Lecture 4: Applications of Lagrangian Mechanics

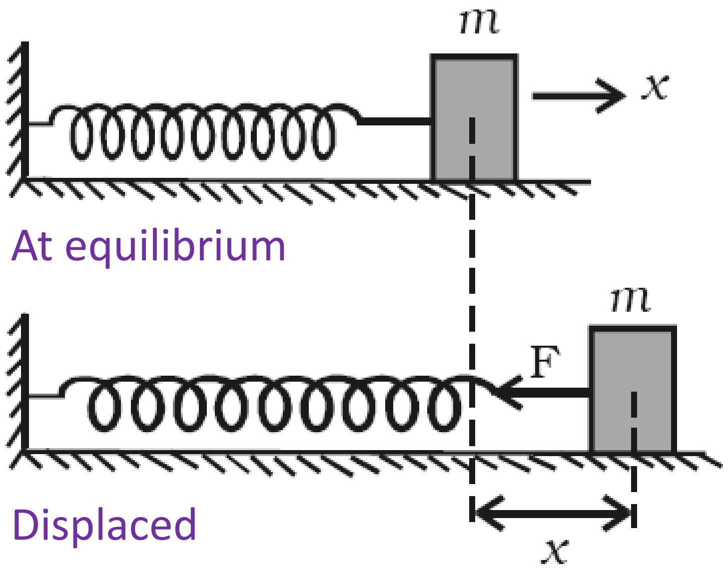
- Key examples: Linear oscillator, multidimensional oscillator
- Mass-Stiffness-Damping model
- Oscillation modes and waves
- Attenuation coefficient
- *Q*-factor: meaning, definitions, resonance and forced-oscillation *Q*, multiple forms
    - Frequency dependence of *Q*
- Effective viscoelasticity and effective density

- Reading: Chapter 4 in the text

Goals of this lecture

- You have likely heard about the "*Q*-factor" for Earth materials
- Is *Q* the same thing as "attenuation" and "energy dissipation"?
- <u>This is not quite so</u>, and we are going to elucidate these concepts

Linear harmonic oscillator with damping



- Linear harmonic oscillator is the simplest mechanical system containing all key features of laboratory experiments and waves
  - This may be a good model of low-frequency mechanical testing of a rock core in the laboratory
  - This is a system with a single observable variable x and elastic force $\mathbf{F}$ proportional to $\mathbf{x}$ ('linear')
  - Three "material properties": mass m, natural frequency ω_0, and damping constant ξ
- The Lagrangian (kinetic energy E_k minus potential energy E_p) is:

$$L(\mathbf{r},\dot{\mathbf{r}}) = \frac{m}{2}\dot{\mathbf{r}}^2 - \frac{m\omega_0^2}{2}\mathbf{r}^2 ,$$

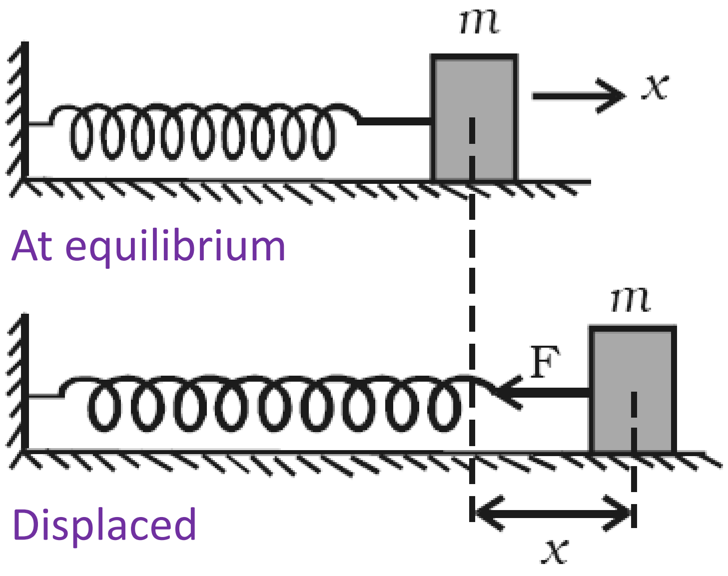
where $m\omega_0^2 = k$ is the spring constant

- The dissipation function is proportional to the kinetic energy:

$$D = \xi\omega_0 E_k = \xi\omega_0 \frac{m}{2}\dot{\mathbf{r}}^2$$

Linear oscillator

- From the Lagrangian:
  - Momentum: $$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}}$$
  - Mechanical energy (denoted H in Lecture 3): $$E_{mech} = \dot{\mathbf{r}}\frac{\partial L}{\partial \dot{\mathbf{r}}} - L = \frac{m}{2}\dot{\mathbf{r}}^2 + \frac{m\omega_0^2}{2}\mathbf{r}^2$$
  - Elastic force (denoted Q before): $$\mathbf{f} = \frac{\partial L}{\partial \mathbf{r}} = -m\omega_0^2\mathbf{r} = -k\mathbf{r}$$

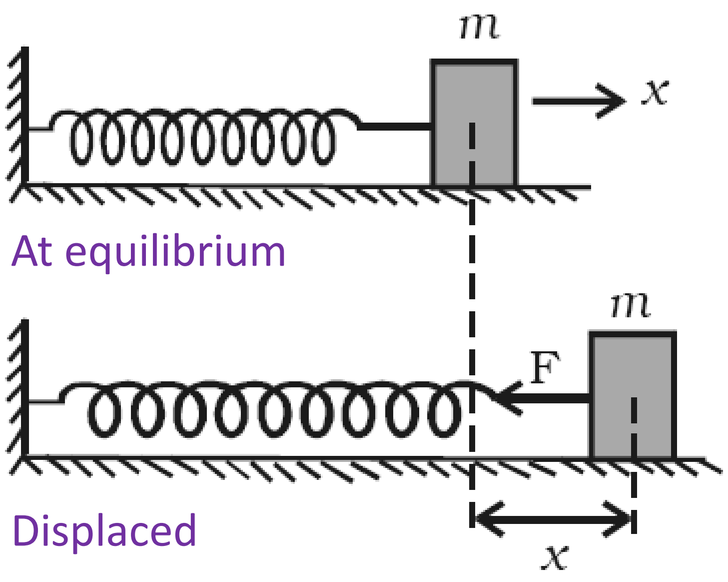


- From the dissipation function:
  - Force of friction (R before): $$\mathbf{f}_D = -\frac{\partial D}{\partial \dot{\mathbf{r}}} = -\xi m\omega_0\dot{\mathbf{r}}$$
  - Mechanical-energy rate of change: $$\frac{dE_{mech}}{dt} = -\dot{\mathbf{r}}\frac{\partial D}{\partial \dot{\mathbf{r}}} = -\xi m\omega_0\dot{\mathbf{r}}^2 = -2\xi\omega_0 E_k$$

Note that the force is proportional to velocity (due to linear viscosity)

Note that energy dissipation occurs "from the kinetic energy" rather than from the "stored" elastic energy as argued in seismology or materials-science texts

Linear oscillator



- Euler-Lagrange equation gives Newton's equation of motion:

$$m\ddot{\mathbf{r}} = -m\omega_0^2\mathbf{r} - \xi m\omega_0\dot{\mathbf{r}}$$

which simplifies to the "damped linear oscillator" equation:

$$\ddot{\mathbf{r}} + \xi\omega_0\dot{\mathbf{r}} + \omega_0^2\mathbf{r} = 0$$

This is why factor 'm' was included in all terms in L and D

- With $\xi << 1$ (weak damping), the general solution is:

Case $\xi = 1$ is called "critical damping"

$$\mathbf{r}(t) = \mathrm{Re}\left[\mathbf{A}\exp\left(-i\omega_0^* t\right)\right]$$

where

$$\omega_0^* \approx \omega_0\left(\pm 1 - \frac{i\xi}{2}\right) = \pm\omega_0 - i\chi$$, and $$\chi = \omega_0\frac{\xi}{2}$$ is the attenuation coefficient (discussed below)

and so:

$$\mathbf{r}(t) = \mathrm{Re}\left[\mathbf{A}\exp\left(\pm i\omega_0 t\right)\right]\exp\left(-\chi t\right)$$

Oscillations at ω_0 with amplitude decreasing with time

Q of a linear oscillator

- Let us consider two questions:
  1. What is the meaning of the Q-factor for an oscillator?
  2. Does this quantity apply to forced oscillations?
- The answer to the first question is simple: Q is a positive number measuring the relative width of the resonance peak in the amplitude response, taken at $1/\sqrt{2}$ of the peak level
  - The same width is more directly measured by parameter χ

Equation for displacement:

$$A(\omega) = \frac{f(\omega)}{m}\Lambda(\omega)$$

$f(\omega)$ is the spectrum of the force, $A(\omega)$ is the oscillation amplitude, $\Lambda(\omega)$ is the response function

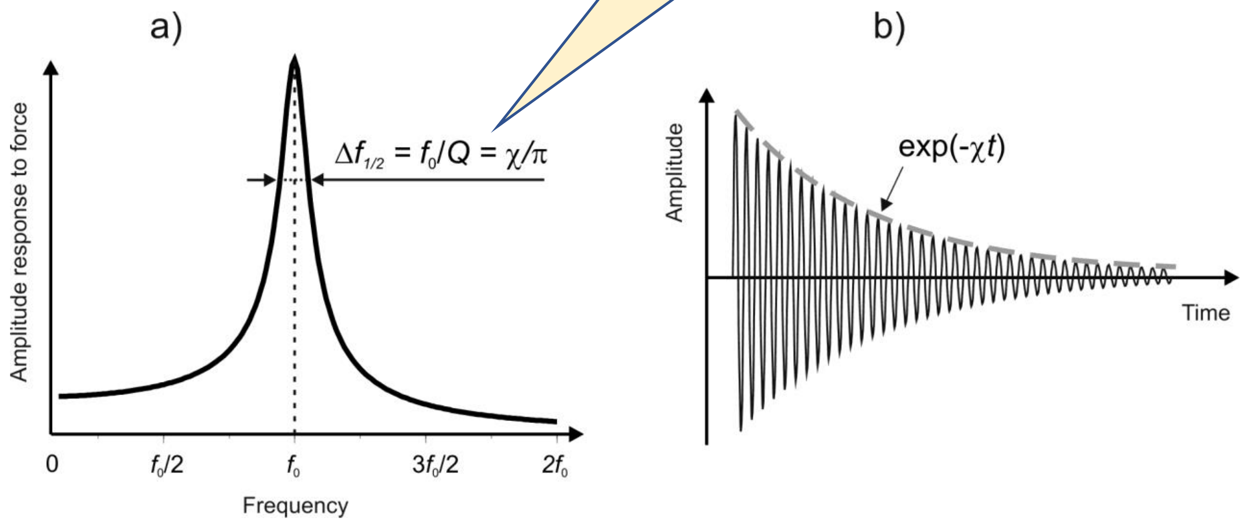
Complex response function:

$$\Lambda(\omega) = \frac{1}{\omega_0^2 - 2i\chi\omega - \omega^2}$$

Power:

$$\left|\tilde{\Lambda}(\omega)\right|^2 = \frac{1}{\left(\dfrac{\omega_0^2 - \omega^2}{2\omega_0\chi}\right)^2 + 1}$$

This is the meaning of the Q for oscillator (resonator)



Q of a linear oscillator

- Thus, the Q-factor is half of the ratio of the frequency of a resonance peak to its spectral width (which is the attenuation coefficient):

$$Q = \frac{\pi f_0}{\chi} = \frac{\omega_0}{2\chi} = \frac{1}{\xi}$$

- Note that this quantity:
  - Only refers to the resonance peak. It does not exist for non-resonant systems such as boundless rock
  - It should not be related to a modification of the spring constant k or mass m
  - It cannot be frequency-dependent (because the resonance contains only one frequency ω_0). The Q of a resonator is simply a constant equal $1/\xi$.

- So, what is meant when measuring a frequency-dependent Q from strain-stress relations in a forced-oscillation experiment? (answers will be given below)

Quiz #1:

- Consider a cylindrical rock sample, of length L and cross-sectional area A *(Figure).* The cylinder is mounted on one end, and force $\mathbf{F}(t)$ is applied to the other end. The density of the cylinder is ρ, and the Young's modulus is E. In addition to elastic modulus, the sample also has extensional viscosity η_E.
- Write the equation of motion for variable ε.
- Determine the natural frequency of oscillation and *Q*-factor of the sample

Attenuation coefficient

- When discussing "attenuation," we need to look at the **attenuation coefficient** first
- Term "attenuation" usually refers to observing a decrease of amplitude with time t (for oscillation of a finite body) or with travel distance x (for a wave):

$$A(t) = A(0)e^{-\chi t}$$

For a wave, the 't' in the exponent is the travel time: $t = \frac{x}{c}$

- Parameter χ in this relation is the attenuation coefficient
- From an observed decay of $A(t)$ with time, χ can be obtained in two ways:
  1. By measuring the decrease of mechanical energy averaged over a period: $\bar{E}_{mech}(t) = \bar{E}_{mech}(0)e^{-2\chi t}$

     therefore: $$\chi = -\frac{\dot{\bar{E}}_{mech}}{2\bar{E}_{mech}}$$

  2. By measuring the displacement-acceleration phase lag: if $u(t) = A(t)e^{-i\omega t} = A(0)e^{-i\omega^* t}$, (usually $\chi \ll \omega$) where $\omega^* = \omega - i\chi$, then $\ddot{u}(t) = -\omega^{*2}u(t)$, and its phase advance relative to $u(t)$ equals

$$\delta = \arctan\left(\frac{2\chi}{\omega}\right)$$

this phase lag is viewed as strain-stress phase lag in the viscoelastic approach

this $\frac{2\chi}{\omega}$ is usually denoted Q^{-1}

(but this notation should be understood carefully)

Definitions and meanings of Q

- It is always better to use the inverse Q^{-1} (which equals zero when there is no energy dissipation) instead of the "quality factor" Q. Recall that for a resonance peak (oscillator), the Q^{-1} is simply ξ:

$$Q^{-1} = \frac{2\chi}{\omega_0} \approx \xi$$

Physically, this quantity can be understood as the ratio of the energy dissipation rate to the kinetic energy of oscillation:

$$Q^{-1} = \xi = \frac{-\dot{E}_{mech}}{2\omega_0 E_k} \qquad \text{(taken at any time } t\text{)} \qquad (***)$$

- However, in seismic and lab applications, different types of Q^{-1} are used. Let us consider them

- For waves (Aki and Richards, 2002), the wave-period average energy-dissipation rate are measured as in eq. (***): $\bar{\dot{E}}_{mech} = \text{peak}\left(\dot{E}_{mech}\right)/2$, but in the denominator, the peak kinetic or elastic energies are taken. Thus, the definition of **seismic Q^{-1}** is

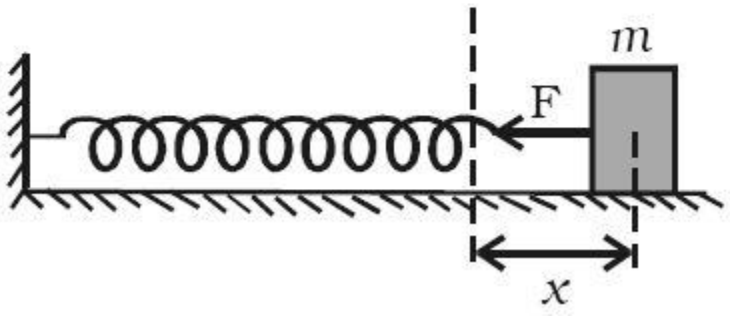
$$Q^{-1} = \frac{-\bar{\dot{E}}_{mech}}{\omega E_{ref}}$$

where the reference energy level:

$$E_{ref} = \text{peak}\left(E_k\right) = \text{peak}\left(E_p\right) = \frac{1}{2}\text{peak}\left(E_{mech}\right)$$

Q-factors for forced oscillations

$F(t) = F_0 \cos(\omega t)$

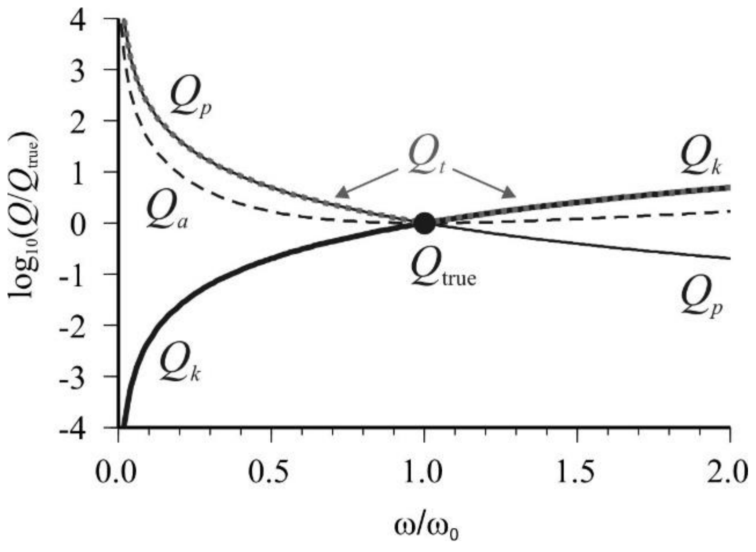


- For forced oscillations at variable frequency, the same definition of Q^{-1} is used, <u>but ω_0 (fixed) is replaced with the variable frequency ω</u> of the applied force:

$$Q^{-1} = \frac{-\dot{E}_{mech}}{\omega E_{ref}}$$

- This replacement makes Q^{-1} frequency-dependent, and also dependent on how one specifies E_{ref}. Consider several approaches giving different "Q-factors" (graphs on the right):

1. If we think that the kinetic energy is the one which "dissipates", then $E_{ref} = \mathrm{peak}(E_k)$ and Q increases with ω: $$Q_k = \frac{\omega}{\omega_0} Q$$
2. If we consider the potential energy as the dissipating one, then Q decreases with ω: $$Q_p = \frac{\omega_0}{\omega} Q$$
3. We can also assume that the peak total mechanical energy dissipates, and then: $$Q_t = \begin{cases} Q_p & \text{for } \omega < \omega_0, \\ Q_k & \text{for } \omega \geq \omega_0, \end{cases}$$
4. Also, some people define Q^{-1} using <u>the total mechanical energy averaged over a period.</u> In this case: $$Q_a = \frac{Q_k + Q_p}{2}$$



Phase-lag *Q*

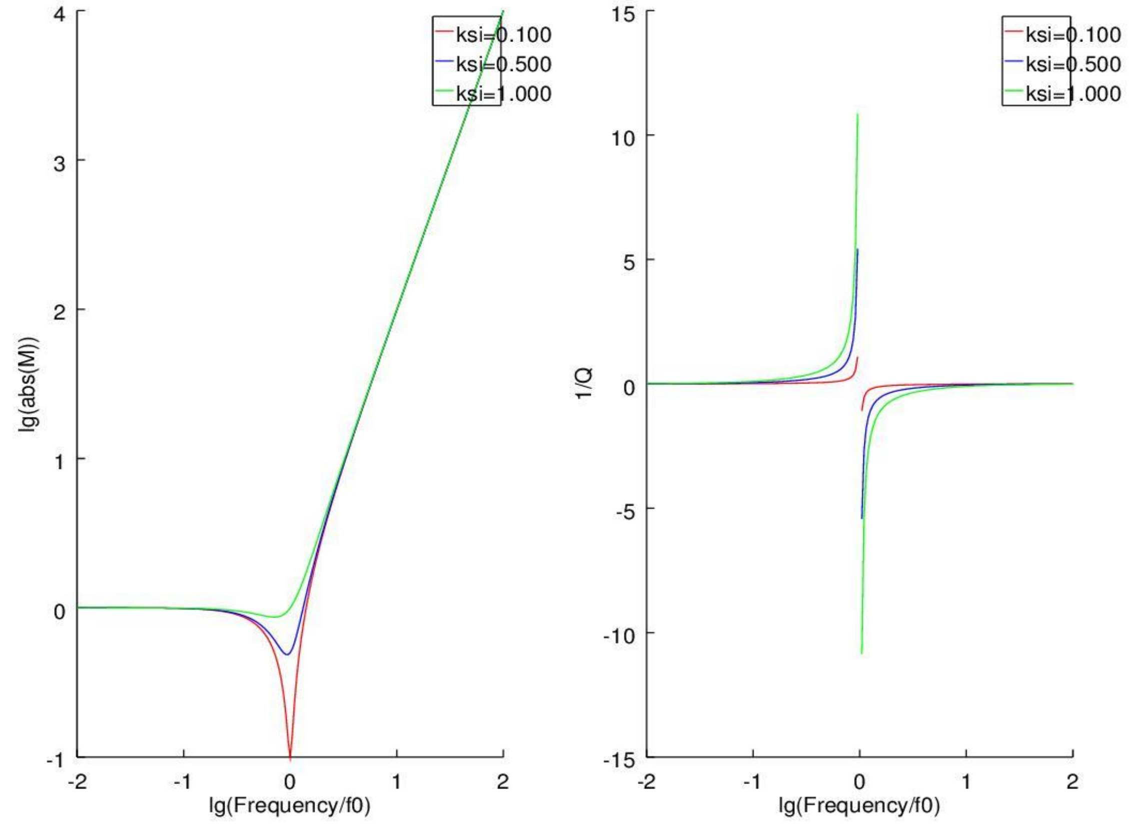
- The above *Q*-factors measured from the dissipated power come from studies with seismic waves (Aki and Richards, 2002), but in the laboratory, the stress-strain (displacement-force) phase-lag *Q* is typically used: $Q_\delta^{-1} \equiv \tan\delta$

- The complex-valued displacement equals: $u(\omega) = \frac{\hat{f}/m}{\omega_0^2 - 2i\chi\omega - \omega^2}$

- Therefore, the phase lag is also related to the attenuation coefficient χ:

$$\delta = Arg\left(\omega_0^2 - \omega^2 + 2i\chi\omega\right) = \arctan\frac{2\chi\omega}{\omega_0^2 - \omega^2}$$

$M(f)$ and $Q^{-1}(f)$ for an oscillator

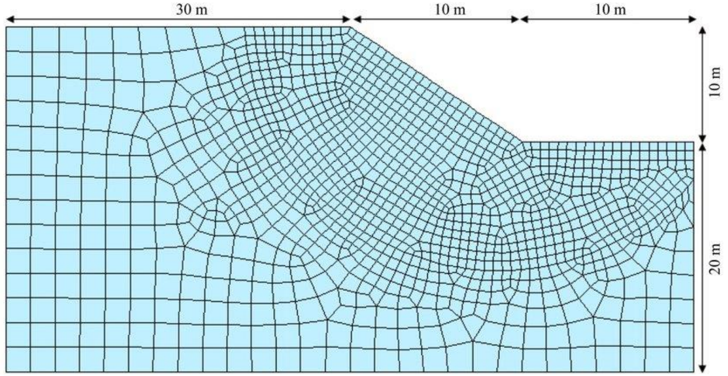


- At $\omega \ll \omega_0$, this *Q* is close to Q_p and inversely proportional to frequency:

$$Q_\delta = \frac{\omega_0^2 - \omega^2}{2\chi\omega} \approx \frac{\omega_0^2}{2\chi\omega} = Q_p$$

- At $\omega \gg \omega_0$, the phase lag is near 180°, and the *Q* is negative: $Q_\delta < 0$

This low-frequency limit is used in practice



Systems with many oscillation/relaxation modes: Mass-Stiffness-Damping (MSD) method

- Now let us consider an arbitrarily complex but finite linear mechanical system
  - For example, this system can be a 3-D volume of subsurface gridded on a regular grid or a variable mesh
- All generalized coordinates of the system can be combined in a single vector of variables:

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \vdots \\ q_N \end{pmatrix}$$

- Then, in the most general linear case, the dynamics of the system is completely described by three matrices: "mass" $\mathbf{M}$, "stiffness" $\mathbf{S}$, and "damping" $\mathbf{D}$:

$$\begin{cases} L = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^T \mathbf{S}\mathbf{q}, \\ D = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{D}\dot{\mathbf{q}}, \end{cases}$$

The stiffness matrix is often denoted $\mathbf{K}$, but we reserve this symbol for *bulk modulus*

- These matrices are symmetric and non-negative definite (i.e., for example, $\mathbf{q}^T \mathbf{S}\mathbf{q} \geq 0$ for any vector $\mathbf{q}$)

MSD equations and solutions

- The Euler-Lagrange equations with external force $\mathbf{f}$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} + \frac{\partial D}{\partial \dot{\mathbf{q}}} = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{S}\mathbf{q} = \mathbf{f}$$

- In full-waveform inversion or modeling forced oscillations of a rock sample in the lab, this time-spatial domain equation is often transformed into the frequency-spatial ("f-x") domain (time derivatives replaced with factors $-i\omega$):

$$\left(-\omega^2\mathbf{M} - i\omega\mathbf{D} + \mathbf{S}\right)\mathbf{q} = \mathbf{f}\left(\omega\right)$$

- This equation can be solved with some form of matrix inverse:

$$\mathbf{q} = \mathbf{G}\mathbf{f} \qquad \text{where} \qquad \mathbf{G} = \left(-\omega^2\mathbf{M} - i\omega\mathbf{D} + \mathbf{S}\right)^{-1}$$

is the frequency-domain response function for the mechanical system

- At low frequencies and short distances, the effect of $\mathbf{M}$ may be negligible compared to $\mathbf{D}$ and $\mathbf{S}$ (like when modeling a rock sample in the lab), and the quasi-static approximation can be used:

$$\mathbf{q} \approx \left(-i\omega\mathbf{D} + \mathbf{S}\right)^{-1}\mathbf{f}$$

Static equilibrium under force $\mathbf{f}$: $\mathbf{q} = \mathbf{S}^{-1}\mathbf{f}$

Effective viscoelasticity and/or effective mass

- In the above equation:

$$\left(-\omega^2\mathbf{M} - i\omega\mathbf{D} + \mathbf{S}\right)\mathbf{q} = \mathbf{f}\left(\omega\right)$$

- The term $-i\omega\mathbf{D}$ can be included in one of the other ones, giving two possible, alternate interpretations:
  1. As part of an effective, complex-valued “viscoelastic modulus (stiffness)” matrix: $\mathbf{S}^* = \mathbf{S} - i\omega\mathbf{D}$
     - This approach is taken in viscoelastic interpretations
     - It works better for lower ω. In this case, $\mathbf{D}$ gives a small correction to the ‘elastic’ stiffness $\mathbf{S}$. This modification illustrates the correspondence principle.
  2. As part of a complex-valued “effective density”: $\mathbf{M}^* = \mathbf{M} + i\dfrac{\mathbf{D}}{\omega}$
     - This approach is used to explain empirical-modulus relaxation in Biot’s poroelasticity and in models of oscillations of granular media
     - This interpretation seems more appropriate at higher ω

Oscillation modes and waves

- Consider external force $\mathbf{f} = \mathbf{0}$ in the above equation (i.e., homogenous equation):

$$\left(-\omega^2\mathbf{M} - i\omega\mathbf{D} + \mathbf{S}\right)\mathbf{q} = \mathbf{0}$$

- This equation describes free oscillations or waves (in a spatially unbounded model). Its solution consists of multiple mode vectors $\mathbf{q}^{(n)}$ satisfying a <u>generalized eigenvalue problem:</u>

$$\omega_n^2\mathbf{M}\mathbf{q}^{(n)} = \left(\mathbf{S} - i\omega_n\mathbf{D}\right)\mathbf{q}^{(n)}$$

- Vector $\mathbf{q}^{(n)}$ here gives the spatial distribution of the oscillation or wave mode, and ω_n is its frequency
- Because of the matrix character of this equation, it is only satisfied by certain vectors $\mathbf{q}^{(n)}$ (eigenvectors, or eigenmodes), and ω_n^2 above is the eigenvalue corresponding to the n^{th} mode

Mechanical-energy conservation and equipartitioning

- In the absence of friction ($\mathbf{D} = \mathbf{0}$), the spatially-averaged kinetic and elastic energies equal each other:

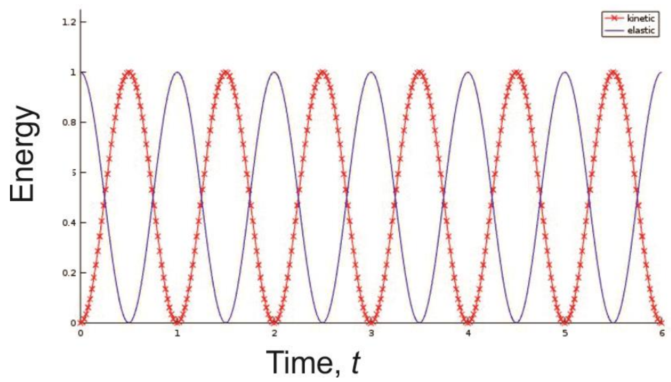
$$\left(-\omega^2\mathbf{M}+\mathbf{S}\right)\mathbf{q}=\mathbf{0} \quad \text{, or} \quad \omega^2\mathbf{Mq}=\mathbf{Sq} \quad \text{, or} \quad \frac{\omega^2}{2}\mathbf{q}^\dagger\mathbf{Mq}=\frac{1}{2}\mathbf{q}^\dagger\mathbf{Sq}$$

(using Hermitian conjugate instead of transpose for complex-valued $\mathbf{q}$)

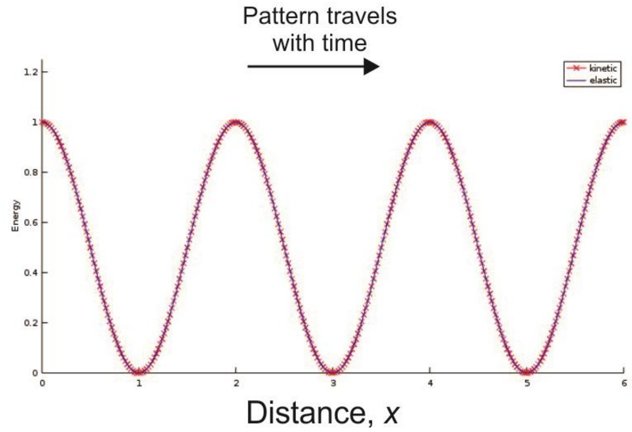
- This equation is known as the equipartitioning of energy
  - It can be used, for example, for determining the frequency of a surface waves from a given wavenumber: $\omega_0 = \sqrt{\dfrac{\mathbf{q}^\dagger\mathbf{Sq}}{\mathbf{q}^\dagger\mathbf{Mq}}}$

- In an elastic body, the energies oscillate at double frequency. Oscillations of elastic and kinetic energies are phase-shifted by 180°, and their sum is constant in time (plot a))
- In a wave, the energies are in-phase, and the pattern of total-energy highs and lows travels in space (plot b))

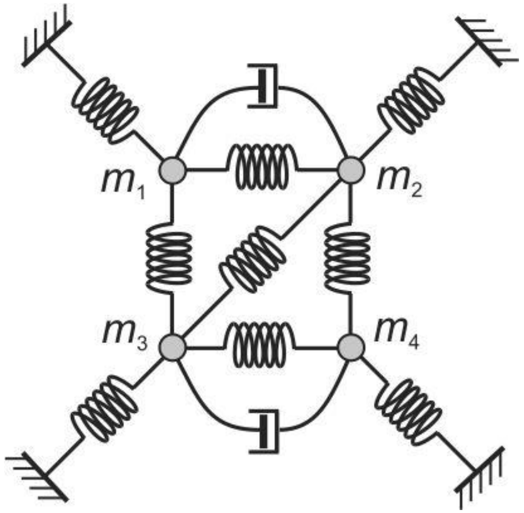
a) Time dependence in a body



b) Spatial dependence in a wave



Linked oscillators



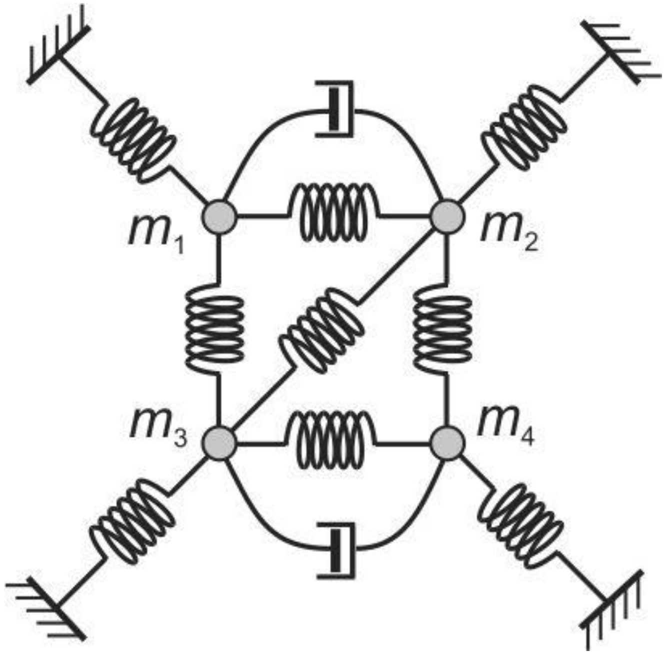
- A system of interlinked linear oscillators (multidimensional oscillator) represents a close analogy to the elastic medium
- It helps understanding the behavior of a rock sample in the laboratory or multiple types of waves in the field. Such models can also be viewed as a simplified (but insightful) approach to "digital rock"

$$E_k = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} \qquad E_p = \frac{1}{2}\mathbf{q}^T\mathbf{S}\mathbf{q} \qquad D = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}}$$

- With $\mathbf{D} = 0$, the elastic system would show harmonic oscillations satisfying: $\omega_n^2\mathbf{M}\mathbf{q}^{(n)} = \mathbf{S}\mathbf{q}^{(n)}$
- The time-averaged total mechanical energy equals $\bar{E}_{mech} = \frac{1}{2}\hat{\mathbf{q}}^\dagger\left(\omega^2\mathbf{M}+\mathbf{S}\right)\hat{\mathbf{q}}$
- …the energy dissipation rate: $\dot{\bar{E}}_{mech} = -\frac{1}{t}\int_0^t \dot{q}_i\frac{\partial D}{\partial \dot{q}_i}d\tau = -\frac{\omega^2}{2}\hat{\mathbf{q}}^\dagger\mathbf{D}\hat{\mathbf{q}}$
- …therefore, attenuation coefficient: $\chi = \frac{-\dot{\bar{E}}_{mech}}{2\bar{E}_{mech}} = \frac{\omega^2}{2}\frac{\hat{\mathbf{q}}^\dagger\mathbf{D}\hat{\mathbf{q}}}{\hat{\mathbf{q}}^\dagger\left(\omega^2\mathbf{M}+\mathbf{S}\right)\hat{\mathbf{q}}} \approx \frac{\omega^2}{2}\frac{\hat{\mathbf{q}}^\dagger\mathbf{D}\hat{\mathbf{q}}}{\hat{\mathbf{q}}^\dagger\mathbf{S}\hat{\mathbf{q}}}$ (quasi-static approximation)
- …and finally, Q for the mode: $Q = \frac{\omega}{2\chi}$

Quiz #2

- Taking X and Y coordinates of the masses in the preceding slide (repeated here) as generalized variables, write elements of matrices $\mathbf{M}$, $\mathbf{S}$, and $\mathbf{D}$ related, for example, to mass m_1



Conclusions about the frequency-dependent Q and χ

- A well-defined and meaningful Q-factor only exists for an oscillator (resonator), or oscillation mode for a complex mechanical system
- For forced oscillations or waves at variable frequencies, values of Q and their frequency dependencies are sensitive to the adopted definitions and measurement procedures
- Conventionally, the phase-lag Q_δ is used in laboratory experiments
  - When using this quantity, note that with this attenuation Q^{-1} contains a built-in increase proportional to the frequency even for the simplest mechanical system (oscillator)
- The measured $Q^{-1}(f)$ can be transformed to a more useful and mechanically meaningful quantity - the frequency-dependent (empirical) attenuation coefficient:

$$\chi(\omega) = \frac{\omega Q^{-1}}{2} = \pi f Q^{-1}$$

  - This $\chi(\omega)$ directly relates to the rate of amplitude decay in a seismic wave:

$$A(t) = A(0)\exp\left(-\frac{\omega Q^{-1}}{2}t\right) = A(0)\exp\left[-\chi(\omega)t\right]$$

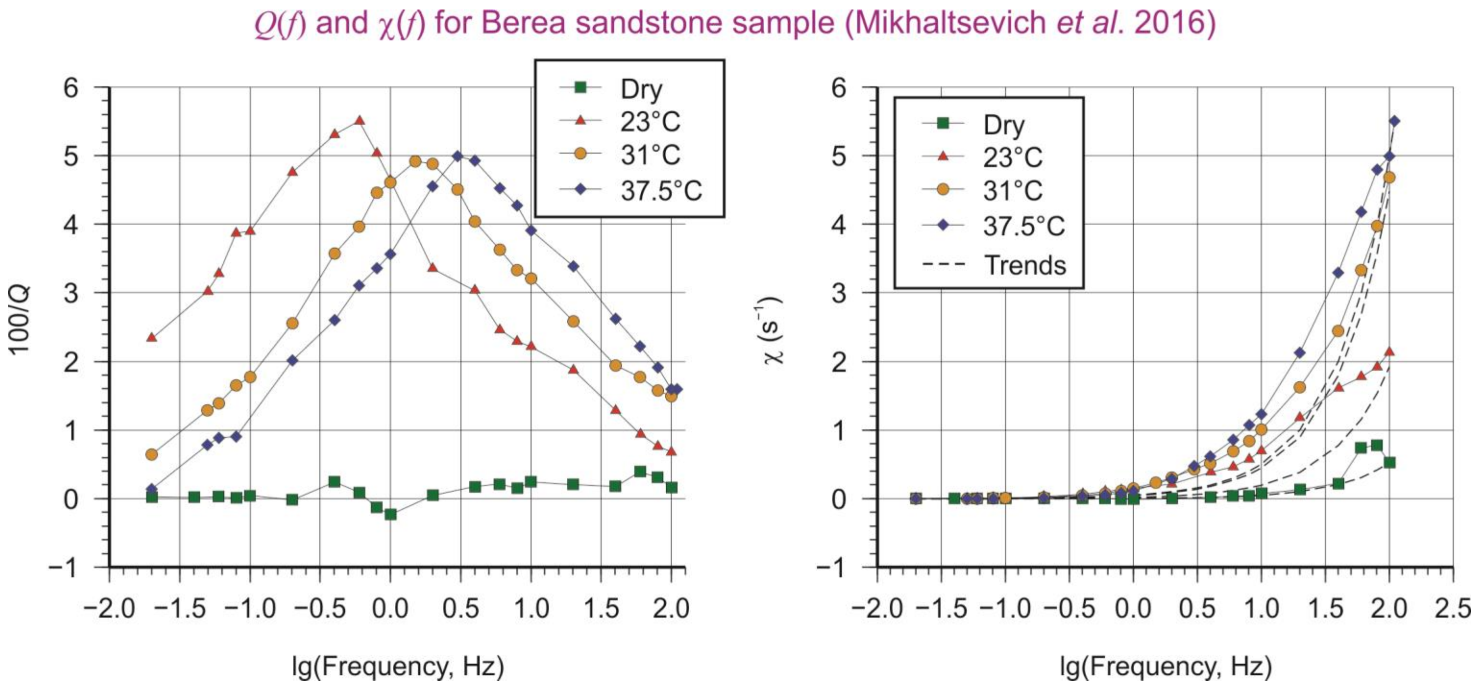
  - It has a simple frequency dependence with often nonzero $\chi(0)$:

$$\chi(\omega) \approx \gamma + \omega\frac{Q_e^{-1}}{2} + \omega^2\frac{\tau}{2} + \ldots$$

These first two terms are most important in $\chi(f)$, but they are practically lost when looking at the data in $Q(f)$ form

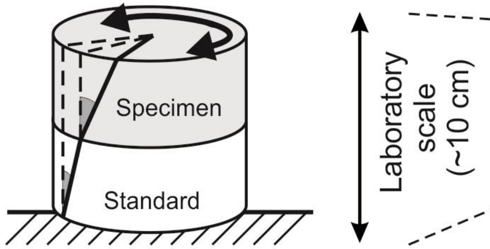
Conclusions about the frequency-dependent Q and χ

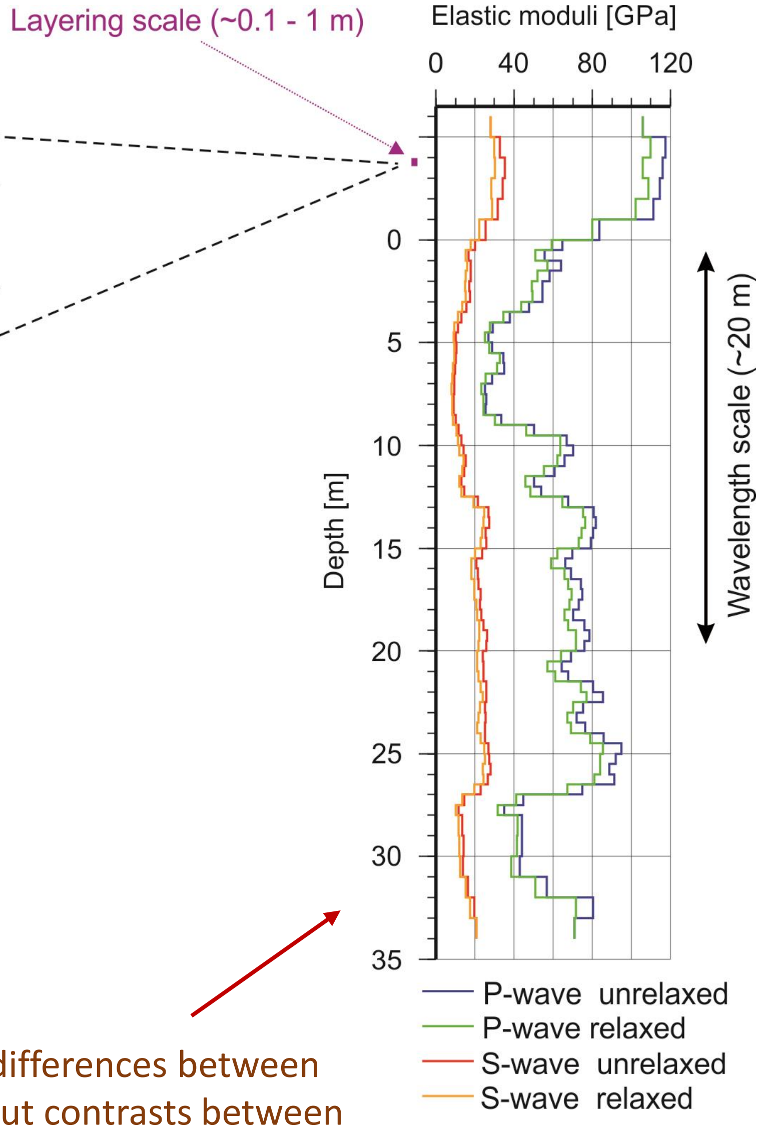
- For example, compare $Q(f)$ and $\chi(f)$ plots for a Berea sandstone sample at variable temperatures:



- Note that the major effects are the variable increases of attenuation with frequency (seen in $\chi(f)$ form *on the right*)
- … but presentation of the data by $Q(f)$ (*left*) misses these effects and only emphasizes the small bumps in $\chi(f)$

Scalability of χ or Q^{-1}





- Attributes $\chi(f)$ and $Q^{-1}(f)$ also differ in terms of scalability between laboratory and field experiments
  - $Q^{-1}(f)$ (phase lag) is an attribute sensitive to the phase of the signal (caused by scattering, layers)
    - It downplays the low-frequency parameters γ, q_e and τ_e
  - Quantity $\chi(f)$ has the same meaning as attenuation coefficient for a traveling wave (relative energy decay rate)
    - Quantities γ, q_e and τ_e should be more reliably scaled from laboratory to the layering and wavelength scales

Note that Q^{-1} of layers represents the differences between the "unrelaxed" and "relaxed" moduli, but contrasts between layers are often stronger.

Thus, the $Q^{-1}(f)$ seen in field data is mostly controlled by layering, which is not seen in lab experiments.