Lecture 6: Fluid-saturated porous rock (Biot’s model)

- Biot’s poroelastic model as a GLS with $N = 2$
- Elastic moduli and Skempton coefficients
- Gassmann’s equation
- Waves
- Extensions of Biot’s model

- Reading: Section 5.4 in the text

Biot's poroelasticity

- Biot's model of porous rock is well known from around 1940's. Its structure and approximations can be clearly seen in matrix GLS form:
  1. The model is macroscopic, i.e. it contains no notions of internal rock or pore structure (pore shapes, distribution, pore connectivity, etc.)
     - However, perfectly connected pores are assumed when considering dynamic processes, so that the pore pressure can equilibrate within the time span of the experiment
  2. The model describes the rock by only two generalized variables: displacement or the whole rock and only one pore fluid. This means that Biot's model is a GLS with $N = 2$.
     - Variable number 1 (u_{Ji} with $J = 1$) represents the observable displacement of the whole rock, and variable *J*=2 is the (hard to measure) average displacement of pore fluid
     - Accordingly, there is only one pair of stresses: pressure applied to the whole rock and to its pore fluid
  3. The elastic energy is only due to stresses of the rock and pore fluid, and not to the relative displacement between them. This excludes matrix ζ from the model.
  4. The pore fluid affects only bulk deformation. This means that the shear elastic modulus matrix μ is of a simple form discussed later.
  5. The pore fluid is viscous, but this viscosity causes only Darcy type (body-force) friction. This constraint means viscous GLS matrices equal zero.

Biot's poroelasticity

- With the above assumptions (simplifications), Biot's model is a special case of a GLS model with $N = 2$:

$$\begin{cases} L = \frac{1}{2}\dot{\mathbf{u}}_i^T \boldsymbol{\rho} \dot{\mathbf{u}}_i - \left(\frac{1}{2} \boldsymbol{\Delta}^T \mathbf{K} \boldsymbol{\Delta} + \tilde{\boldsymbol{\varepsilon}}_{ij}^T \boldsymbol{\mu} \tilde{\boldsymbol{\varepsilon}}_{ij} \right), \\ D = \frac{1}{2}\dot{\mathbf{u}}_i^T \mathbf{d} \dot{\mathbf{u}}_i, \end{cases}$$

where the i^{th} spatial component of the bivector of variables is a vector in 2-D model space (6 variables in total):

$$\mathbf{u}_i = \begin{pmatrix} u_{1i} \\ u_{2i} \end{pmatrix}$$ and $$u_{2i} = -\phi\left(u_{\text{fluid}} - u_{1i}\right)$$

$$\boldsymbol{\rho} = \begin{bmatrix} \rho & -\rho_f \\ -\rho_f & \frac{a}{\phi}\rho_f \end{bmatrix}$$

Density matrix

$$\mathbf{K} = \begin{bmatrix} K_U & -\alpha M \\ -\alpha M & M \end{bmatrix}$$

Bulk modulus matrix

$$\boldsymbol{\mu} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}$$

Shear modulus matrix

$$\mathbf{d} = \begin{bmatrix} 0 & 0 \\ 0 & \eta/\kappa \end{bmatrix}$$

Inverse mobility matrix

- Scalar quantity $\xi = \operatorname{div} u_2$ is called the fluid content

Fluid mobility:

$$M = \frac{\kappa}{\eta}$$

Darcy law:

$$flow = M \nabla p$$

Relations of generalized coordinates to actual displacements

- Our second GLS variable is often called "fluid content" and denoted ξ: $\begin{pmatrix} \Delta \\ \xi \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} \partial_i u_{1i} \\ \partial_i u_{2i} \end{pmatrix}$

- From these vectors of displacements and dilatations, the displacements and dilatation of the solid and fluid phases are obtained by matrix products:

$$\begin{pmatrix} u_{si} \\ u_{fi} \end{pmatrix} = \mathbf{U}\mathbf{u} \equiv \mathbf{U} \begin{pmatrix} u_{1i} \\ u_{2i} \end{pmatrix}, \qquad \begin{pmatrix} \Delta_s \\ \Delta_{fl} \end{pmatrix} = \mathbf{U} \begin{pmatrix} \Delta \\ \xi \end{pmatrix} \text{, where matrix } \quad \mathbf{U} = \begin{bmatrix} 1 & -1 \\ 1 & -1/\phi \end{bmatrix}$$

Selections of matrix forms for material properties

- Among the above material-property matrices, only $\mathbf{K}$ and $\boldsymbol{\rho}$ require further comments (others are simply definitions of K_U, μ, M, α, and m)
- In matrices $\mathbf{K}$ and $\boldsymbol{\rho}$, the off-diagonal elements are negative. Why?
- The general answer is in the following observations:
  - Note the meaning of matrix expressions for energy, for example:

$$\frac{1}{2}\boldsymbol{\Delta}^T\mathbf{K}\boldsymbol{\Delta} = \frac{1}{2}K_U\Delta_1^2 - \alpha M\Delta_1\Delta_2 + \frac{1}{2}M\Delta_2^2$$

1. The diagonal elements represent the values of energy in the end-member cases $u_1 = 0$ or $u_2 = 0$.
2. The variables are defined so that if u_1 and u_2 are nonzero and of the same sign, then the resulting energy is lower (because of the negative -aM) than the sum of energies due to the variables alone.
3. Thus, from energy standpoint, the system will prefer deformations with nonzero values of u_1 and u_2 of the same sign.

Density matrix

- Elements of the density matrix can be understood as follows:
  - This matrix means the following expression for kinetic energy:

$$E_k = \frac{1}{2}\dot{\mathbf{u}}_i^T \boldsymbol{\rho} \dot{\mathbf{u}}_i = \frac{\rho}{2}\dot{u}_{1i}\dot{u}_{1i} + \frac{a}{\phi}\frac{\rho_f}{2}\dot{u}_{2i}\dot{u}_{2i} - \rho_f \dot{u}_{1i}\dot{u}_{2i}$$

- So, if $u_2 = 0$ (pore fluid not moving relative to the rock), $E_k \frac{1}{2}\dot{\mathbf{u}}_i^T \boldsymbol{\rho} \dot{\mathbf{u}}_i = \frac{\rho}{2}\dot{u}_{1i}\dot{u}_{1i}$. This means that ρ is the density of the average rock:

$$\rho = (1-\phi)\rho_s + \phi\rho_f$$

- If $u_1 = 0$ (rock is not moving), then $u_{2i} = \phi\left(u_{1i} - u_{fi}\right) = -\phi u_{fi}$, and therefore $E_k = \phi a \frac{\rho_f}{2}\dot{u}_{fi}\dot{u}_{fi}$ This is the definition of tortuosity a (ratio of the actual kinetic energy in a flow to the energy of its averaged macroscopic equivalent)
- The third coefficient in term $\left(-\rho_f \dot{u}_{1i}\dot{u}_{2i}\right)$ is merely a Lagrangian representation of the force exerted on a solid body with coordinate $u_1(t)$ placed within a fluid flowing at velocity $\dot{u}_{2i}$.

Elastic parameters

- Elements of the bulk-modulus matrix $\mathbf{K} = \begin{bmatrix} K_U & -\alpha M \\ -\alpha M & M \end{bmatrix}$ can also be understood by considering various experiments with porous rock. For static (time-independent) experiments, the applied stress σ and pore pressure p are related to deformations as:

$$\begin{pmatrix} \sigma \\ -p \end{pmatrix} = \mathbf{K} \begin{pmatrix} \Delta \\ \xi \end{pmatrix}$$, and inverse $$\begin{pmatrix} \Delta \\ \xi \end{pmatrix} = \mathbf{J} \begin{pmatrix} \sigma \\ -p \end{pmatrix}$$, where compliance matrix $$\mathbf{J} = \mathbf{K}^{-1} = \frac{1}{K_D} \begin{bmatrix} 1 & \alpha \\ \alpha & K_U/M \end{bmatrix}$$

where $$K_D = \frac{\det \mathbf{K}}{M} = K_U - \alpha^2 M$$

- Under 'undrained' compression (zero pore flow, sealed sample, $\xi = 0$), $\sigma = K_U \Delta$, and therefore K_U is the 'undrained' modulus
- Under 'drained' compression (zero pore pressure variation, pores freely communicating with the outside space, $p = 0$), $\Delta = \frac{\sigma}{K_D}$, and therefore K_D is called the 'drained' modulus
- Under the same drained conditions, $\xi = \alpha\Delta$, and therefore the Biot-Willis parameter can be measured experimentally as ratio $\alpha = \left.\frac{\xi}{\Delta}\right|_{p=0}$.

Skempton coefficients

- Skempton coefficients are important ratios of experimental observations useful for obtaining poroelastic moduli
  - Skempton A is the same as Biot-Willis α -- ratio of change of fluid content to rock volume change under drained condition:

$$A = \alpha = \left.\frac{\xi}{\Delta}\right|_{p=0}$$

  - Skempton B is the ratio of the induced pore pressure to the change of stress loading under undrained condition

$$B = \left.\frac{p}{\sigma}\right|_{\xi=0} = \frac{\alpha M}{K_U}$$

- These relations give a way to measure poroelastic modulus M:

$$M = K_U \frac{B}{\alpha}$$

Elastic parameters

- To find the elastic modulus of the material of solid matrix ("solid grains"), again consider drained experiment ($p = 0$) and look at the expansion of the solid and fluid:

$$\begin{pmatrix} \Delta_s \\ \Delta_{fl} \end{pmatrix} = \mathbf{U} \begin{pmatrix} \Delta \\ \xi \end{pmatrix} = \frac{\sigma}{K_D} \begin{pmatrix} 1-\alpha \\ 1-\alpha/\phi \end{pmatrix}$$

, and therefore

$$\Delta_s = \frac{\sigma}{K_D}(1-\alpha)$$

- Because the fluid exerts no pressure in this experiment, all stress s should be supported by the solid frame. Therefore, the modulus of the material of the frame equals:

$$K_s = \frac{K_D}{1-\alpha}$$

- … or conversely, the Biot-Willis parameter is determined from the ratio of the modulus of the empty porous matrix to the modulus of its material

$$\alpha = 1 - \frac{K_D}{K_s}$$

- From these relations, α must satisfy: $\phi \le \alpha \le 1$

Elastic parameters

- To find the elastic modulus of the pore fluid in matrix $\mathbf{K}$, consider an experiment in which the rock frame and the fluid are held at equal pressures ($\sigma = -p$). This is an experiment with <u>unjacketed</u> rock sample
  - Then, the deformations of the solid and fluid phases are:

$$\begin{pmatrix} \Delta_s \\ \Delta_f \end{pmatrix} = \mathbf{UJ} \begin{pmatrix} -p \\ p \end{pmatrix} = \frac{p}{K_D} \begin{pmatrix} 2\alpha - \dfrac{K_U}{M} - 1 \\ \alpha - \dfrac{K_U}{\phi M} - 1 + \dfrac{\alpha}{\phi} \end{pmatrix}$$

- Δ_f here should equal $\Delta_f = \dfrac{-p}{K_f}$, and after some simplifications, K_f can be expressed:

$$\frac{\phi}{K_f} = \frac{1}{M} - \frac{\alpha - \phi}{K_s}$$

Gassmann's equation

- Gassmann's equation (next slide) predicts the bulk modulus of fluid-saturated rock (K_U) from porosity ϕ and the moduli of:
  - Solid grains (K_s)
  - Drained (empty) porous matrix (K_D)
  - Pore fluid (K_f)
- Because of the ability to use different K_f with the same rock frame, this equation is often used for fluid substitution (modeling the behavior of rock with different saturating fluids)

- Gassmann's equation $K_U = K_U(\phi, K_s, K_D, K_f)$ is just one relation satisfied by the static (elastic) limit of Biot's or any other model with single porosity

Gassmann's equation

- Gassmann's equation is simply an identity relation satisfied by the elastic matrix $\mathbf{K}$
- Matrix $\mathbf{K}$ contains only three independent elastic constants in (K_U, α, and M), but four empirical moduli K_U, K_D, K_s, K_f are measured from it in various experiments
  - Therefore, the empirical moduli must be mutually related. This relation is the Gassmann's equation
  - The simplest form of this relation was given above:

$$K_D = \frac{\det \mathbf{K}}{M} = K_U - \alpha^2 M \quad \text{, therefore} \quad K_U = K_D + \alpha^2 M$$

  - There are several forms of final relations excluding α and M using K_s and K_f. The following one looks most elegant, in the form of a Reuss average of the solid and drained compliances:

$$K_U^{-1} = \frac{K_s^{-1} + \phi' K_D^{-1}}{1+\phi'} \quad \text{, where} \quad \phi' \equiv \phi \frac{K_f^{-1} - K_s^{-1}}{K_D^{-1} - K_s^{-1}}$$

- One of the commonly used forms of Gassmann's equation is:

$$\boxed{K_U = K_D + \frac{\left(1 - \dfrac{K_D}{K_s}\right)^2}{\phi\left(\dfrac{1}{K_f} - \dfrac{1}{K_s}\right) + \dfrac{1}{K_s}\left(1 - \dfrac{K_D}{K_s}\right)}}$$

Waves in porous media ("global flow")

$$\boldsymbol{\rho}^* \equiv \boldsymbol{\rho} + i\frac{\mathbf{d}}{\omega}$$

$$k^* \equiv k + i\alpha$$

$$\gamma \equiv \frac{\left(k^*\right)^2}{\omega^2}$$

- Plane harmonic waves are obtained from the same GLS eigenvalue equation (matrix $\mathbf{M}$ is real-valued in Biot's model):

$$\boldsymbol{\rho}^* \boldsymbol{\upsilon}^{(n)} = \gamma^{(n)} \mathbf{M} \boldsymbol{\upsilon}^{(n)}$$

- This equation gives:
  - Two P-wave modes (primary wave and a diffusive "slow" secondary wave)
  - One S-wave mode for a given polarization)
- Biot's characteristic frequency:

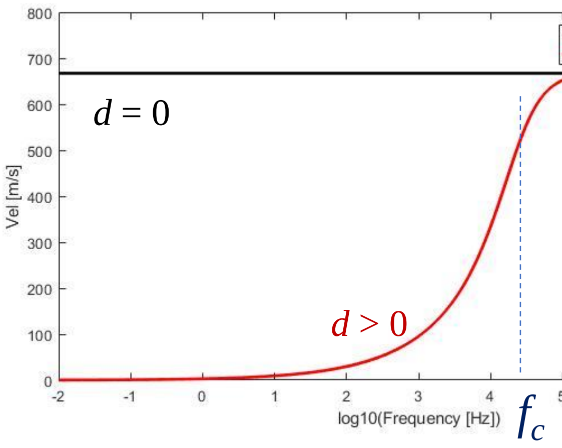
$$f_c = \frac{\phi\eta}{2\pi\rho_f \kappa}$$

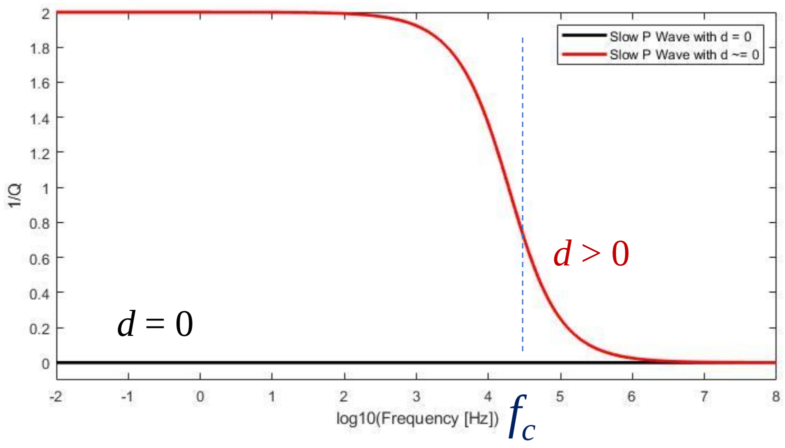
Note that the frequency increases with viscosity

  - At this frequency, there is a peak of $Q^{-1}(f)$
  - However, f_c is usually very high (30 kHz to 1 GHz for water), and therefore, the attenuation and dispersion due to this primary-pore ("Biot's", or "global") flow is usually weak

Wave velocity dispersion and attenuation (Q^{-1})

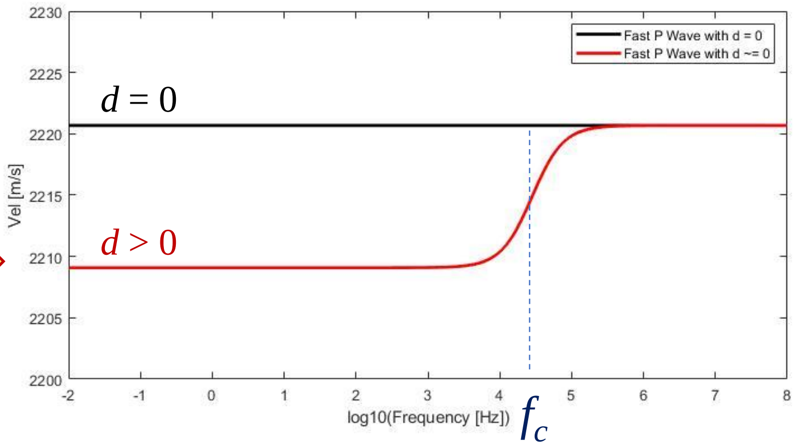
Secondary ("slow") P wave

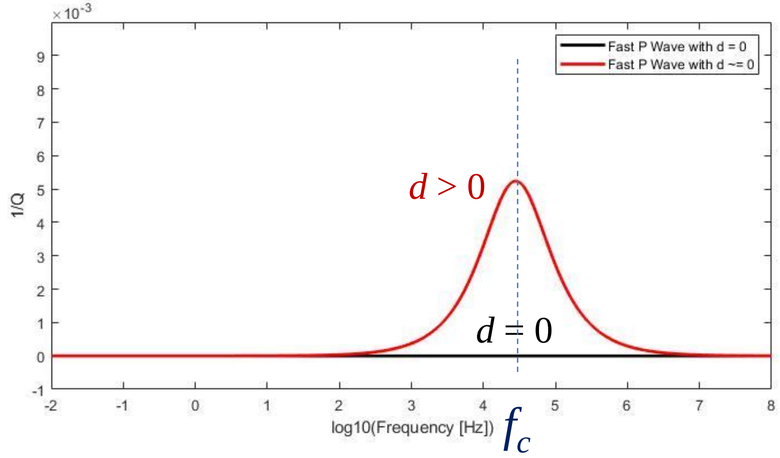




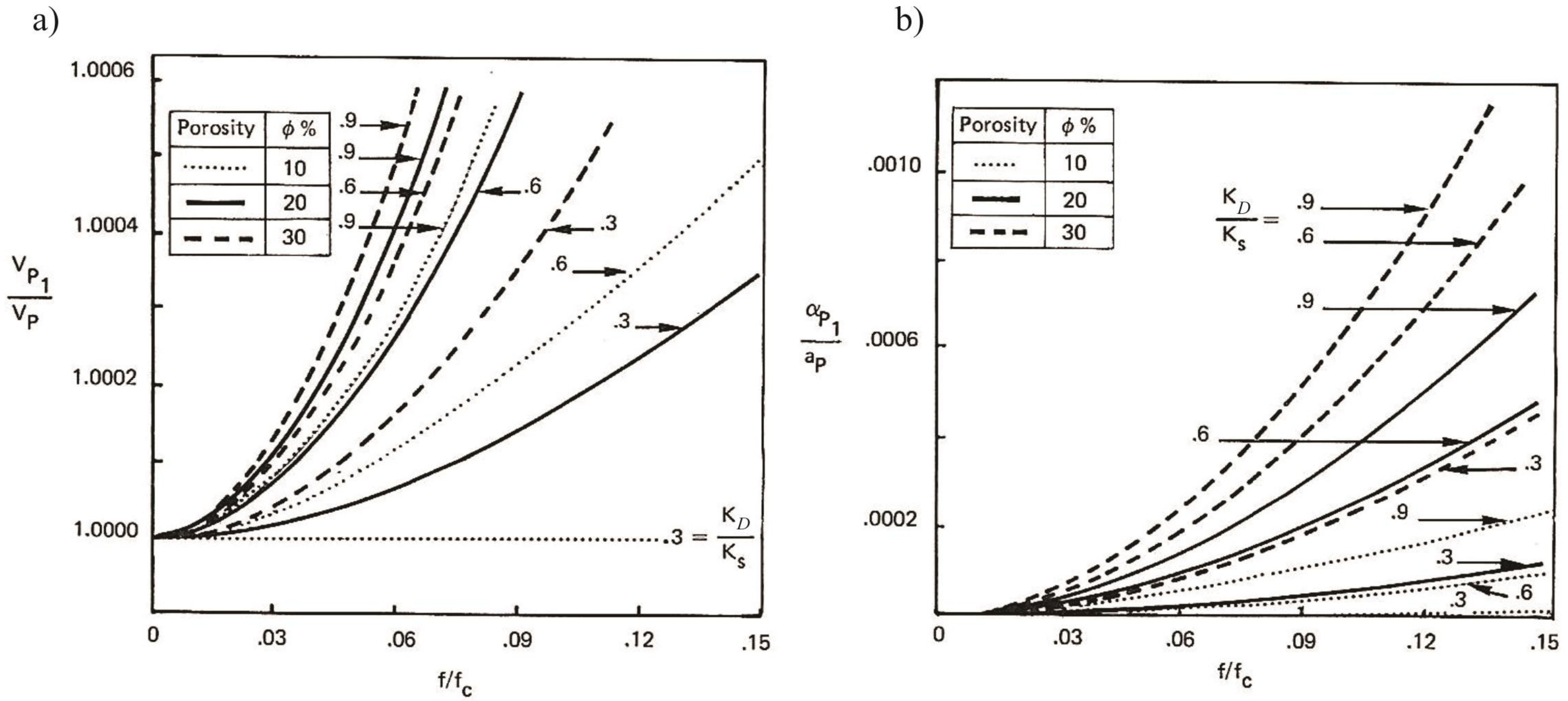
Primary ("fast") P wave

Effect of "complex density" ρ^*





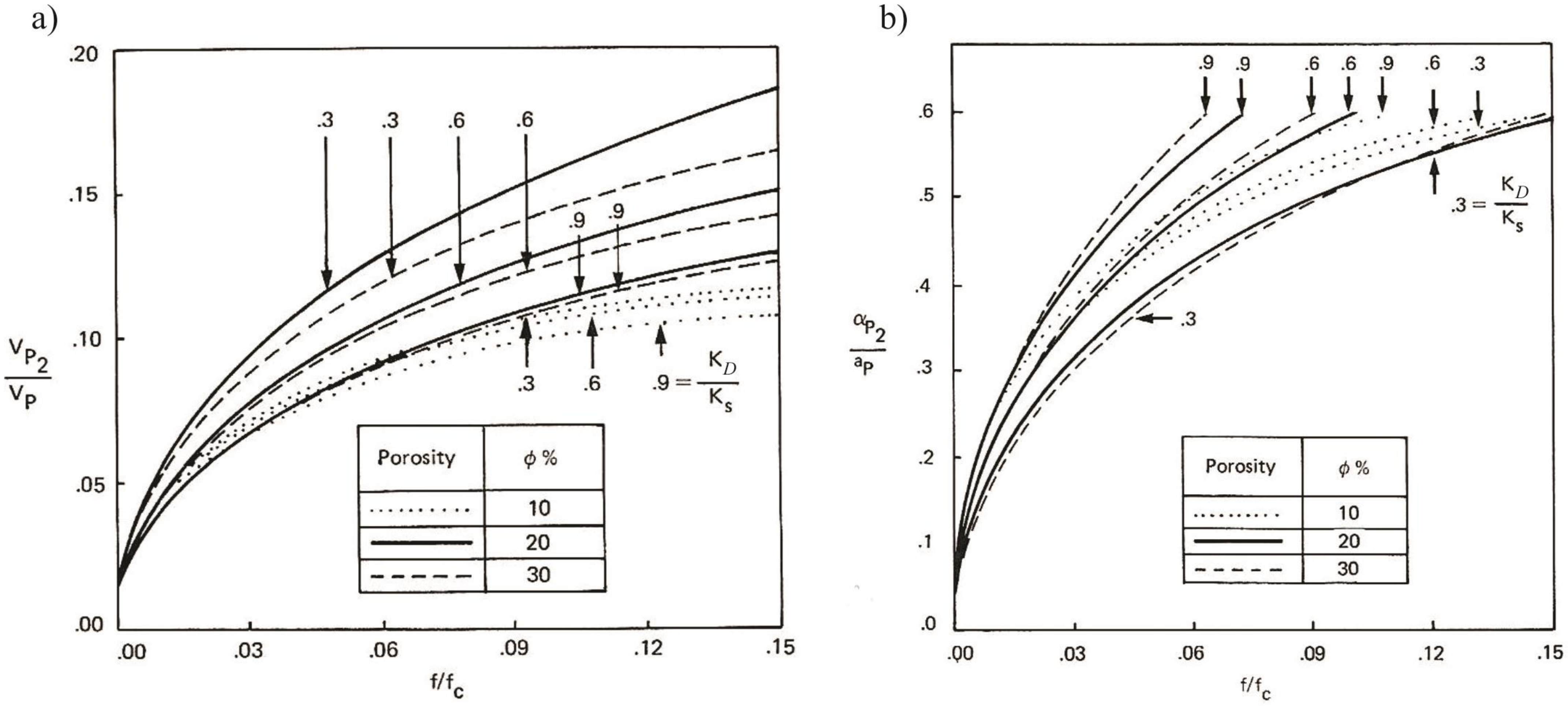
Primary P wave



- Attenuation $\alpha(f)$ scaled by $\alpha_P = \dfrac{2\pi f_c}{V_P}$

From Bourbié et al., 1987

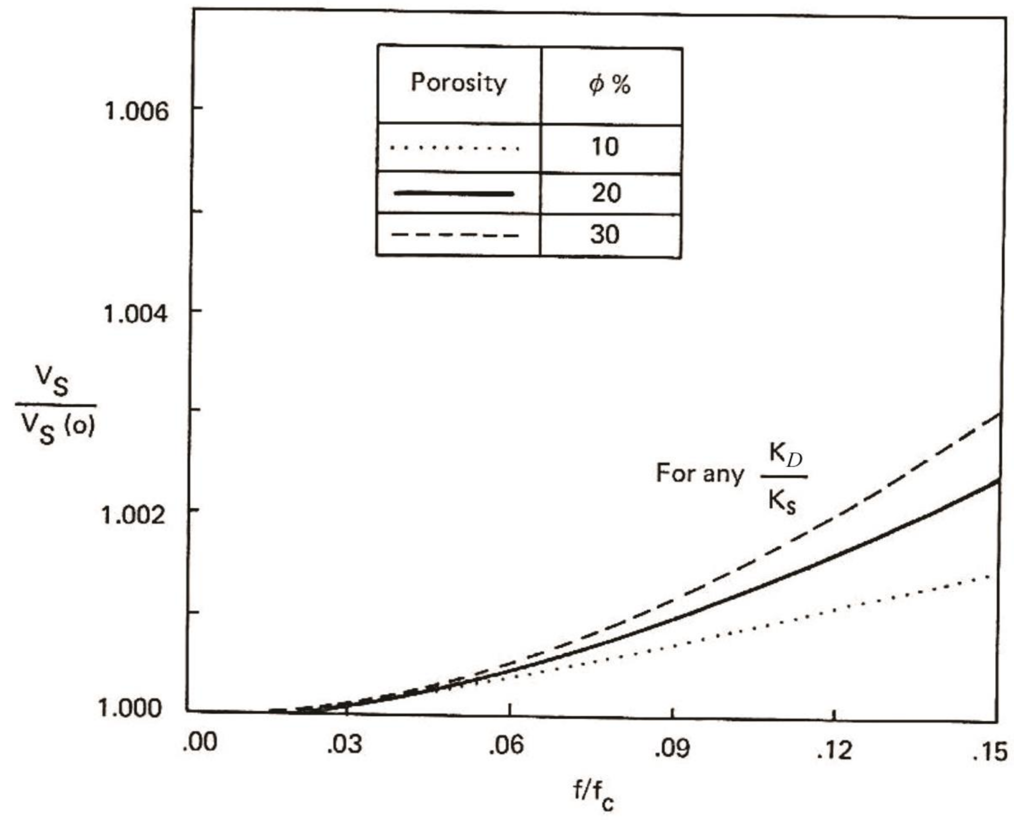
Secondary P wave



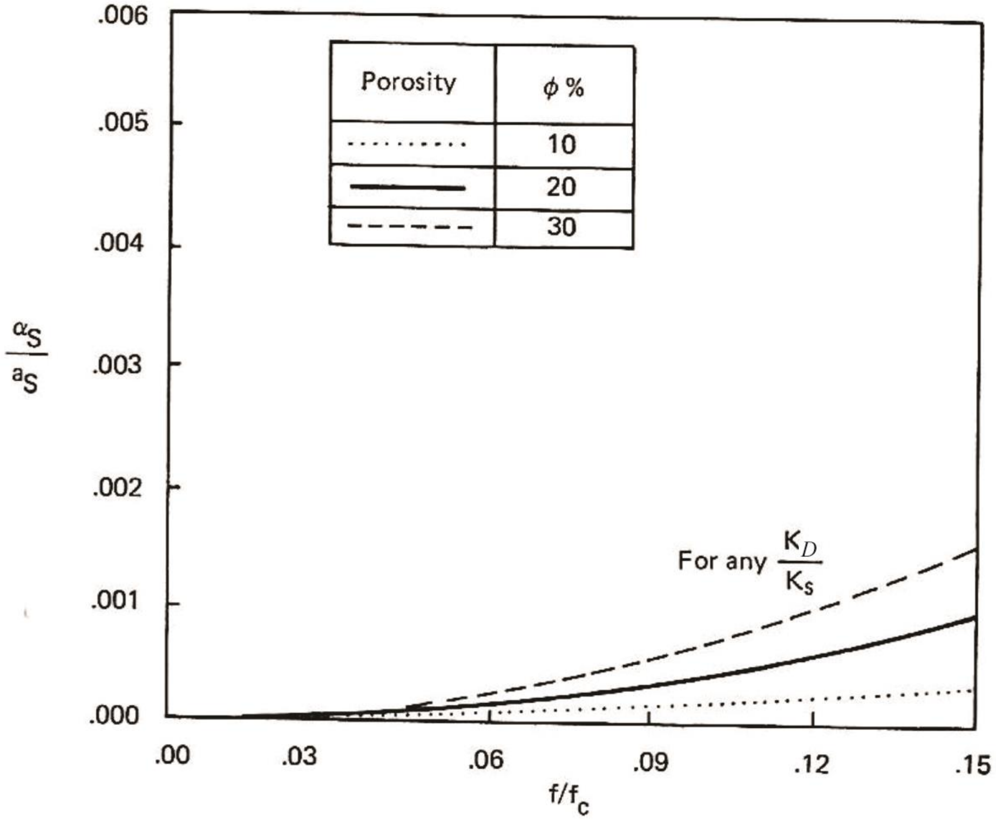
- Attenuation $\alpha(f)$ scaled by $\alpha_P = \frac{2\pi f_c}{V_P}$

S wave (only one)

a)



b)



- Attenuation $\alpha(f)$ scaled by $\alpha_S = \frac{2\pi f_c}{V_S}$

Extensions of Biot's model

- Consider all possible extensions of Biot's model of poroelasticity without additional porosities, nonlinearity, and anisotropy (which were sort of missed by Biot)
- These extensions are seen in the complete GLS model with $N = 2$:

$$\begin{cases} L = \frac{1}{2}\dot{\mathbf{u}}_i^T \boldsymbol{\rho} \dot{\mathbf{u}}_i - \left(\frac{1}{2}\mathbf{u}_i^T \boldsymbol{\zeta} \mathbf{u}_i + \frac{1}{2}\boldsymbol{\Delta}^T \mathbf{K} \boldsymbol{\Delta} + \tilde{\boldsymbol{\varepsilon}}_{ij}^T \boldsymbol{\mu} \tilde{\boldsymbol{\varepsilon}}_{ij} \right), \\ D = \frac{1}{2}\dot{\mathbf{u}}_i^T \mathbf{d} \dot{\mathbf{u}}_i + \left(\frac{1}{2}\dot{\boldsymbol{\Delta}}^T \boldsymbol{\eta}_K \dot{\boldsymbol{\Delta}} + \dot{\tilde{\boldsymbol{\varepsilon}}}_{ij}^T \boldsymbol{\eta}_\mu \dot{\tilde{\boldsymbol{\varepsilon}}}_{ij} \right). \end{cases}$$

Red shows terms omitted in Biot's model, or matrix $\boldsymbol{\mu}$ was selected in a too simple form

- Using $N > 2$ gives many models of multiple porosities (e.g., "squirt flow" models)
- Replacing the terms highlighted in red in the case with $N = 2$ also gives interesting extensions:
  - Including wettability (capillary forces) effects – elastic term for pore fluid: $\frac{1}{2}\mathbf{u}_i^T \boldsymbol{\zeta} \mathbf{u}_i$
    - Note that capillary forces are related not to strain ($\boldsymbol{\varepsilon}_{ij}$) but to relative displacement of the pore fluid (u_{2i})
  - Including viscosity of the frame and pore fluid – terms $\left(\frac{1}{2}\dot{\boldsymbol{\Delta}}^T \boldsymbol{\eta}_K \dot{\boldsymbol{\Delta}} + \dot{\tilde{\boldsymbol{\varepsilon}}}_{ij}^T \boldsymbol{\eta}_\mu \dot{\tilde{\boldsymbol{\varepsilon}}}_{ij} \right)$
  - Partially solid saturation material (bitumen?) – should give a more complete structure of the shear modulus matrix $\boldsymbol{\mu}$