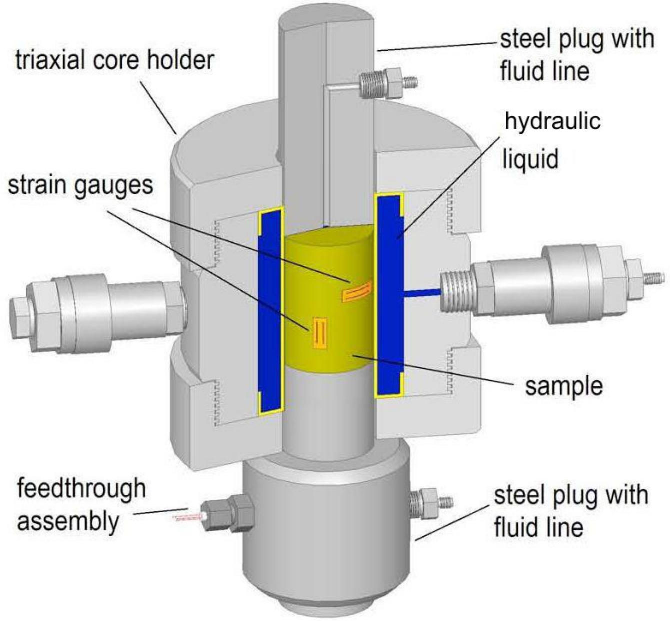
- Note that for $f << f_{\text{peak}}$, $Q^{-1} \propto f$, and for $f >> f_{\text{peak}}$, $Q^{-1} \propto 1/f$

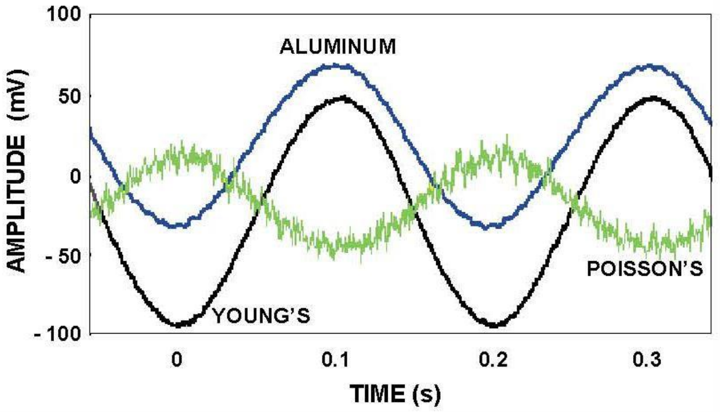
Phenomenology #3: Modulus dispersion

- The frequency band of increased $Q^{-1}(f)$ also "modulus dispersion band" – the measured modulus changing from M_R ("relaxed") to M_U ("unrelaxed")
- The typical modulus dispersion (in "Zener's body") looks like this:



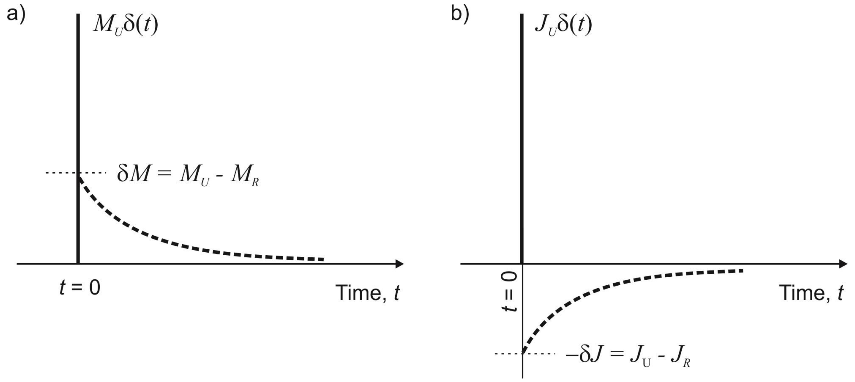
- Note that the shape of $\mathrm{Im}M(f)$ is close that of $Q^{-1}(f)$
- This is because $M = M'\left(1 - iQ^{-1}\right)$





Causality

- Regardless of the physical mechanisms, the data measured in any of the above mechanical experiments **must satisfy causality constraints**
- This requirement means that the data (even sine functions recorded in a frequency-domain experiment) represent a record obtained from some impulsive source
  - For example, modulus $M(t)$ recorded after a pulse of strain, or compliance $J(t)$ recorded from a pulse of stress:
  - This should be the case for any mechanical system like rock-physics apparatus



- The causality requirement simply means that $M(t) = 0$ and $J(t) = 0$ for all $t < 0$. The response cannot appear before the source at $t = 0$.
- Mathematically, this condition is written as: $M(t) = M(t)\theta(t)$ and $J(t) = J(t)\theta(t)$

  where $\theta(t)$ is the Heaviside step function: $\theta(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

$$\theta(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ 1/2 & \text{for} \quad t = 0, \\ 1 & \text{for} \quad t > 0. \end{cases}$$

Kramers-Krönig (causality) relations

- Simple time-domain properties of recorded signals have somewhat more complex forms in the frequency domain:

1) The time-domain signal is **real-valued**. In frequency domain, this means the following symmetry of the real part of the complex spectrum and anti-symmetry of the imaginary part:

$$\begin{cases} \operatorname{Re} M(-\omega) = \operatorname{Re} M(\omega), \\ \operatorname{Im} M(-\omega) = -\operatorname{Im} M(\omega). \end{cases}$$

2) Causality ($M(t<0)=0$) means the integral Kramers-Krönig relations between the real and imaginary parts of $M(\omega)$:

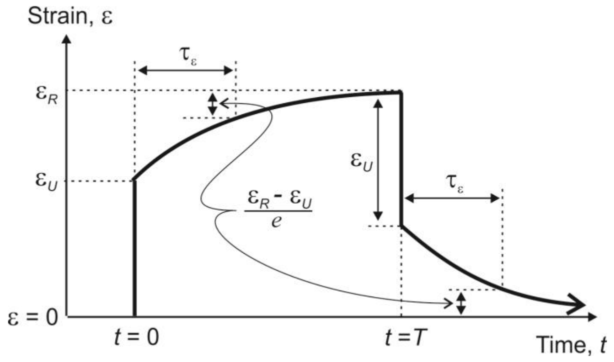
$$\begin{cases} \operatorname{Re} M(\omega) = \frac{2}{\pi} \mathrm{P.v.} \int_0^\infty d\omega' \frac{\operatorname{Im} M(\omega')}{\omega'^2 - \omega^2}, \\ \operatorname{Im} M(\omega) = -\frac{2\omega}{\pi} \mathrm{P.v.} \int_0^\infty d\omega' \frac{\operatorname{Re} M(\omega')}{\omega'^2 - \omega^2}. \end{cases}$$

'P.v.' means the "Cauchy principal value" of the integrals taken at point $\omega' = \omega$

- For spectra with band-limited attenuation ($\operatorname{Im} M$), the integral Kramers-Krönig relation can be approximated by a differential one:

$$\frac{2}{\pi} Q^{-1} \approx \frac{d\left[\ln\left(\operatorname{Re} M\right)\right]}{d\left[\ln \omega\right]} = \frac{d\left(\operatorname{Re} M\right)}{d\omega} \frac{\omega}{\operatorname{Re} M}$$

Kinetic equation



- Now let us look for a phenomenological differential equation that can describe the shapes of relaxation processes like $\varepsilon(t)$ again shown on the right:
- A simple and general law of this kind is given by the kinetic equation
  - This equation is commonly used to describe diffusion, heating, or variations of concentrations of elements during chemical reactions
- For a given (thermodynamic) variable ξ (such as strain), assume that there exists some equilibrium ('relaxed') value $\bar{\xi}$ depending on some external factors like σ (stress) and temperature T: $\bar{\xi}(\sigma, T)$
- Then, if ξ is currently shifted from this equilibrium, then the **kinetic equation expresses its rate of change**:

$$\dot{\xi} = -\frac{1}{\tau_r}\left[\xi - \bar{\xi}(\sigma, T)\right]$$

Note the characteristic time τ_r

which means that $\xi(t)$ approaches the equilibrium at a rate proportional to the deviation from it.

- If $\bar{\xi}(\sigma, T)$ stays constant with time, then this approach to the equilibrium is exponential:

$$\xi(t) = \bar{\xi} + C \exp\left(-\frac{t}{\tau_r}\right)$$

Zener's (Standard Linear Solid) equation

- The kinetic equation can also be written as: $\xi + \tau_r \dot{\xi} = \bar{\xi}(\sigma, T)$
- Zener's equation generalizes this relation to ξ being strain and stress:

$$\sigma + \tau_\sigma \dot{\sigma} = M_R (\varepsilon + \tau_\varepsilon \dot{\varepsilon})$$

  - With constant stress ($\dot{\sigma} = 0$), this is a kinetic equation for strain (taking $\bar{\xi} = \sigma / M_R$), with relaxation time τ_ε
  - Vice versa, with constant strain $\dot{\varepsilon} = 0$, this is a kinetic equation for stress $\sigma(t)$, with relaxation time τ_σ

- Another useful way to write this equation is by using differential operators:

$$\left(1 + \tau_\sigma \frac{d}{dt}\right) \sigma = M_R \left(1 + \tau_\varepsilon \frac{d}{dt}\right) \varepsilon$$

- For a harmonic oscillation with $\sigma(t) = \sigma_0 e^{-i\omega t}$ and $\varepsilon(t) = \varepsilon_0 e^{-i\omega t}$, this equation becomes very simple and directly gives the complex modulus:

$$M(\omega) \overset{\text{def}}{=} \frac{\sigma_0}{\varepsilon_0} = M_R \frac{1 - i\omega\tau_\varepsilon}{1 - i\omega\tau_\sigma}$$

We will look at the graphs when discussing "Zener's body" later

Generalized Zener’s equation

- Zener’s equation can be generalized using additional time derivatives:

$$\left(1+\sum_{n=1}^{N_\sigma}\tau_{\sigma n}\frac{d^n}{dt^n}\right)\sigma = M_R\left(1+\sum_{k=1}^{N_\varepsilon}\tau_{\varepsilon k}\frac{d^k}{dt^k}\right)\varepsilon$$

  - … and the derivatives can be further generalized as fractional (non-integer orders)…

- These equations describe “viscoelastic bodies” with spectra $M(\omega)$ being (almost) arbitrary rational functions of ($-i\omega$) with real coefficients:

$$M(\omega) = M_R\frac{1+\sum_{k=1}^{N_\varepsilon}\tau_{\varepsilon k}(-i\omega)^k}{1+\sum_{n=1}^{N_\sigma}\tau_{\sigma n}(-i\omega)^n}$$

Interpretation of M_R, M_U, τ_ε, and τ_σ in Zener's equation

- As you see, it is often useful to first write the most general equation and then look for the meanings of its parameters. The same approach will be taken in Lagrangian mechanics.
- What are the meanings of "material-property" parameters in Zener's equation?
  - M_R is the relaxed modulus (modulus observed at equilibrium, when both $\dot{\sigma}=0$ and $\dot{\varepsilon}=0$)
  - The unrelaxed modulus is "hidden" but still present there. If we consider very fast deformations, then the time-rate terms will dominate, and therefore the unrelaxed modulus

$$M_U = \frac{\dot{\sigma}}{\dot{\varepsilon}} = M_R \frac{\tau_\varepsilon}{\tau_\sigma}$$

  - Therefore, M_U is contained in the ratio of the two taus:

$$M_U = M_R \frac{\tau_\varepsilon}{\tau_\sigma}$$

Note that therefore always $\tau_\varepsilon > \tau_\sigma$

  - And so, it turns out that Zener's equation contains **only one** characteristic relaxation time:

$$\tau_r = \sqrt{\tau_\sigma \tau_\varepsilon}$$

Spring-dashpot diagrams

- Graphical spring-dashpot diagrams give a convenient way for representing equations of the kinetic or Zener's type
- The elements are combined to achieve:
  - Desired low-frequency response (M_R)
  - Desired high-frequency response (M_U)
  - Desired frequency (or a broader band) of transition from M_R to M_U

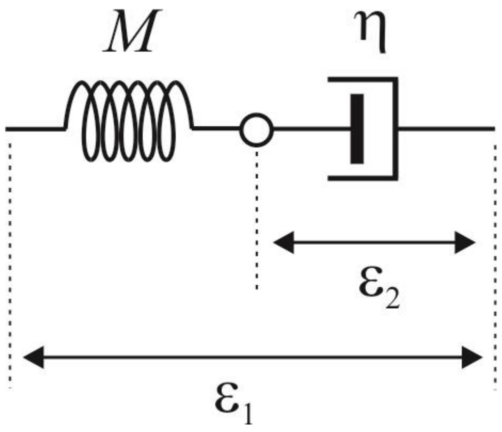
Rules for spring-dashpot arrangements

- For elements connected in series, the stress is common to all of them. Therefore, the stress can be used as an independent variable, and the strains derived from it.
- For elements connected in parallel, the strains are equal, and they can be used for parameterizing the deformation.
- For a spring element, the strain and stress on it are related by $\sigma = M\varepsilon$
- For a dashpot element, the strain rate and stress are related by $\sigma = \eta\dot{\varepsilon}$
- Therefore, any stresses is generally represented by linear combinations of strains and strain rates in the various elements:

$$\sigma = \sum_{\text{springs } i} M_i \varepsilon_i + \sum_{\text{dashpots } j} \eta_j \dot{\varepsilon}_j$$

- Let us consider several standard diagrams known as linear bodies (or linear solids)

Maxwell’s body

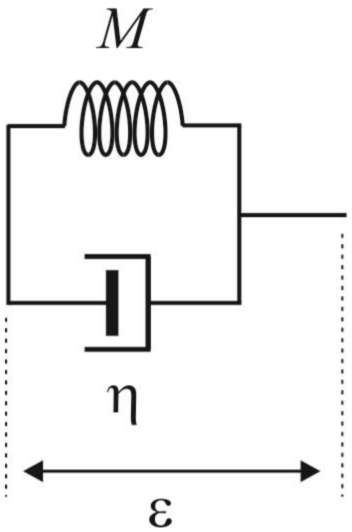


- Maxwell’s body consists of a spring and a dashpot connected in series:
- Using σ (common to both elements) as an independent variable, Zener’s type equation for this body is obtained (try this!):

$$\sigma + \frac{\eta}{M}\dot{\sigma} = \eta\dot{\varepsilon}$$

- Thus, comparing to Zener’s equation:
  - $M_U = M$
  - $M_R \to 0$ (plastic deformation)
  - Strain relaxation time $\to \infty$ as $\tau_\varepsilon = \frac{\eta}{M_R}$
  - Stress relaxation time $\tau_\sigma = \frac{\eta}{M}$
  - Complex modulus $M(\omega) = \frac{\sigma(\omega)}{\varepsilon(\omega)} = \frac{-i\omega\eta}{1 - i\omega\tau_\sigma}$

Kelvin-Voigt's body

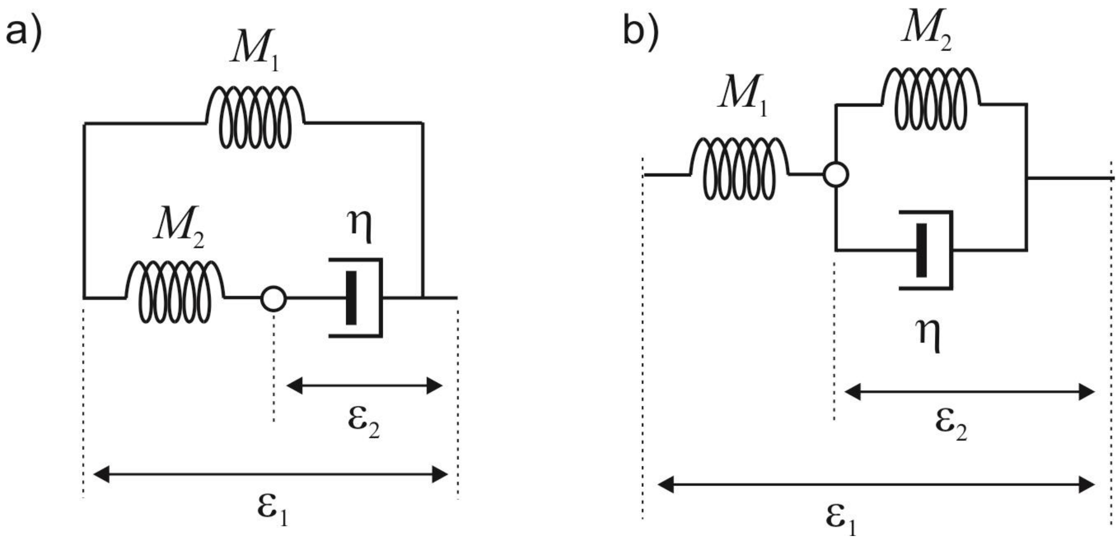


- Kelvin-Voigt's body consists of a spring and a dashpot connected in parallel:
- The strain ε (common to both elements) can be used as an independent variable, and Zener's type equation for this body is (verify this!):

$$\sigma = M\varepsilon + \eta\dot{\varepsilon}$$

- Comparing to Zener's equation:
  - $M_U = \infty$
  - $M_R = M$ (elastic deformation)
  - Strain relaxation time is finite: $\tau_\varepsilon = \dfrac{\eta}{M}$
  - Stress relaxation time $\tau_\sigma = 0$
  - Complex modulus $M(\omega) = M(1 - i\omega\tau_\varepsilon) = M - i\omega\eta$

Zener’s body (Standard Linear Solid, SLS)



- The Zener’s body has two graphical forms:
- Both forms lead to the same Zener’s equation for the observed strain and stress:

$$\sigma + \tau_\sigma \dot{\sigma} = M_R \left(\varepsilon + \tau_\varepsilon \dot{\varepsilon} \right)$$

  - …but different relations for internal variables

- For diagram a):
  - $M_U = M_1 \frac{\tau_\varepsilon}{\tau_\sigma}$ (from basic relation and τ_ε and τ_σ below)
  - $M_R = M_1$ (only this spring is deformed in relaxed state)
  - Strain relaxation time: $\tau_\varepsilon = \eta \left(\frac{1}{M_1} + \frac{1}{M_2} \right)$
  - Stress relaxation time $\tau_\sigma = \frac{\eta}{M_2}$
  - Complex modulus $M(\omega) = M_1 \frac{1 - i\omega\tau_\varepsilon}{1 - i\omega\tau_\sigma}$

Properties of the Standard Linear Solid

- Let us write the Zener's equation with constant stress σ at $t \geq 0$: $$\varepsilon + \tau_{\varepsilon}\dot{\varepsilon} = \frac{\sigma}{M_R}$$
- This is a kinetic equation (see above), with solution: $$\varepsilon(t) = \frac{\sigma}{M_R} + C\exp\left(-\frac{t}{\tau_{\varepsilon}}\right)$$
- Constant C here is selected so that the initial strain is "unrelaxed": $$\varepsilon(0) = \frac{\sigma}{M_U}$$
- and therefore: $$C = \sigma\left(\frac{1}{M_U} - \frac{1}{M_R}\right)$$ (note that $C < 0$)
- Creep function (defined by $\frac{\varepsilon(t)}{\sigma} = \frac{1}{M_U}\left[1+\phi(t)\right]$) is then: $$\phi(t) = \left(\frac{\tau_{\varepsilon}}{\tau_{\sigma}} - 1\right)\left(1 - e^{-t/\tau_{\varepsilon}}\right)$$

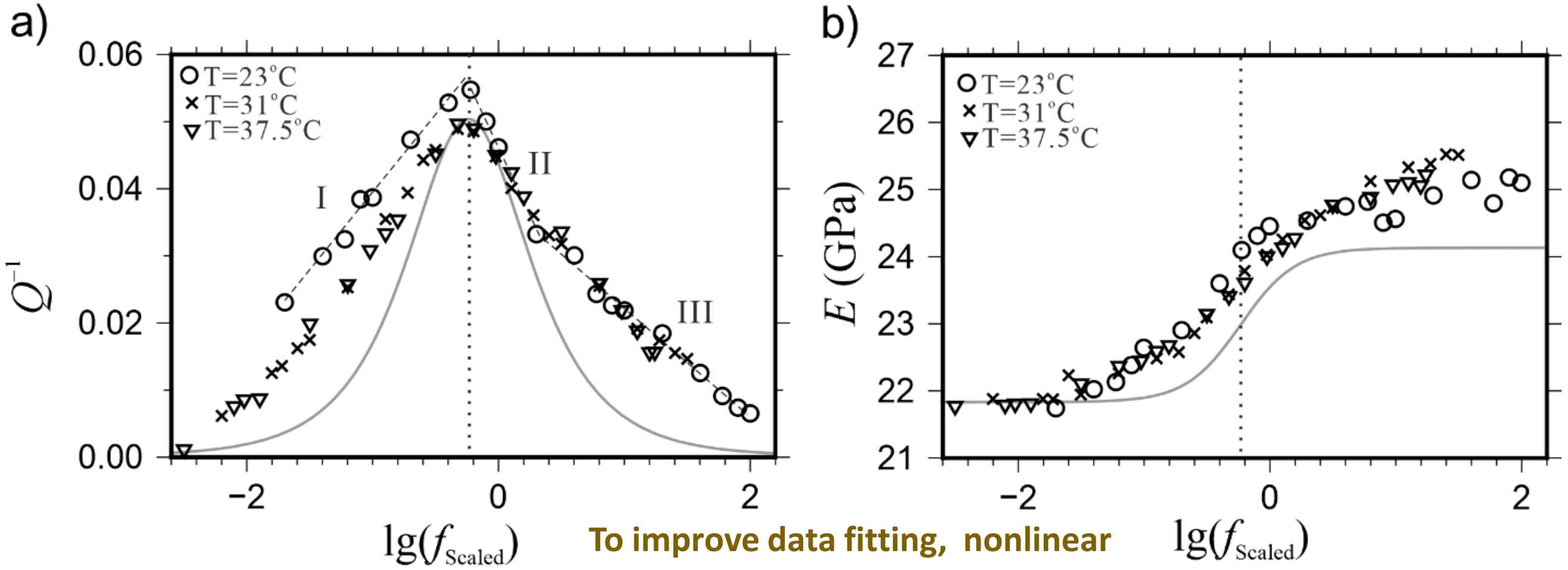
Properties of the Standard Linear Solid (cont.)

- From the complex modulus spectrum: $$M(\omega) = M_R \frac{1 - i\omega\tau_\varepsilon}{1 - i\omega\tau_\sigma}$$
- Attenuation factor: $$Q^{-1}(\omega) = \frac{\omega(\tau_\varepsilon - \tau_\sigma)}{1 + \left(\frac{\omega}{\omega_m}\right)^2}$$
- where the angular frequency of the peak in $Q^{-1}(f)$: $$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{\tau_\varepsilon \tau_\sigma}} = \frac{1}{\eta} \frac{M_2}{\sqrt{1 + \frac{M_2}{M_1}}}$$

Note that the peak frequency is inversely proportional to η
Thus, frequency of the peak is a true indicator of viscosity
(internal friction).
The magnitude (height) of the $Q^{-1}(\omega)$ peak is an elastic property
(next slide, and recall the Kramers-Krönig relations)

Interpretation of $M(f)$ and $Q^{-1}(f)$ data

- The elastic and anelastic parameters of a rock can be obtained by approximating the measured Re$M(f)$ and $Q^{-1}(f)$ spectra **by the spectra of an SLS** and estimating M_R, M_U, τ_ε, and τ_σ :
  - Plateaus in the Re$M(f)$ spectrum give M_R and M_U (elastic properties)
  - From M_R and M_U, you can obtain the ratio of τ_σ and τ_ε (convenient to denote by β^2): $$\frac{\tau_\sigma}{\tau_\varepsilon} = \frac{M_R}{M_U} = \beta^2$$
  - Then, verify that the height of the SLS $Q^{-1}(f)$ peak equals $$Q_m^{-1} = \frac{\beta^{-1} - \beta}{2} = \frac{M_U - M_R}{2\sqrt{M_R M_U}}$$ (so, the height of the peak does not add much to the interpretation!)
  - The frequency $\omega_0 = 2\pi f_0$ of the peak in $Q^{-1}(f)$ gives τ_ε and τ_σ (only one anelastic property): $$\tau_\varepsilon = \frac{1}{\beta\omega_0}, \quad \tau_\sigma = \frac{\beta}{\omega_0}$$

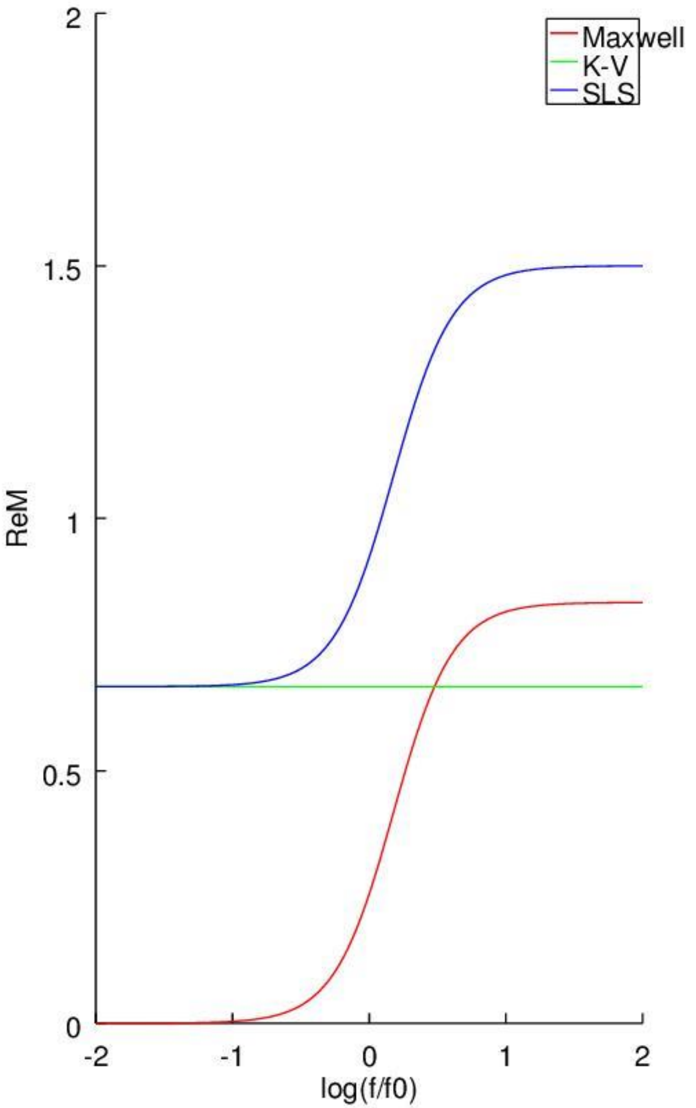


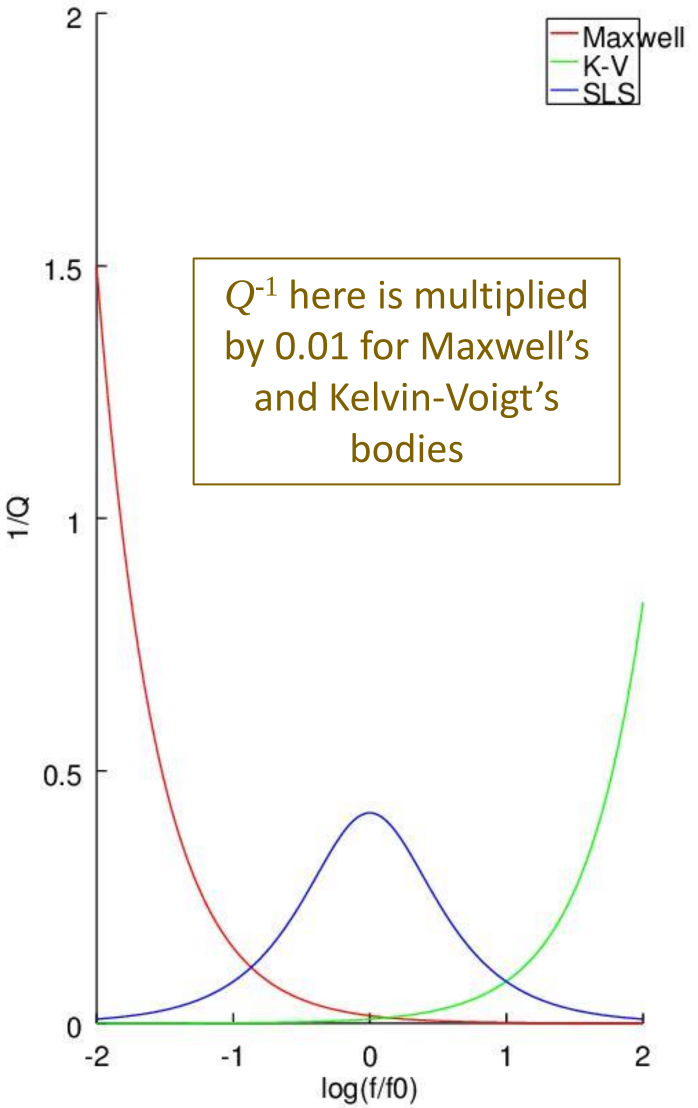
To improve data fitting, nonlinear models can be used (Cole-Cole, etc.)

Young's modulus dispersion and attenuation (Q^{-1}) measurements in glycerol-saturated Berea sandstone at three temperatures and their approximation by an SLS (lines)

Dispersion and attenuation spectra

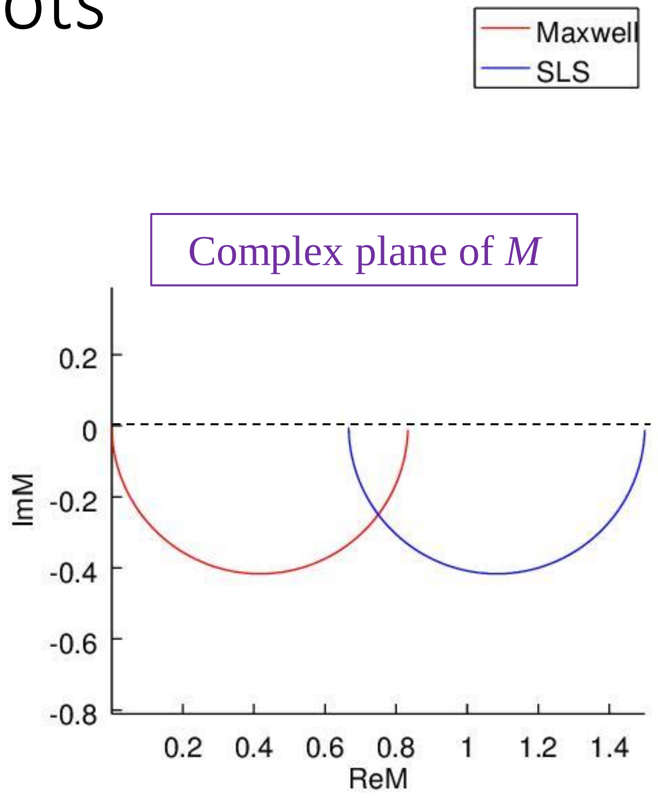
- Normalized Re$M(f)$ and $Q^{-1}(f)$ spectra for the three basic linear bodies
  - Frequency scaled to make $f_0 = 1$ for SLS (Zener)
  - ReM scaled to make the average $\bar{M} = \sqrt{M_R M_U} = 1$ in this plot, with $\beta = 1.5$
  - Viscosity η in Maxwell's and Kelvin-Voigt (K-V) bodies taken the same as in the SLS
  - M_R in K-V body is taken the same as in the SLS

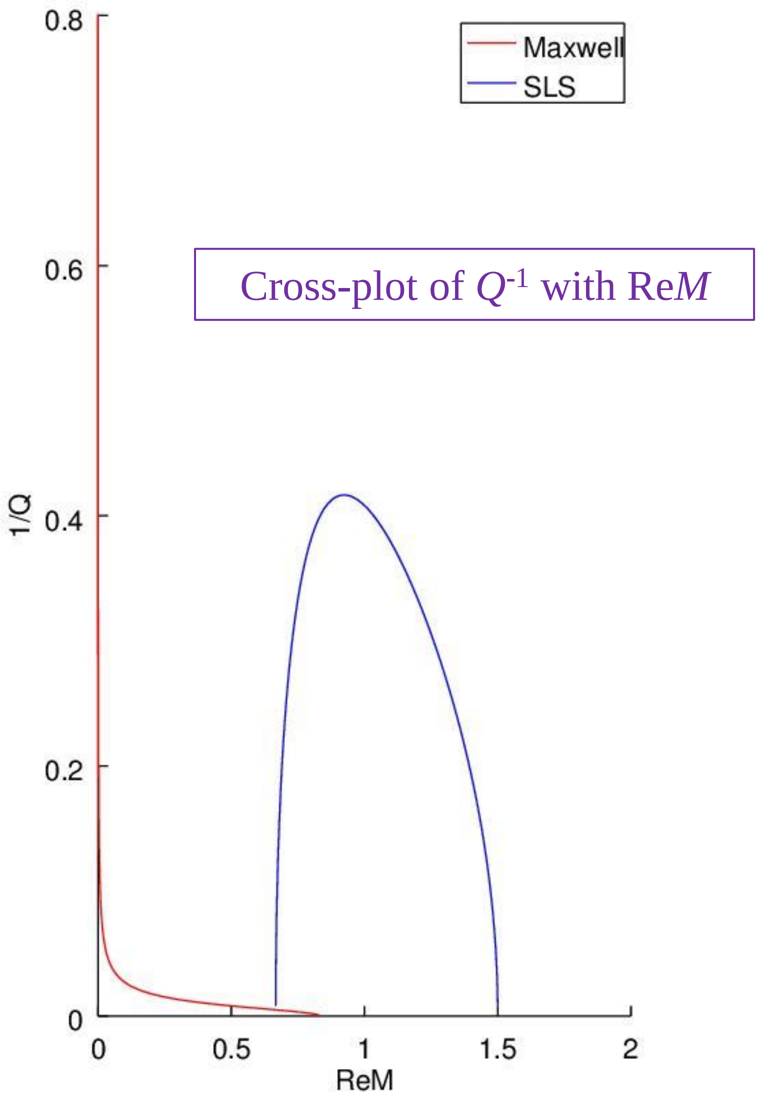




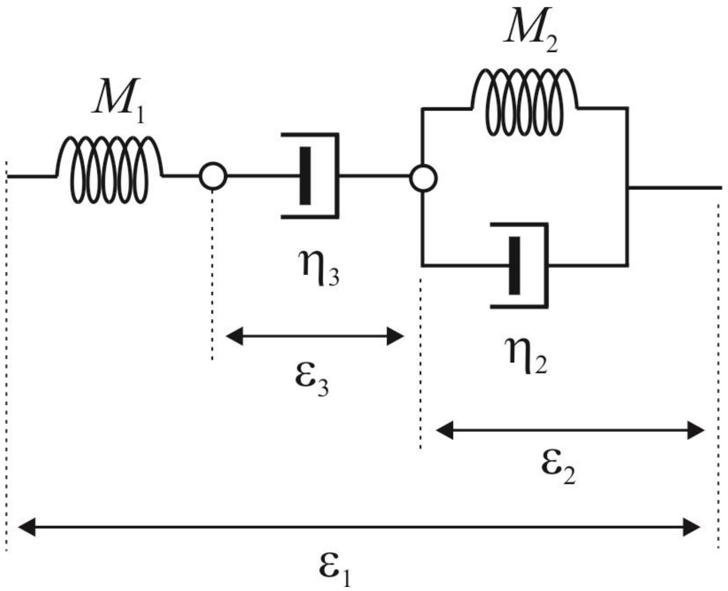
Q^{-1} here is multiplied by 0.01 for Maxwell's and Kelvin-Voigt's bodies

Cross-plots


Complex plane of M


Cross-plot of Q^{-1} with ReM

Burgers' body



- This is a Zener's body with an additional dashpot connected in series for plastic deformation:
- This system contains two internal degrees of freedom and therefore does not reduce to Zener's equation
  - Example of the "generalized Zener's equation"
- Complex modulus can be simply obtained by viewing this system as a Maxwell's and Kelvin-Voigt bodies connected in series:

$$M(\omega) = \left(\frac{1}{M_{\text{Maxwell}}} + \frac{1}{M_{\text{Kelvin-Voigt}}} \right)^{-1} = \left(\frac{1 - i\omega \dfrac{\eta_1}{M_1}}{-i\omega\eta_1} + \frac{1}{M_2 - i\omega\eta_2} \right)^{-1}$$

- Note that this $M(\omega)$ is a ratio of <u>second-order</u> polynomials of $(-i\omega)$:

$$M(\omega) = \frac{-i\omega\eta_1 \left(M_2 - i\omega\eta_2 \right)}{M_2 + a(-i\omega) + b(-i\omega)^2}$$

Second-order polynomials because of the <u>two internal variables</u>

Exercise

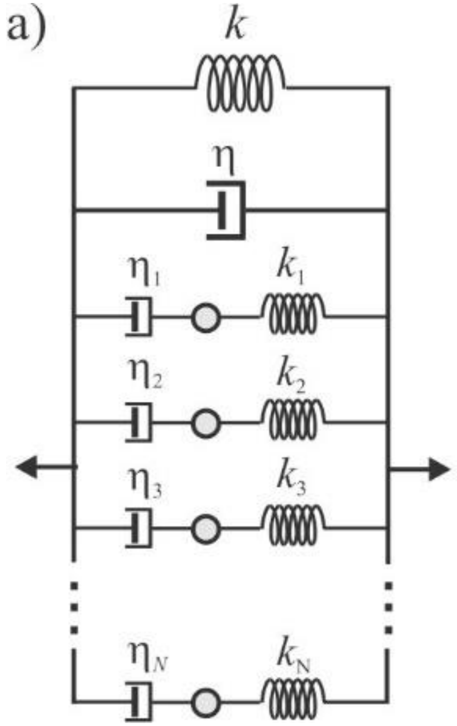
- Write a Matlab or Octave program to plot the spectra of $\mathrm{Re}M(f)$ and $Q^{-1}(f)$ for Burgers body using this equation:

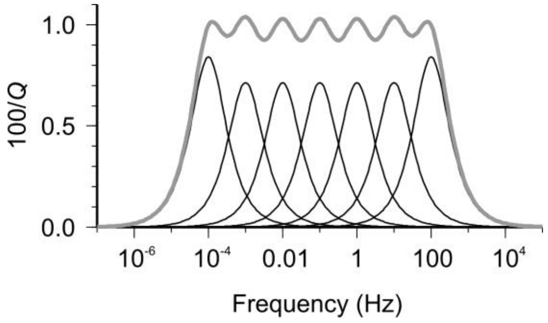
$$M(\omega)=\left(\frac{1-i\omega\dfrac{\eta_1}{M_1}}{-i\omega\eta_1}+\frac{1}{M_2-i\omega\eta_2}\right)^{-1}$$

Generalized Linear Solids

- These systems are constructed to provide broad attenuation ($Q^{-1}(f)$) and dispersion frequency bands
  - Broad bands are achieved by using multiple internal variables
  - Each of these variables is activated (starts moving) at different characteristic frequency $\omega_c = k/\eta$ (k denotes moduli M in this figure)

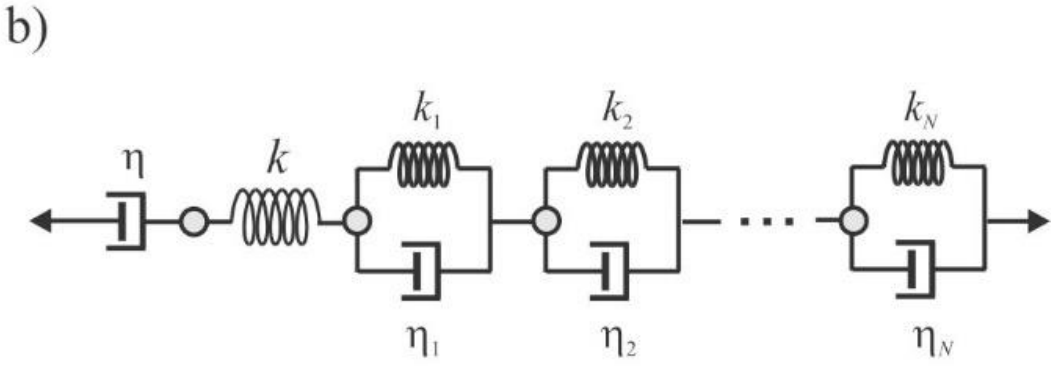
Generalized Maxwell solid
(with $\eta = 0$, this is also the **Wiechert**, **Generalized Standard Linear Solid**, or **Generalized Zener body**)





Attenuation spectrum is a superposition of peaks for individual modes

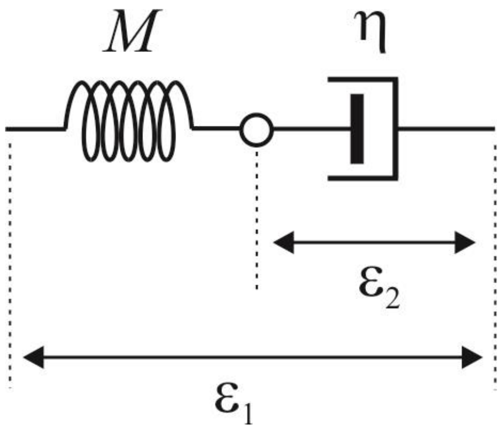
Generalized Voigt solid



Problems of the Viscoelastic model in seismology

- The VE model is broadly used and considered good for describing real media (rock samples, subsurface, Earth's mantle, and the whole Earth!)
  - It reasonably well explains most data
  - It works well for quasi-homogeneous media or for whole rock samples tested in the laboratory
- However, as a theory of seismic waves, this model suffers from several important problems:
  - In physics, there exists no definite strain-stress relation. Equations of physics are for acceleration, not for strain rates like in kinetic or Zener's equations
  - Also specific problems summarized in the next slides:
    - The VE model is quasi-static, and so it is designed only for slow deformations of samples in the laboratory. It is therefore unknown how accurate this model is for seismic waves.
    - It replaces effects of heterogeneity with those of time dependences, which is not always sufficient;
    - It does not include body-force friction (only surface stress was denoted σ above and called "stress"). Such body forces come, for example, from primary-pore fluid friction;
    - It disregard boundary conditions between internal variables;
    - It is also known to produce some nonphysical wave solutions in layered media.

VE problem #1: Quasi-static character



- The VE model attributes zero mass to the internal variables, like the one shown by the white dot in Maxwell's body here
  - This results, for example, in the absence of additional wave modes in media with complex structures, such as the Standard Linear Solid
- However, in physics, all components of materials (matter) must have mass
  - Additional wave modes should also appear in media with internal structures
  - The internal masses can be small, but this is not guaranteed
  - For example, in porous rock, there exists the so-called (Biot's) secondary P wave. This wave disappears when the rock is approximated by a VE model

VE problem #2: Time dependence instead of Heterogeneity

- The viscoelastic model was originally designed for explaining quasi-static deformations with negligible spatial variations, such as creep of the whole rock sample in a lab experiment (preceding section). In this case, the time is the only significant independent variable, and there exists only one complex-valued function $M(t)$ or $J(t)$ describing the whole system.
- However, in physics, heterogeneous deformations are always driven by spatial gradients of pressure, which lead to multiple types of waves and material and heat flows.
  - For example, stress σ at point $\mathbf{x}$ and time t is affected by the deformations of adjacent points at earlier times, and not by the earlier strain values at the same point $\mathbf{x}$.
  - At time *t* after an application of stress, the strain at point $\mathbf{x}$ is affected by boundaries and material-property contrasts at distance ct from this point, where c is the wave or some flow velocity within the medium.

VE problem #3: Non-surface stress forces

- Dispersion and attenuation effects can be caused by "global" pore-fluid flows within rock
- An example of a "drained-to-undrained transition" is shown here. This transition is caused by pore fluid flowing across the entire sample and into the tubes of the measurement apparatus ("dead volume")
- Effects of this flow lead to <u>body-force friction</u> which is not described by a stress tensor σ
  - Therefore, they cannot be described by the VE theory
  - Nevertheless, the shapes of the observed $M(f)$ and $Q^{-1}(f)$ are generally close to those of Zener's body
  - But this body is the whole apparatus

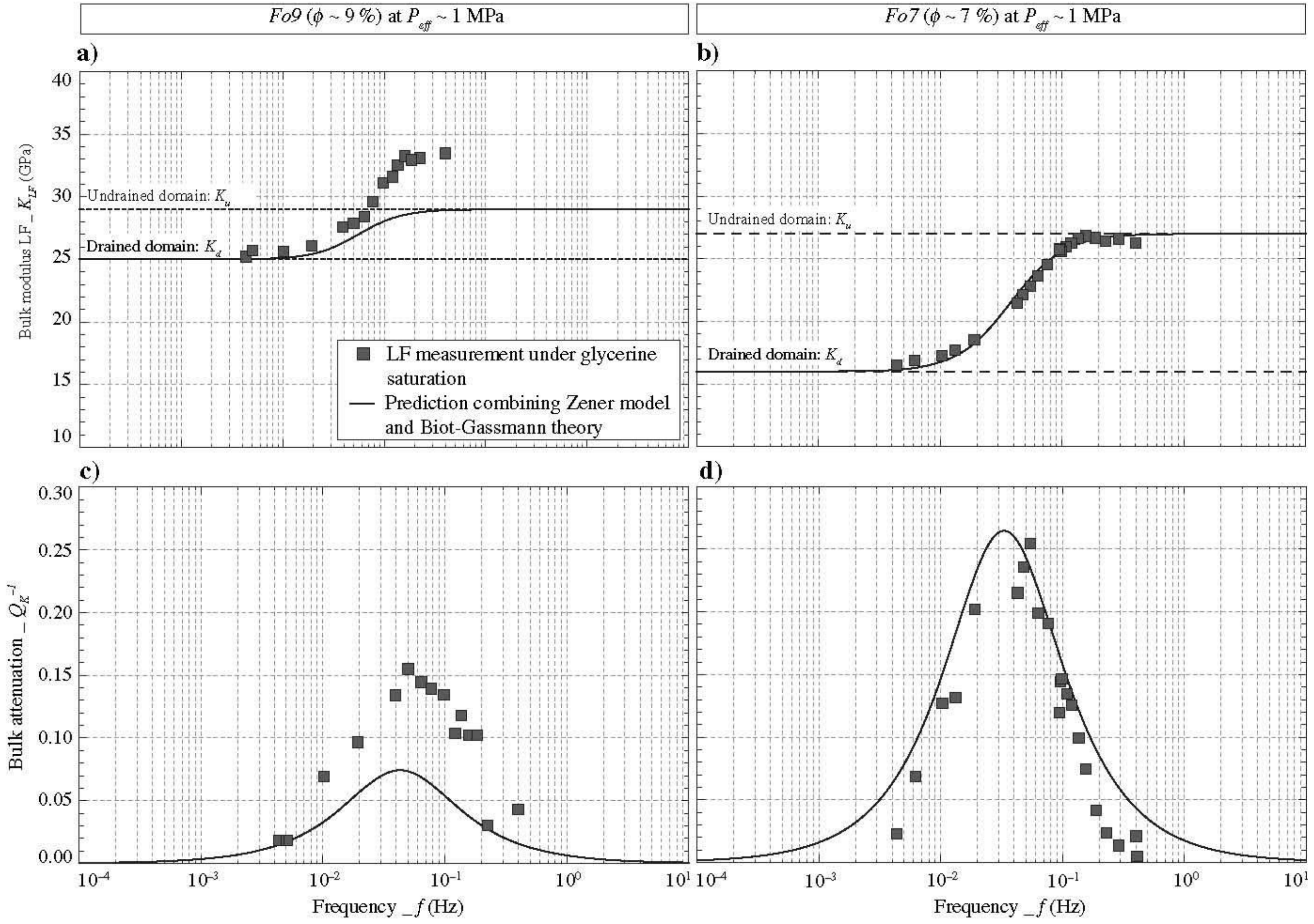
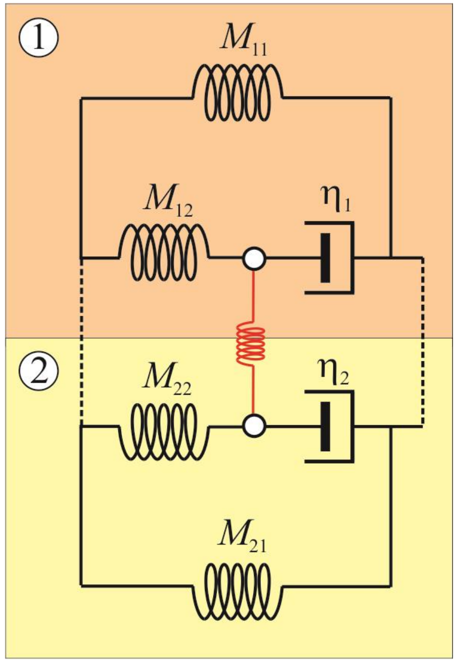


Figure 18. Comparison between measured and predicted (a) bulk modulus and (b) attenuation variations with frequency in the glycerin-saturated Fo7. The prediction is obtained by combining the Zener rheological model and the Biot-Gassmann theory.

Pimienta et al. (2015)

VE problem #4: Missing boundary conditions

- Imagine two layers of rock in a welded contact, for example two layers of brine-saturated and oil-saturated sandstone. Let us approximate each of these layers by an SLS (Zener's) rheology (figure on the right)
- Question: Do there exist mechanical elements connecting across the boundary? In particular, elements connecting the internal-variable points (red here)?
    - This is the big question. The VE model contains no connections like this and assumes all internal variables quasi-static and non-interacting.



VE problem #5: Nonphysical wave solutions

- Another serios problem with application of the viscoelastic model to seismic waves is in producing nonphysical wave solutions in layered media, or upon reflection from interfaces. Such nonphysical solutions have to be avoided by special mathematical conventions. For more on this subject, see, for example, Ruud (2006), Krebes and Daley (2007), and Vavryčuk (2010).

Thus, to what cases can the VE model be safely applied in seismic applications?

This question is not easy to answer

1) To model laboratory experiments in which **only a single pair** of $\sigma(t) \leftrightarrow \varepsilon(t)$ or $\sigma(f) \leftrightarrow \varepsilon(f)$ relations needs to be explained
   - However, these relations may be different for the same rock in *in-situ* conditions

2) To model waves in heterogeneous rock *in situ* **under certain conditions**. These conditions should ensure that the material can be viewed as locally homogeneous, and the non-VE wave modes are weak:
   - The scale of observation $L_{\mathrm{REV}} >> L_{\mathrm{structure}}$ (scale or micro- or meso-structure of the rock)
   - Simultaneously, distance from any boundaries $D >> \lambda_{\text{non-VE\_waves}}$ (lengths of secondary waves not accounted for by the VE model, such as Biot's slow waves)
     - However, this conditions is difficult to satisfy in practice (~20-m long waves, ~10-cm scale layering)
   - "Global" pore-fluid flows (unidirectional or divergent pore flows across the entire REV are insignificant
     - This condition may also be difficult to enforce in laboratory experiments with "dead volumes" of pore fluids outside of the sample