Lecture 7: Mechanical viscoelasticity

- In this lecture, my goal is to show what part of the viscoelastic model for rocks (with time-dependent interactions) can be represented by rigorous continuum mechanics
  - The answer to this question is: "mechanically-implementable" viscoelastic models (represented by spring-dashpot diagrams) can be modeled by mechanics
    - Such models are likely all that matter among the VE models
  - However, there also exist many other mechanical models which cannot be described by spring-dashpots diagrams or by the VE model

Lecture 7: Mechanical viscoelasticity

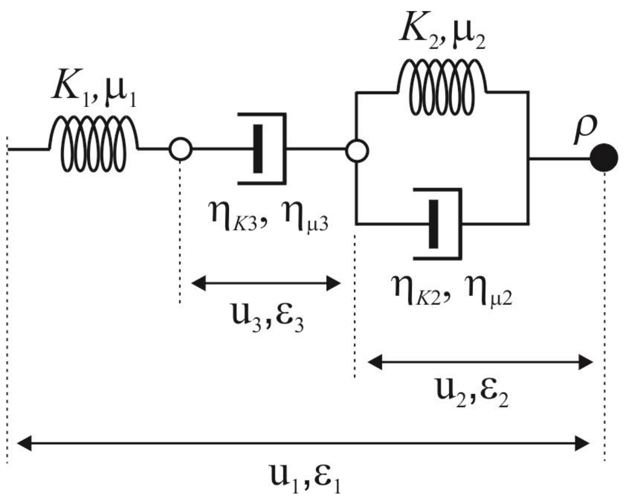
- A different meaning of spring-dashpot mechanical diagrams
- Lagrangian and dissipation functions for linear solids
- Generalized standard linear solid (GSLS)
- Extended GSLS
- Conclusion of the course

- Reading: Sections 5.5 – 5.7 in the text

Spring-dashpot diagrams

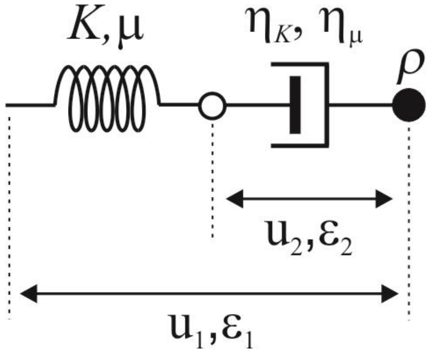
- In the viscoelastic (VE) theory spring-dashpot diagrams are used to illustrate mechanically-implementable stress-strain relations.
- Recall that VE diagrams are only designed to implement certain relations between time-dependent functions $\varepsilon(t) \leftrightarrow \sigma(t)$. Therefore, the VE mechanical diagrams are limited to a certain specific form:
  - They have a form of chains of elements with one pair of $\varepsilon(t)$ and $\sigma(t)$ measured at the ends
  - They contain only springs or dashpots which are related to stress and strain tensors
  - Different diagrams may correspond to the same $\varepsilon(t) \leftrightarrow \sigma(t)$ relation

- However, it is easier and unambiguous to view springs-dashpot diagrams as illustrations of the construction of the Lagrangian and dissipation functions
  - Each connector is an internal variable
  - Each element is an elastic term in L or viscous term in D
  - The end connector (black here) is the kinetic-energy term in L



Maxwell's body

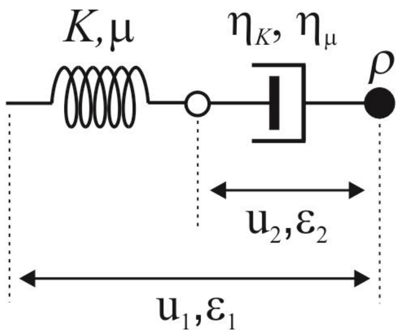
- Maxwell's body contains one internal variable, therefore $N = 1$
- There are three material-property constants:



$$\boldsymbol{\rho} = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix}$$

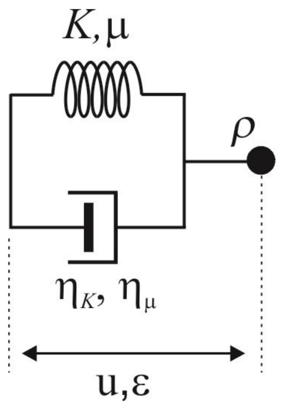
Maxwell's and Kelvin-Voigt's bodies

- Maxwell's body contains one internal variable, therefore $N = 2$
- There are three material-property constants:



$$\boldsymbol{\rho} = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix}$$

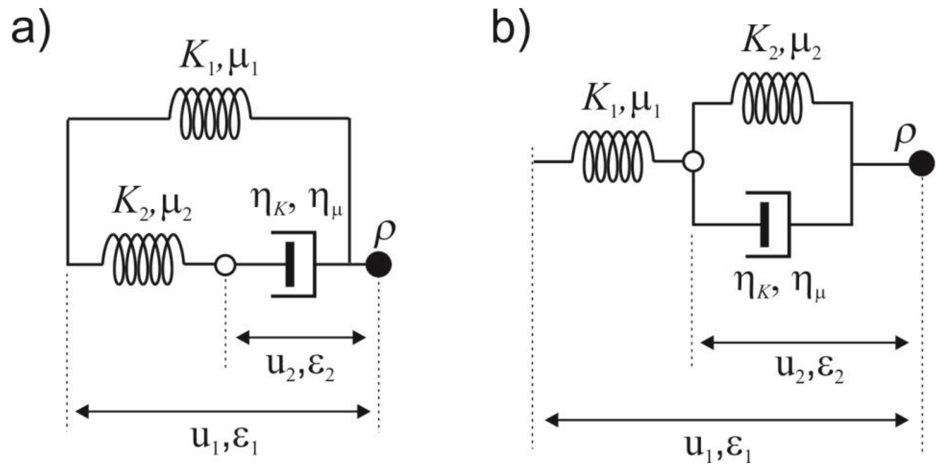
- For Kelvin-Voigt's body, $N = 1$, and also three material properties:



$$\boldsymbol{\rho} = [\rho] \qquad \mathbf{M} = [M] \qquad \boldsymbol{\eta} = [\eta]$$

Zener's body (standard linear solid, SLS)

- Two possible diagrams shown here
  - The difference is in the movement of the internal variable
- $N = 2$ and four material-property constants
- For model a) here:



$$\boldsymbol{\rho} = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} M_1 + M_2 & -M_2 \\ -M_2 & M_2 \end{bmatrix} \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix}$$

- For model b):

$$\mathbf{M}' = \begin{bmatrix} M_1' & -M_1' \\ -M_1' & M_1' + M_2' \end{bmatrix} \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 & 0 \\ 0 & \eta' \end{bmatrix}$$

See the text for hints to obtain relations between parameters of the two SLS models

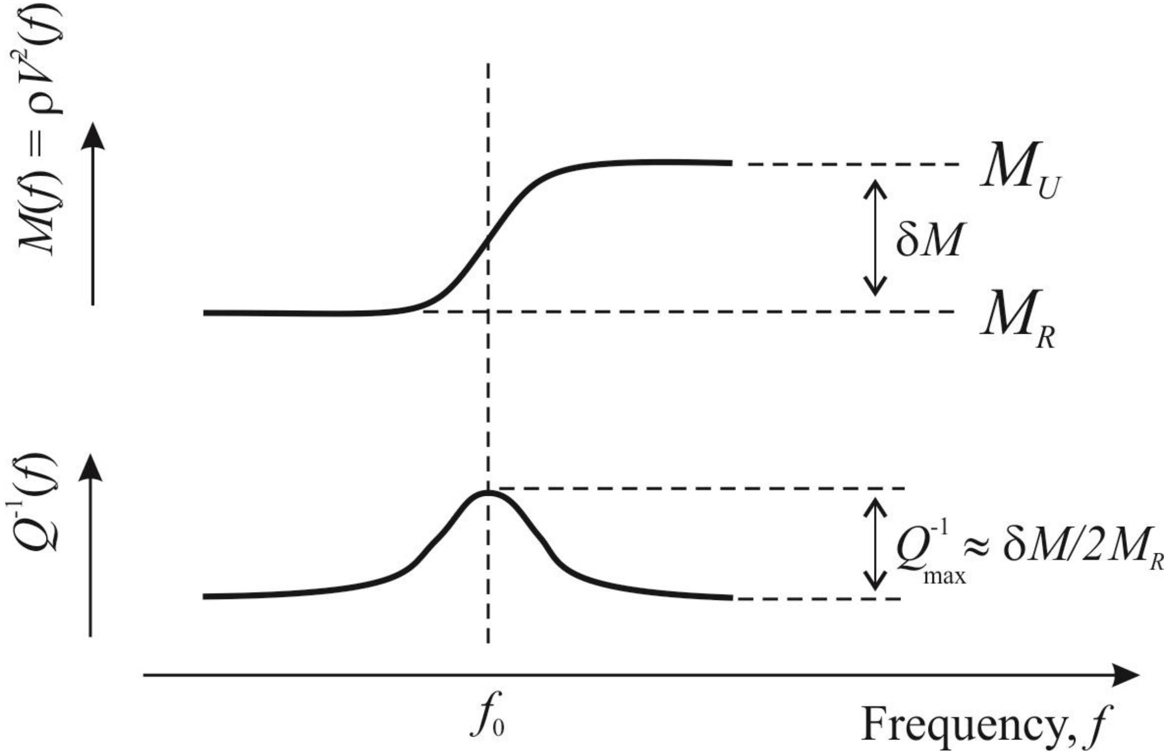
Wave velocity dispersion and attenuation for SLS)

- Eigenmode equation from Lecture #5: $\boldsymbol{\rho}\boldsymbol{\upsilon}^{(n)} = \gamma^{(n)}\mathbf{M}^{*}\boldsymbol{\upsilon}^{(n)}$
  gives one wave mode with a characteristic peak in $Q^{-1}(f)$ and increase in velocity from "relaxed" to "unrelaxed" level:

$$\mathbf{M}^{*} \equiv \mathbf{M} - i\omega\boldsymbol{\eta}_{M}$$

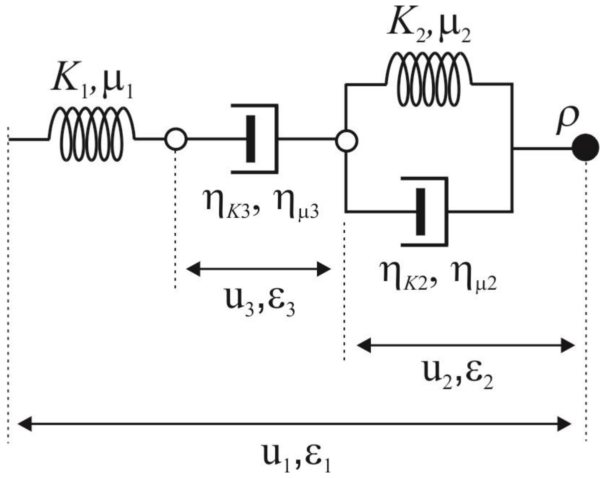
$$k^{*} \equiv k + i\alpha$$

$$\gamma \equiv \frac{\left(k^{*}\right)^{2}}{\omega^{2}}$$



Burgers' body

- For Burgers' body, $N = 3$ and three material-property constants:



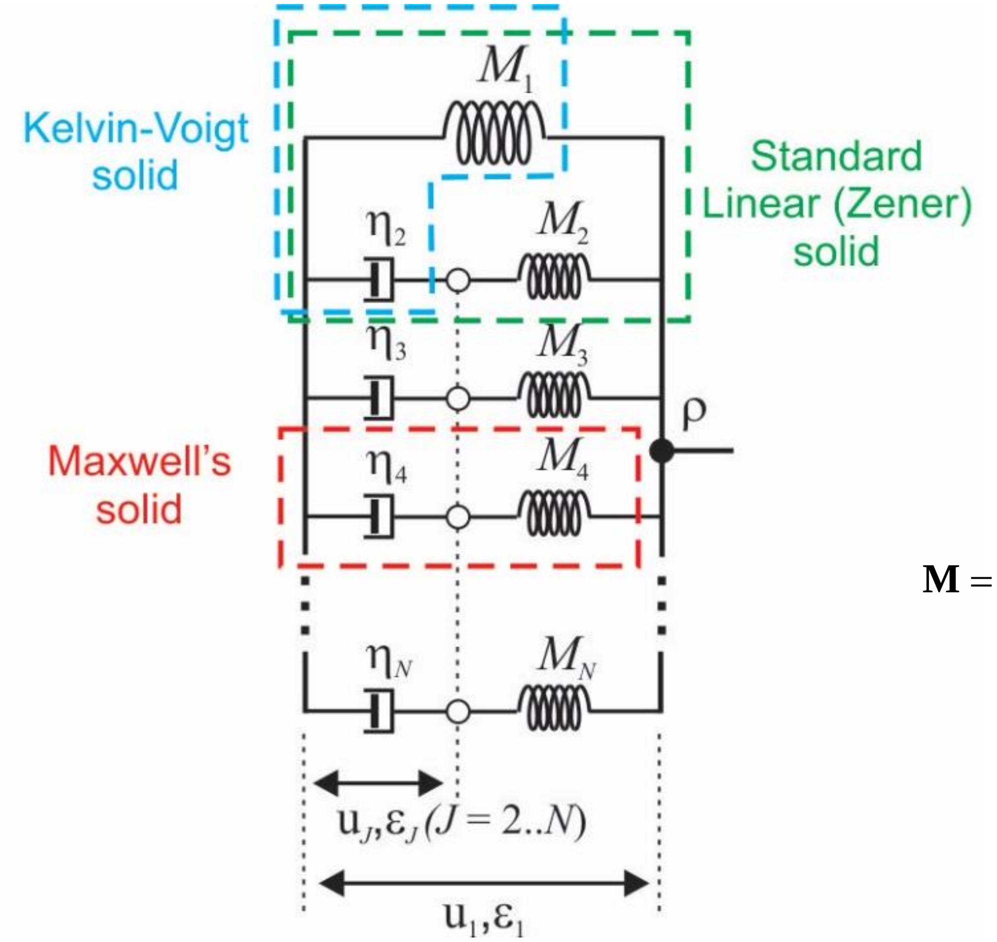
$$\boldsymbol{\rho} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} M_1 & -M_1 & -M_1 \\ -M_1 & M_1 + M_2 & -M_1 \\ -M_1 & -M_1 & M_1 \end{bmatrix}$$

$$\boldsymbol{\eta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{bmatrix}$$

Generalized Standard Linear (Zener) Solid (GSLS)

- The GSLS consists of an elastic element (term M_1 in matrix element M_{11}) and a series of Maxwell-type 4-element blocks for Maxwell chains



$$\boldsymbol{\rho} = \begin{bmatrix} \rho & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \sum_{J=1}^{N} M_J & -M_2 & -M_3 & \cdots & -M_N \\ -M_2 & M_2 & 0 & \cdots & 0 \\ -M_3 & 0 & M_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -M_N & 0 & 0 & \cdots & M_N \end{bmatrix}$$

$$\boldsymbol{\eta} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \eta_2 & 0 & \cdots & 0 \\ 0 & 0 & \eta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta_N \end{bmatrix}$$

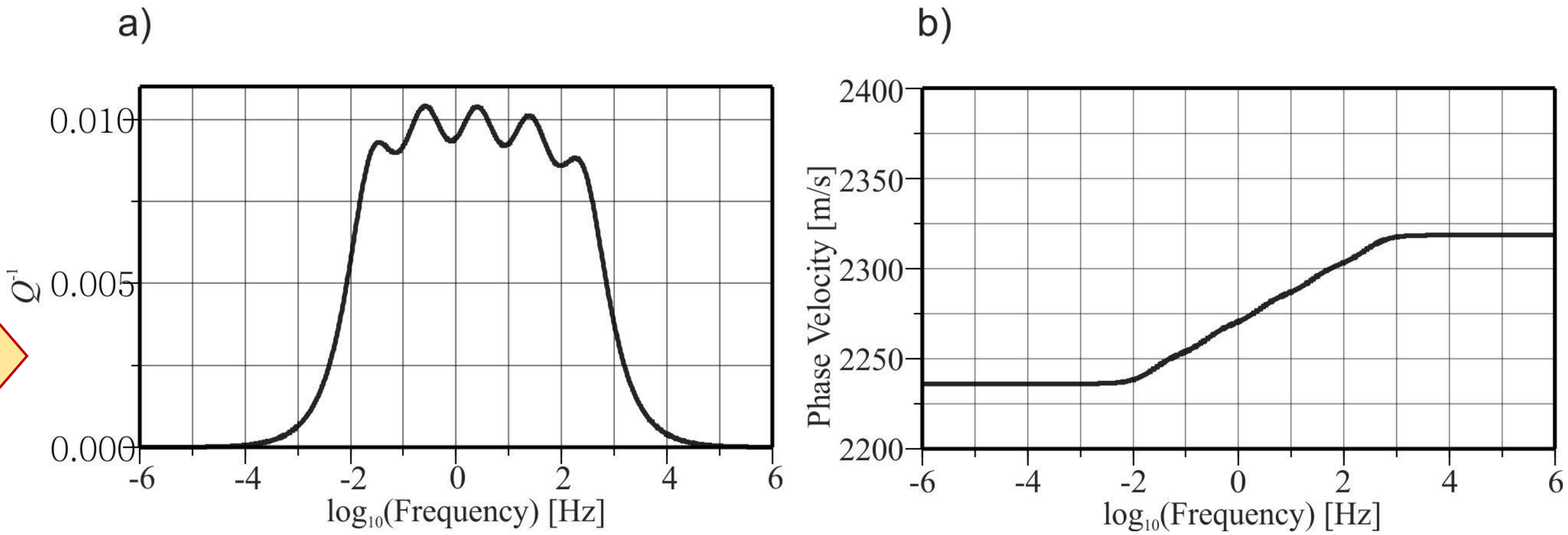
Waves in a GSLS medium

- Recall from Lecture #5 that plane waves of any kind are found by solving eigenvalue equation (with the appropriate type of modulus $\mathbf{M}$):

$$\boldsymbol{\rho}^{*}\mathbf{v}^{(n)} = \gamma^{(n)}\mathbf{M}^{*}\mathbf{v}^{(n)} \quad \text{, where} \quad \gamma \equiv \frac{\left(k^{*}\right)^{2}}{\omega^{2}}$$

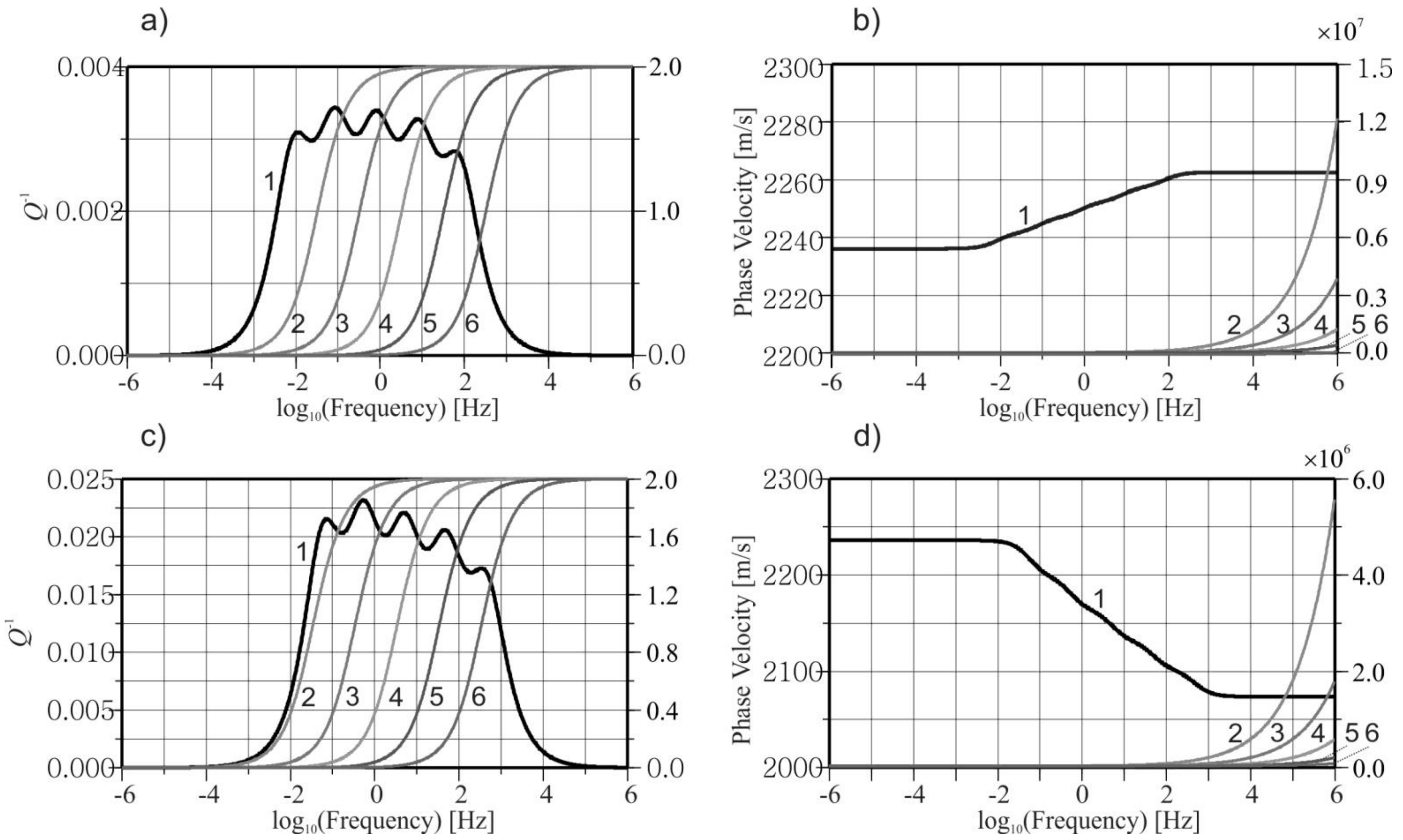
- Because GSLS contains zero mass densities for all internal variables, there is only one wave mode (with finite velocity). Its velocity and Q^{-1} are shown here:

GSLS is commonly used in waveform modeling software to produce such attenuation spectra



Waves in a GSLS medium

- If we consider nonzero densities for internal variables, additional wave modes appear (gray lines):



Internal densities equal 1% of the main one
Note the **Inverse dispersion** of primary mode (increase of velocity with frequency)

Internal densities equal 0.5% of the main one
Note the **Normal dispersion** of primary mode (decrease of velocity with frequency)

Extended GSLS

- Let us extend the GSLS by using lessons from Biot's theory of porous rock
- Compare elastic moduli matrices with $N = 2$ for a Zener's body (SLS) and Biot's model:

$$\mathbf{M} = \begin{bmatrix} M_1 + M_2 & -M_2 \\ -M_2 & M_2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} K_U & -\alpha M \\ -\alpha M & M \end{bmatrix}, \text{ where } \quad K_U = K_D + \alpha^2 M$$

  - In the SLS, M_1 is analogous to K_D, and M_2 is analogous to M
  - Biot's matrix contains one additional parameter α ($\phi \leq \alpha \leq 1$). For Zener's model, $\alpha = 1$.
- Therefore, it should be useful to generalize the GSLS by adding parameters α_J to Maxwell's chains
  - The model becomes more like bulk modulus of porous rock with multiple porosities:

$$\mathbf{M} = \begin{bmatrix} M_1 + \sum_{J=2}^{N} \alpha_J^2 M_J & -\alpha_2 M_2 & -\alpha_3 M_3 & \cdots & -\alpha_N M_N \\ -\alpha_2 M_2 & M_2 & 0 & \cdots & 0 \\ -\alpha_3 M_3 & 0 & M_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\alpha_N M_N & 0 & 0 & \cdots & M_N \end{bmatrix}$$

… and the same diagonal matrix η as for GSLS

… and matrix ρ can also be modified similar to poroelastic one…

This model cannot be drawn as a spring-dashpot diagram,
but it appears to correspond to rocks more accurately

Conclusion

- Returning to the question of our course:

"How does rock deformation work“?

- Answer:

  - It works by means of mechanics!

(also some thermodynamics, electrostatics, magnetics,… etc.)